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COMPACTNESS OF THE PARETO SETS IN MULTI-OBJECTIVE OPTIMIZATION WITH QUASI-CONCAVE FUNCTIONS*

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In this paper we consider Pareto-optimal and Pareto-front sets in multi-objective optimization with several objective functions, and convex and compact feasible set. It is proved that there exists an upper semi-continuous mapping from the feasible set into the Pareto-optimal set, as well as the compactness of the Pareto sets, if the objective functions are continuous and quasi-concave is established.

1. Introduction. In a general form, the multi-objective optimization problem $MOP\,(X,F)$ is to find $x \in X \subset R^m$, $m \geq 1$, so as to maximize $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))$, subject to $x \in X$, provided the feasible set X is nonempty and compact, $J = \{1, 2, \ldots, n\}$ is the index set, $n \geq 2$, $f_j : X \to R$ is a given continuous objective function for all $j \in J$.

Definitions of the Pareto-optimal solutions can be formally stated as follows:

(a) A point $x \in X$ is called Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i\,(y) \geq f_i\,(x)$ for all $i \in J$ and $f_k\,(y) > f_k\,(x)$ for some $k \in J$. The set of the Pareto-optimal solutions of X is denoted by $Max\,(X,F)$ and it is called Pareto-optimal set. The set $F\,(Max\,(X,F)) = Eff\,(F\,(X))$ is called Pareto-front set or efficient set.

(b) A point $x \in X$ is called weakly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i\,(y) > f_i\,(x)$ for all $i \in J$. The set of the weakly Pareto-optimal solutions of X is denoted by $WMax\,(X,F)$ and it is called weakly Pareto-optimal set. The set $F\,(WMax\,(X,F)) = WEff\,(F\,(X))$ is called weakly Pareto-front set or weakly efficient set.

One of the most important $MOP\,(X,F)$ is the investigation of the compactness of the Pareto-optimal and Pareto-front sets (Pareto sets).

As it is well known, the Pareto-optimal set $Max\,(X,F)$ is nonempty, the weakly Pareto-optimal set $WMax\,(X,F)$ is a nonempty compact set $Max\,(X,F) \subset WMax\,(X,F)$ and $Eff\,(F\,(X)) \subset WEff\,(F\,(X))$, see [2] and [5, Theorem 5]. It can be shown that both sets $Eff\,(F\,(X))$ and $WEff\,(F\,(X))$ lie on the boundary of the set $F\,(X)$.

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Let X be convex and $i \in J$. A function f_i is quasi-concave on X if and only if for any $x, y \in X$ and $t \in [0;1]$, $f_i(tx + (1-t)y) \geq \min(f_i(x), f_i(y))$. A function f_i is strictly quasi-concave on X if and only if for any $x, y \in X$, $x \neq y$ and $t \in (0;1)$, $f_i(tx + (1-t)y) > \min(f_i(x), f_i(y))$. A function f_i is concave on X if and only if for any $x, y \in X$ and $t \in [0;1]$, $f_i(tx + (1-t)y) \geq tf_i(x) + (1-t)f_i(y)$.

It is also known that if X is convex and the functions $\{f_j\}_{j=1}^n$ are strictly quasi-concave on X, then $Max(X,F) = WMax(X,F)$, see [2] and [5, Theorem 3]. Then, under these assumptions the Pareto-optimal set $Max(X,F)$ is compact.

The aim of this paper is to prove that:

– There exists an upper semi-continuous point-to-set mapping $\varphi : X \rightrightarrows Max(X,F)$ such that $\varphi(X) = Max(X,F)$;

– If the functions $\{f_j\}_{j=1}^n$ are quasi-concave on the convex set X, then the sets $Max(X,F)$ and $Eff(F(X))$ are compact.

2. The main result. In this section, let the functions $\{f_j\}_{j=1}^n$ be quasi-concave on the convex set X.

Now, under these assumptions we discuss the compactness of the Pareto-optimal and Pareto-front sets.

For fixed $x \in X$ and $i \in J$, let $R_i(x) = \{y \in X | f_i(y) \geq f_i(x)\}$. It is easy to check that the sets $\{R_j(x)\}_{j=1}^n$ are nonempty, convex and compact subset of X. This allows us to define the point-to-set mapping $\rho : X \rightrightarrows X$ by $\rho(x) = \left\{y \in X | y \in \bigcap_{j=1}^n R_j(x)\right\}$ for all $x \in X$. It can be shown that $\rho(x)$ is a nonempty, convex and compact set for all $x \in X$ and there is $x \in \rho(x)$. Hence, the point-to-set mapping ρ is convex-valued and compact-valued on X.

Define the function $f : X \to R$ by $f(x) = \sum_{j=1}^n f_j(x)$ for all $x \in X$. It is easy to show that the function f is continuous on X and $Arg\max(X,f) \subset Max(X,F)$.

