МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
*Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006*

RANDOMLY INDEXED GALTON-WATSON BRANCHING PROCESSES

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A subcritical Galton-Watson branching process with random time is defined. The asymptotic formulas for the first two factorial moments and for the probability of non-extinction are proved. A non-degenerate limit distribution is also obtained.

1. Introduction. A randomly indexed branching process was introduced by T. Epps [2] for modeling of stock prices. He considered Galton-Watson branching processes with four particular distributions of the offspring of a particle, indexed by a Poisson process. The results obtained there are restricted to the asymptotic of the moments needed for the model.

In the present note we consider subcritical case. We have found the asymptotic formulas for the moments in some more general setting, namely, when the distribution of the offspring is arbitrary, and the indexing process is an ordinary renewal process. The asymptotic of the probability for non-extinction and the limit theorem are proved under the condition when the offspring distribution is geometric one.

2. Definition. Let on the probability space $(\Omega, \mathcal{A}, \Pr)$ be given:

1. The Galton-Watson branching process

$$Z_0 = 1, \text{ a.s. }, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_i(n+1), \quad n = 0, 1, 2, \dots, \tag{1}$$

with pgf of the offspring of a particle

$$f(s) = E[s^{X_i(n)}] = \sum_{k=0}^{\infty} p_k s^k, \quad s \in [0, 1]. \tag{2}$$

2. The renewal process $(N(t), t \geq 0)$ with cdf $F(t) = \Pr(T_n \leq t)$ of interarrival times T_n.

The processes $Z_n, n = 0, 1, \dots$ and $N(t), t \geq 0$ are independent.

Definition 1. *A continuous time branching process* $Z(t), t \geq 0$ *is defined by*

$$Z(0) = Z_0, \quad Z(t) = Z_{N(t)}, \quad t > 0. \tag{3}$$

Further, we will suppose that the following conditions hold:

Condition 1. *The offspring probability generating function (pgf) satisfies*

$$0 < m = f'(1) < \infty, \quad 0 < b = f''(1) < \infty.$$

Condition 2. *The cumulative distribution function (cdf) $F(t)$ is continuous $F(0+) = 0$, and*

$$0 < \mu = \int_0^\infty x dF(x) < \infty \quad 0 < \nu^2 = \int_0^\infty (x-\mu)^2 dF(x) < \infty.$$

The pgf $f_n(s)$ of Z_n is the n-fold iteration of $f(s)$, i.e. $f_n(s) = f(f_{n-1}(s))$, $f_1(s) = f(s)$, $f_0(s) = s$.

Denote by $H(t) = E[N(t)] = \sum_{n=0}^\infty F^{n*}(t), \quad t \geq 0$ the renewal function and by $P_k(t) = \Pr(N(t) = k), k = 0, 1, 2, \ldots$. Denote also the pgf of the process $N(t)$ by $\Psi(t;s) = E[s^{N(t)}] = \sum_{k=0}^\infty P_k(t)s^k$.

By the independence of $Z_n, n \geq 0$ and $N(t), t \geq 0$, it is easy to see that the pgf of the process $Z(t), t \geq 0$ has the form

$$\Phi(t;s) = E[s^{Z(t)}|Z(0) = 1] = \sum_{k=0}^\infty P_k(t)E[s^{Z_k}|Z_0 = 1] = \sum_{k=0}^\infty P_k(t)f_k(s). \tag{4}$$

Differentiating (4) and setting $s = 1$ yields

$$\Phi'_s(t;1) = \sum_{k=0}^\infty P_k(t)f'_k(1), \quad \Phi''_{ss}(t;1) = \sum_{k=0}^\infty P_k(t)f''_k(1).$$

