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ENUMERATION OF 2-(15,7,6) DESIGNS WITH AUTOMORPHISMS OF ORDER 7 OR 5

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All nonisomorphic 2-(15,7,6) designs with automorphisms of order 7 or 5 are found, and the orders of their groups of automorphisms are determined. There are 180 nonisomorphic 2-(15,7,6) designs with automorphisms of order 7 and 29 with automorphisms of order 5.

1. Introduction. A 2-(v,k,λ) design is a collection of k-element subsets (*blocks*) of a set of v elements (*points*), such that each pair of points is contained in exactly λ blocks.

For the basic concepts and notations concerning combinatorial designs we refer for instance to [1], [2], [3], [6].

Let b denote the number of the blocks of the design, and r – the number of blocks in which a given point is contained. An incidence matrix of the design is a matrix of v rows and b columns which contains a 1 in the i-th row and j-th column if the i-th point is contained in the j-th block, and 0 if not.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence, i.e. if the incidence matrix of the first design can be obtained from the incidence matrix of the second one by permuting rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks. The set of all automorphisms of a design is a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Each 2-(v,k,λ) design determines a 2-$(v,k,m\lambda)$ design for every integer $m > 1$. These 2-$(v,k,m\lambda)$ designs are called *quasimultiples* of the 2-(v,k,λ) design. A quasimultiple 2-$(v,k,m\lambda)$ is *reducible* into m 2-(v,k,λ) designs if there is a partition of its blocks into m subcollections each of which forms a 2-(v,k,λ) design. For $m = 2$ quasimultiple designs are called *quasidoubles*, and the reducible quasidouble designs are called *doubles*.

The 2-(15,7,6) design is quasidouble of the 2-(15,7,3) design. According to computer estimations [5] there exist 5 nonisomorphic 2-(15,7,3) designs and at least 57810 nonisomorphic 2-(15,7,6) designs. It is of interest to classify the double 2-(15,7,6) designs and

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establish how their number depends on the automorphism groups of the constituent 2-(15,7,3) designs. The first step towards this classification implies the construction and study of 2-(15,7,6) designs with nontrivial automorphims.

The five 2-(15,7,3) designs have automorphism groups of orders 20160, 576, 96, 168, 168, respectively. They possess automorphisms of prime orders 7, 5, 3, and 2. A double design can have automorphisms of order 2 and automorphisms which preserve the two constituent designs (see for instance [4]). Thus to study the reducible 2-(15,7,6) designs, we have to start with the classification of reducible 2-(15,7,6) with automorphisms of orders 7, 5, 3, and 2.

This work deals with the classification of possibly reducible 2-(15,7,6) designs with the greatest possible automorphism orders, i.e. designs possessing automorphisms of order 7 or 5.

2. Designs with automorphisms of order 7. Let D be a 2-(15,7,6) design with an automorphism φ of order 7 fixing 1 point and 2 blocks. Without loss of generality we can assume that φ acts as follows:
$\varphi = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15)$ on the points,
$\varphi = (1,2,3,4,5,6,7)(8,9,10,11,12,13,14)\dots(22,23,24,25,26,27,28)(29)(30)$ on the blocks. Then, there are two possibilities for the incidence matrix A of D:

case 1)
$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & Z^T & Z^T \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & U^T & U^T \\ Z & Z & U & U & 0 & 0 \end{pmatrix}$$

and

case 2)
$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & U^T & Z^T \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & Z^T & U^T \\ Z & Z & U & U & 0 & 0 \end{pmatrix},$$

where $A_{i,j}$, $i=1,2$, $j=1,2,3,4$ are circulant matrices of order 7, $U=(1,1,1,1,1,1,1)$, $Z=(0,0,0,0,0,0,0)$.

Let $m_{i,j}$, $i=1,2$, $j=1,2,3,4$ be equal to the number of 1's in a row of $A_{i,j}$. The following equations hold for the two possibilities for the matrix $M=(m_{i,j})_{2\times 4}$

case 1)

(1) $$\sum_{j=1}^{4} m_{1,j} = 14, \qquad \sum_{j=1}^{4} m_{2,j} = 12$$

(2) $$\sum_{j=1}^{4} m_{1,j}^2 = 50, \qquad \sum_{j=1}^{4} m_{2,j}^2 = 36$$

(3) $$\sum_{j=1}^{4} m_{1,j}m_{2,j} = 42.$$

Thus
$$M = \begin{pmatrix} 4 & 4 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

After replacement with circulants in the above given matrix and addition of the fixed point and blocks, 88 nonisomorphic 2-(15,7,6) designs are obtained, and the orders of their automorphism groups are computed. Only one of these designs possesses an automorphism of order 5. The number of reducible designs among them is 32. The results are illustrated in Table 1.