Theorem. *There exists an upper semi-continuous point-to-set mapping* $\varphi : X \rightrightarrows Max(X,F)$ *such that* $\varphi(x) = Arg\max\left(f, \bigcap_{j=1}^n R_j(x)\right)$ *for all* $x \in X$ *and* $\varphi(X) = Max(X,F)$.

At first, we prove some lemmas.

Lemma 1. *If* $\{x_k\}_{k=1}^\infty$,$\{y_k\}_{k=1}^\infty \subset X$ *is a pair of sequences such that* $\lim\limits_{k\to\infty} x_k = x_0 \in X$ *and* $y_k \in \rho(x_k)$ *for all* $k \in N$, *then there exists a convergent subsequence of* $\{y_k\}_{k=1}^\infty$ *whose limit belongs to* $\rho(x_0)$.

Proof. The assumption $y_k \in \rho(x_k)$ for all $k \in N$ implies $f_i(y_k) \geq f_i(x_k)$ for all $k \in N$ and all $i \in J$. From the condition $\{y_k\}_{k=1}^\infty \subset X$ it follows that there exists a convergent subsequence $\{y'_k\}_{k=1}^\infty \subset \{y_k\}_{k=1}^\infty$ such that $\lim\limits_{k\to\infty} y'_k = y_0 \in X$. Therefore, there exists a convergent subsequence $\{x'_k\}_{k=1}^\infty \subset \{x_k\}_{k=1}^\infty$ such that $y'_k \in \rho(x'_k)$ and $\lim\limits_{k\to\infty} x'_k = x_0$. Thus, we have that $f_i(y'_k) \geq f_i(x'_k)$ for all $k \in N$ and for all $i \in J$. Taking the limit as $k \to \infty$, we obtain $f_i(y_0) \geq f_i(x_0)$ for all $i \in J$. This implies$y_o \in \rho(x_0)$. The lemma is proved.

Lemma 2. *If the sequence* $\{x_k\}_{k=1}^\infty \subset X$ *converges to* $x_0 \in X$ *and* $y_0 \in \rho(x_0)$, *then there exists a sequence* $\{y_k\}_{k=1}^\infty \subset X$ *such that* $y_k \in \rho(x_k)$ *for all* $k \in N$ *and* $\lim\limits_{k\to\infty} y_k = y_0$.

Proof. Let denote the distance between y_0 and $x \in X$ by $dis\,(y_0, x)$ and the distance between y_0 and $\rho\,(x_k)$ by $d_k = \inf\,\{dis\,(y_0, x)\,|x \in \rho\,(x_k)\}$. By the hypothesis that the set $\rho\,(x_k)$ is nonempty, convex and compact it follows that if $y_0 \notin \rho\,(x_k)$, then there exists unique $\bar{y} \in \rho\,(x_k)$ such that $d_k = d\,(\bar{y}, y_k)$.

It is obvious that there are two cases as follows:

Firstly, if $y_0 \in \rho\,(x_k)$, then $d_k = 0$ and let $y_k = y_0$.

Secondly, if $y_0 \notin \rho\,(x_k)$, then $d_k > 0$ and let $y_k = \bar{y}$.

Finally, we obtain a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \rho\,(x_k)$ for all $k \in N$ and $d_k = dis\,(y_0, y_k)$. Further, $\lim\limits_{k\to\infty} x_k = x_0$ implies that the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $\lim\limits_{k\to\infty} d_k = 0$. As a result we have $\lim\limits_{k\to\infty} y_k = y_0$. The lemma is proved.

Lemma 3. *The point-to-set mapping ρ is continuous on X.*

Proof. From Lemma 1 it follows that the point-to-set mapping ρ is upper semi-continuous on X [3]. On the other hand, from Lemma 2 it follows that ρ is lower semi-continuous on X [3]. Hence, ρ is continuous on X. The lemma is proved.

Lemma 4 ([1] [6, Theorem 9.14])**.** *Let $X \subset R^m$ be compact, $f : X \to R$ be a continuous function and $\rho : X \rightrightarrows X$ be a continuous compact-valued point-to-set mapping. Then, the function $m : X \to R$, defined by $m\,(x) = \max\,\{f\,(y)\,|y \in \rho\,(x)\}$, is continuous on X, and the point-to-set mapping $\varphi : X \rightrightarrows X$, defined by $\varphi\,(x) = \{y \in \rho\,(x)\,|\,f\,(y) = m\,(y)\}$, is upper semi-continuous on X.*

Lemma 5. *If $x \in X$, then $\varphi\,(x) \subset Max\,(X, F)$.*

Proof. Using Lemma 4, it is sufficient to show that $|\varphi\,(x)| \geq 1$. Let $y \in \varphi\,(x)$ and assume that $y \notin Max\,(X, F)$. From $y \notin Max\,(X, F)$ it follows that there exists $z \in X$ such that $f_i\,(z) \geq f_i\,(y)$ for all $i \in J$ and $f_k\,(z) > f_k\,(y)$ for some $k \in J$. As a result we have that $z \in \rho\,(x)$ and $f\,(z) > f\,(y)$. This leads to a contradiction, hence, $y \in Max\,(X, F)$. The lemma is proved.