Using the well-known formulas for the first two moments of Z_n (see e.g. [3], p. 45) after simple calculations one can see that

$$M(t) = E[Z(t)|Z(0) = 1] = \begin{cases} \Psi(t;m), & m \neq 1, \\ 1, & m = 1 \end{cases} \tag{5}$$

and

$$B(t) = E[Z(t)(Z(t)-1)] = \begin{cases} \dfrac{b}{m(1-m)}(\Psi(t;m) - \Psi(t;m^2)), & m \neq 1, \\ bH(t), & m = 1. \end{cases} \tag{6}$$

Definition 2. *The process $Z(t)$, $t \geq 0$ is said to be* **subcritical (critical, supercritical)**, *when $m < 1$ ($m = 1$, $m > 1$).*

Condition 3. *To the end of the paper we will assume that the process is subcritical, i.e. $0 < m < 1$.*

3. Moments and probability of extinction. In this section we will prove the asymptotic formulas for the first two factorial moments $M(t)$ and $B(t)$, and for the probability for non-extinction $\Pr(Z(t) > 0 | Z(0) = 1)$. For this we need the following lemma.

Lemma 1. *For $s \in (0, R)$ the following representations hold*

$$\Psi(t; s) = [1 - (1-s)H_s(t)]/s, \tag{7}$$

$$\Psi(t; s) = (1-s)\sum_{n=1}^{\infty} s^{n-1}(1 - F^{*n}(t)), \quad s \in [0, 1), \tag{8}$$

where $R > 0$ is the radius of convergence of $\Psi(t, s)$ and $H_s(t) = \sum_{k=0}^{\infty} s^k F^{k*}(t)$ is a renewal function generated by (possibly improper) cdf's $F(t)$.

The proof of the lemma is simple but long and we omit it.

Condition 4. *The solutions ζ_1 and ζ_2 of the equations*

$$m\int_0^{\infty} e^{-\zeta_1 t} dF(t) = 1, \quad m^2\int_0^{\infty} e^{-\zeta_2 t} dF(t) = 1, \tag{9}$$

exist.

Obviously, under the Conditions 3 and 4 both ζ_1, ζ_2 must be negative and also $0 < -\zeta_1 < -\zeta_2$.

Theorem 1. *Let the conditions $1-4$ be satisfied.*
1. *If for the cdf $F_1(t) = m\int_0^t e^{-\zeta_1 u} dF(u)$ there exists*

$$\mu_1 = \int_0^{\infty} t dF_1(t) = m\int_0^{\infty} te^{-\zeta_1 t} dF(t) < \infty,$$

then

$$M(t) \sim \frac{1-m}{m|\zeta_1|\mu_1} e^{\zeta_1 t}, \ t \to \infty. \tag{10}$$

2. *If for the cdf $F_2(t) = m^2\int_0^t e^{-\zeta_2 u} dF(u)$ there exists*

$$\mu_2 = \int_0^{\infty} t dF_2(t) = m^2\int_0^{\infty} te^{-\zeta_2 t} dF(t) < \infty,$$

then

$$B(t) \sim \frac{b}{m^2|\zeta_1|\mu_1} e^{\zeta_1 t}, \ t \to \infty. \tag{11}$$

Proof. From (5) and (7) it follows that

$$M(t) = \Psi(t; m) = [1 - (1-m)H_m(t)]/m. \tag{12}$$

Since $mF(t) \to m < 1$, $t \to \infty$, then $H_m(t) \to (1-m)^{-1}$, $t \to \infty$. Now from [4], Theorem 2, p. 427, formulas (6.3) and (6.16), under the conditions of the theorem we obtain

$$1-(1-m)H_m(t) \sim \frac{1-m}{|\zeta_1|\mu_1} e^{\zeta_1 t}, \quad t \to \infty,$$

which, together with (7) yields

$$\Psi(t;m) \sim \frac{1-m}{m|\zeta_1|\mu_1} e^{\zeta_1 t}, \quad t \to \infty. \tag{13}$$

The last relation and (12) compelte the proof of (10). The proof of (11) is similar, because under the conditions of the theorem

$$1-(1-m^2)H_{m^2}(t) \sim \frac{1-m^2}{|\zeta_2|\mu_2} e^{\zeta_2 t}, \quad t \to \infty.$$

Therefore (see (7)),

$$\Psi(t,m^2) \sim \frac{1-m^2}{m^2|\zeta_2|\mu_2} e^{\zeta_2 t}, \quad t \to \infty. \tag{14}$$

Combining the last relation, (13) and (6) and taking in view that $0 < -\zeta_1 < -\zeta_2$, we complete the proof of (11).