Table 1. Order of the automorphism group and reducibility of designs with automorphisms of order 7 – case 1

Aut. group of order	7	14	21	42	56	168	20160	All
Designs	67	6	4	2	4	4	1	88
Reducible Designs	11	6	4	2	4	4	1	32

case 2)

(4) $$\sum_{j=1}^{4} m_{i,j} = 13, \qquad i = 1, 2$$

(5) $$\sum_{j=1}^{4} m_{i,j}^2 = 43, \qquad i = 1, 2$$

(6) $$\sum_{j=1}^{4} m_{1,j} m_{2,j} = 42.$$

Thus

$$M = \begin{pmatrix} 4 & 3 & 3 & 3 \\ 3 & 4 & 3 & 3 \end{pmatrix}.$$

After replacement with circulants in this matrix and addition of the fixed point and blocks, 92 nonisomorphic 2-(15,7,6) designs are obtained. None of these designs possess automorphisms of order 5. The number of reducible designs among them is 32. These results are presented in Table 2.

Table 2. Order of the automorphism group and reducibility of designs with automorphisms of order 7 – case 2

Aut. group of order	7	14	21	42	336	2688	All
Designs	67	14	5	3	2	1	92
Reducible Designs	15	6	5	3	2	1	32

3. Designs with automorphisms of order 5. Let D be a 2-(15,7,6) design with an automorphism α of order 5 without fixed points and blocks. Without loss of generality we can assume that α acts as follows:

$\alpha = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10)(11, 12, 13, 14, 15)$ on the points, and

$\alpha = (1, 2, 3, 4, 5)(6, 7, 8, 9, 10) \ldots (26, 27, 28, 29, 30)$ on the blocks.

Then, the incidence matrix of D is:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} & A_{1,5} & A_{1,6} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} & A_{2,5} & A_{2,6} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} & A_{3,5} & A_{3,6} \end{pmatrix},$$

where $A_{i,j}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, 6$, are circulant matrices of order 5.

Let $m_{i,j}$, $i = 1, 2, 3$, $j = 1, 2, \ldots, 6$, be equal to the number of 1's in a row of $A_{i,j}$. The following equations hold for the matrix $M = (m_{i,j})_{3\times 6}$

$$\sum_{j=1}^{6} m_{i,j} = 14, \qquad \sum_{j=1}^{6} m_{i,j}^2 = 38, \quad i = 1, 2 \tag{7}$$

$$\sum_{j=1}^{6} m_{i_1,j} m_{i_2,j} = 30, \qquad 1 \le i_1 < i_2 \le 3. \tag{8}$$

It was found by computer that up to equivalence there is one matrix for which the upper equations hold,

$$M = \begin{pmatrix} 3 & 3 & 3 & 3 & 1 & 1 \\ 3 & 3 & 1 & 1 & 3 & 3 \\ 1 & 1 & 3 & 3 & 3 & 3 \end{pmatrix}.$$

After replacement with circulants in the above given matrix 29 nonisomorphic 2-(15,7,6) designs are obtained. Only one of them has an automorphism of order 7, and was already obtained in section 2, case 1. There are 6 reducible designs. The results are presented in Table 3.

Table 3. Order of the automorphism group and reducibility of designs with automorphisms of order 5

Aut. group of order	5	10	15	20	30	120	20160	All
Designs	9	14	2	1	1	1	1	29
Reducible Designs		4				1	1	6

4. Classification results. Among all the 208 designs with automorphisms of orders 5 or 7, 70 are reducible, and the data about them is of interest as a step towards the full classification of all the reducible 2-(15,7,6). As it was expected, the designs with the biggest automorphism groups are reducible.

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КЛАСИФИКАЦИЯ НА 2-(15,7,6) ДИЗАЙНИ С АВТОМОРФИЗМИ ОТ РЕД 7 ИЛИ 5

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Конструирани са всички неизоморфни 2-(15,7,6) дизайни с автоморфизми от ред 7 или 5 и е определен реда на техните групи от автоморфизми. Установено е, че броят на неизоморфните 2-(15,7,6) дизайни с автоморфизми от ред 7 е 180, а на тези с автоморфизми от ред 5 е 29. Сред тях има 70 дизайна, които са разложими на два 2-(15,7,3) дизайна. Компютърните резултати са получени независимо и с различни програми от авторите.