Lemma 6. *If $x \in Max\,(X, F)$, then $x \in \varphi\,(x)$.*

Proof. Let $x \in Max\,(X, F)$ and assume that $x \notin \varphi\,(x)$. From $|\varphi\,(x)| \geq 1$ it follows that there exists $y \in \varphi\,(x)$. Hence, there are x and $y \in \rho\,(x) \setminus \{x\}$ such that $f\,(y) > f\,(x)$. From $y \in \rho\,(x) \setminus \{x\}$ it follows that $f_i\,(y) \geq f_i\,(x)$ for all $i \in J$. As a result we obtain $f\,(y) \geq f\,(x)$. But $f\,(y) > f\,(x)$, hence, we have that $f_k\,(y) > f_k\,(x)$ for some $k \in J$. This result contradicts the assumption $x \in Max\,(X, F)$, hence $x \in \varphi\,(x)$. The lemma is proved.

Lemma 7. $\varphi\,(X) = Max\,(X, F)$.

Proof. From Lemmas 5 and 6 it follows that $\varphi\,(X) \subset Max\,(X, F)$ and $\varphi\,(Max\,(X, F)) = Max\,(X, F)$. Obviously, $Max\,(X, F) \subset X$ and, hence, $\varphi\,(X) = Max\,(X, F)$. The lemma is proved.

Proof of the Theorem. From Lemmas 4 and 5 it follows that there exists an upper semi-continuous point-to-set mapping $\varphi : X \rightrightarrows Max\,(X, F)$ such that $\varphi\,(x) = Arg\max\left(f, \bigcap_{j=1}^{n} R_j\,(x)\right)$ for all $x \in X$. From Lemma 7 it follows that $\varphi\,(X) = Max\,(X, F)$. This completes the proof of our theorem.

Corollary. *The sets $Max\,(X, F)$ and $Eff\,(F\,(X))$ are compact.*

Proof. As it is well known, the set X is compact and the point-to-set mapping φ is upper semi-continuous on X, and, therefore, the set $\varphi(X) = Max(X, F)$ is compact (see Theorem 1). The function F is continuous and, hence, the set $F(Max(X, F)) = Eff(F(X))$ is compact too. The corollary is proved.

Remark 1. Let the functions $\{f_j\}_{j=1}^n$ be only continuous on arbitrary set X. If the ideal set $IMax(X, F) = \bigcap_{j=1}^n Arg\max(f_j, X)$ is nonempty set, then $\bigcap\limits_{j=1}^n Arg\max(f_j, X) = Max(X, F)$ [2]. Clearly, in this case the set $Max(X, F)$ is nonempty and compact and $|Eff(F(X))| = 1$.

Remark 2. In [5] is considered the optimization problem $MOP(X, F)$, provided the objective functions $\{f_j\}_{j=1}^n$ are continuous, concave and strictly quasi-concave on X. In this case, it is proved that the Pareto-optimal set $Max(X, F)$ is compact and arcwise-connected. Hence, the set $Eff(F(X))$ is compact and arcwise-connected.

Remark 3. In [6] it is considered the full continuity of the point-to-set mapping φ (see Lemmas 4 and 7). It is proved that this mapping is only upper semi-continuous and not necessary lower semi-continuous on X.

[1] C. Berge. Topological Space, Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity, Oliver and Boyd, 1963.

[2] D. Luc. Theory of Vector Optimization, *LNEMS*, **319**, Springer-Verlag, 1989.

[3] A. Mukherjea, K. Pothoven. Real and Functional Analysis, Plenum Press, 1978.

[4] Z. Slavov, C. Evans. On the structure of the efficient set, *Mathematics and Education in Mathematics*, **33** (2004), 247–250.

[5] Z. Slavov. On the Maximal and Efficient Sets in Decision-Making Systems, *Academic Open Internet Journal*, **12** (2004), 1–11.

[6] R. Sundaran. A First Course in Optimization Theory, Cambridge University Press, 1996.

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КОМПАКТНОСТ НА МНОЖЕСТВАТА НА ПАРЕТО В МНОГОЦЕЛЕВАТА ОПТИМИЗАЦИЯ С КВАЗИ-ВДЛЪБНАТИ ФУНКЦИИ

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В статията разглеждаме Парето-оптимално и Парето-фронт множество при многоцелева оптимизация с няколко целеви функции и изпъкнало компактно допустимо множество. Доказва се съществуването на полунепрекъснато отгоре изображение от допустимото множество в Парето-оптимално множество и компактността на множествата на Парето, ако целевите функции са непрекъснати и квази-вдлъбнати.