Definition 3. *The cdf $F(x)$ is said to be subexponential if*

$$\lim_{t\to\infty} \frac{1-F^{*n}(t)}{1-F(t)} \to n, \quad t \to \infty.$$

It is known that for such functions Condition 4 (9) is not satisfied.

Theorem 2. *Assume Conditions $1-3$. Suppose also that the cdf $F(t)$ is subexponential and $\int_0^\infty t dF(t) < \infty$. Then, as $t \to \infty$,*

$$M(t) \sim \frac{m(1-F(t))}{1-m}, \quad B(t) \sim \frac{b(1-F(t))}{(1-m)^2(1+m)}.$$

Proof. From the representation (8) it can be seen that $\Psi(t,s)$ is the tail of the distribution of a random sum of independent and identically distributed (iid) random variables with the cdf $F(t)$, where the number of summands has the geometric distribution $(1-s)s^{k-1}, k = 1, 2, \ldots$. Now the proof follows from the well-known result of Embrechts and Veraverbeke [1], which gives

$$\Psi(t;s) \sim \frac{s}{1-s}(1-F(t)), \quad t \to \infty.$$

The last relation and the equations (5) and (6) prove the theorem.

Remark 1. *The above theorems show that the first two factorial moments of the process $Z(t)$ have asymptotic behavior similar to that of the simple Galton-Watson branching process in case when the constants ζ_1 and ζ_2, exist, i.e. when the tail of $F(t)$*

decreases exponentially. On the other hand, in case of subexponential cdf the asymptotic behavior is different. Thus, if $F(t)$ has regularly varying tail, i.e. $1-F(t)\sim t^{-\alpha}L(t), t\to\infty$, with $\alpha\in(1,2)$, Theorem 2 gives

$$M(t)\sim\frac{m}{1-m}t^{-\alpha}L(t),\quad B(t)\sim\frac{b}{(1-m)^2(1+m)}t^{-\alpha}L(t),\quad t\to\infty.$$

Theorem 3. *Assume the Conditions of Theorem 1. If additionally*

$$f(s)=1-\frac{m(1-s)}{1+\dfrac{b}{2m}(1-s)},\tag{15}$$

then, as $t\to\infty$,

$$\Pr(Z(t)>0|Z(0)=1)\sim\frac{2(1-m)^2}{[b+2m(1-m)]\zeta_1\mu_1}e^{\zeta_1 t}+O(e^{\zeta_2 t}).\tag{16}$$

Proof. It is not difficult to check that if (15) holds then the n-fold iteration has the same form,

$$1-f_n(s)=\frac{m^n(1-s)}{1+\dfrac{b}{2m}\dfrac{1-m^n}{1-m}(1-s)}.$$

Hence

$$\Pr(Z_n>0|Z_0=1)=1-f_n(0)=\frac{m^n}{1+\dfrac{b}{2m}\dfrac{1-m^n}{1-m}}.\tag{17}$$

Using (17) one obtains

$$\begin{aligned}&\Pr(Z(t)>0|Z(0)=1)=1-\Phi(t,0)\\ =&\sum_{n=0}^{\infty}P_n(t)(1-f_n(0))=\sum_{n=0}^{\infty}P_n(t)\frac{m^n}{1+\dfrac{b}{2m}\dfrac{1-m^n}{1-m}}.\end{aligned}$$

Further we have

$$\begin{aligned}&\frac{m^n}{1+b(1-m^n)/(2m(1-m))}-\frac{m^n}{1+b/(2m(1-m))}\\ &=\frac{2bm(1-m)}{[b+2m(1-m)][2m(1-m)+b-bm^n]}m^{2n}.\end{aligned}$$

Hence

$$\begin{aligned}\Pr(Z(t)>0|Z(0)=1)&=\frac{2m(1-m)}{b+2m(1-m)}\Psi(t;m)\\ +&\sum_{n=0}^{\infty}P_n(t)\frac{2bm(1-m)}{[b+2m(1-m)][2m(1-m)+b-bm^n]}m^{2n}.\end{aligned}\tag{18}$$

From (13) (Theorem 1), it follows that

$$\frac{2m(1-m)}{b+2m(1-m)}\Psi(t;m) \sim \frac{2m(1-m)}{b+2m(1-m)}\frac{1-m}{m|\zeta_1|\mu_1}e^{\zeta_1 t}, \quad t\to\infty. \tag{19}$$

For the second sum in (18) we obtain (see (14))

$$\begin{aligned}
&\sum_{n=0}^{\infty} P_n(t)\frac{2bm(1-m)}{[b+2m(1-m)][2m(1-m)+b-bm^n]}m^{2n}\\
\leq\ & \frac{2bm(1-m)}{[b+2m(1-m)][2m(1-m)]}\Psi(t;m^2) \sim \frac{1-m^2}{m^2|\zeta_2|\mu_2}e^{\zeta_2 t}, t\to\infty.
\end{aligned}$$

Combining the last relation and (19) and recalling that $0>\zeta_1>\zeta_2$, we complete the proof of the theorem. □

4. Limit theorem. In this section we prove a limit theorem for the process.

Theorem 4. *Let the conditions* $1-4$ *hold and the pgf* $f(s)$ *has the form* (15). *Then*

$$\lim_{t\to\infty} E(s^{Z(t)}|Z(t)>0) = \frac{R(1-s)+s}{R(1-s)+1} =: \phi(s).$$

where $R = b(2m(1-m))^{-1}$.

Proof. For the conditional pgf $E[s^{Z(t)}|Z(t)>0]$ one has $E(s^{Z(t)}|Z(t)>0) = 1-(1-\Phi(t,s))(1-\Phi(t,0))^{-1}$. Following the same way as in the proof of Theorem 3 it is not difficult to prove that, as $t\to\infty$,

$$1-\Phi(t,s) = \sum_{n=0}^{\infty} P_n(t)\frac{m^n(1-s)}{1+\dfrac{b}{2m}\dfrac{1-m^n}{1-m}(1-s)} \sim \frac{(1-s)(1-\Phi(t,0))}{1+\dfrac{b(1-s)}{2m(1-m)}},$$

for any fixed $s\in[0,1]$. Therefore,

$$\lim_{t\to\infty} E(s^{Z(t)}|Z(t)>0) = 1-\frac{1-s}{1+\dfrac{b(1-s)}{2m(1-m)}} = \frac{\dfrac{b(1-s)}{2m(1-m)}+s}{\dfrac{b(1-s)}{2m(1-m)}+1}.$$

This completes the proof of the theorem. It is not difficult to check that the pgf $\phi(s)$ satisfies the equation $m(1-\phi(s)) = 1-\phi(f(s))$, which characterizes the limiting distribution of the simple Galton-Watson branching process in the subcritical case.

5. Conclusion remarks. The critical and supercritical of the process and the applications for option pricing are under consideration. The work is partially supported by the NFSI contract No. VU-MI-105/2005.

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ПРОЦЕСИ НА ГАЛТОН-УОСЪН СЪС СЛУЧАЕН ИНДЕКС

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В доклада се дефинира разклоняващ се процес на Галтон-Уотсън със случаен индекс. Получени са асимптотични формули за първите два факториални момента и за вероятността за неизраждане. Получено е също неизродено гранично разпределение.