МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2006
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2006
Proceedings of the Thirty Fifth Spring Conference of
the Union of Bulgarian Mathematicians
Borovets, April 5–8, 2006

LOCATION DECISIONS IN PROBLEMS OF INTERTEMPORAL CONSUMPTION*

Andrey A. Vassilev

The present work develops a model of the intertemporal consumption/saving decisions of a consumer attempting to find an optimal location in economic space subject to relocation costs. The paper discusses possible modifications of the model and presents a computational approach to its solution. We use the model to illustrate the type of behaviour obtained for different spatial locations and relocation costs.

1. Introduction. The development of general equilibrium models for investigating spatial economic phenomena remains at the forefront of the research agenda in spatial economics (see [3], Ch.19). One of the standard building blocks of dynamic general equilibrium models in the classical, non-spatial context is the description of household behaviour. The importance of understanding intertemporal household behaviour for general equilibrium modelling is evident from the discussion in the surveys [1] and [2].

The spatial economics literature to date is rather reserved in its discussion of households. Models found in standard references such as [3] and [4] analyze labour location issues or consumption/saving decisions but to our knowledge no model combines the choices of where to work and how much to consume and save in a consistent intertemporal framework that can be used as a building block for larger dynamic general equilibrium models. The present work makes a step in this direction by developing a model of consumer behaviour in which consumers optimally choose their asset holdings and spatial location over time.

2. The model. We study the optimal behaviour of a consumer operating in a discrete time, infinite horizon setting. The consumer maximizes with respect to (c_t, z_t) the utility functional

$$\sum_{t=0}^{\infty} \beta^t (\ln(c_t) - \eta(a_{t+1} - a_t)^2). \tag{1}$$

subject to the intertemporal constraints

$$a_{t+1} = (1+r)a_t + w(x_t) - pc_t - \xi(x_{t+1} - x_t)^2, \quad \text{for a given initial } a_0; \tag{2}$$

$$x_{t+1} = x_t + z_t, \quad \text{for a given initial } x_0. \tag{3}$$

The intertemporal discount factor β is assumed to lie strictly between 0 and 1. We use c_t to denote consumption in real terms in period t, with p being the price of a unit

***2000 Mathematics Subject Classification:** 91B42, 91B72, 91B62.

of the consumption good. As p is constant in our setup, it is simply a scaling factor in the model. We do not normalize with respect to it in order to highlight the possible generalization to a time-varying or space-varying price. The variable $a_t \geq 0$ stands for the stock of nominal assets held by the consumer at time t. Unlike traditional models, we impose a nonnegativity constraint to rule out the possibility of borrowing.

Our consumer derives per-period utility from consumption through a standard logarithmic utility term. The model also allows for a certain degree of habit formation in saving decisions by punishing nominal deviations from the current level of the asset stock through the parameter $\eta \geq 0$. The specific quadratic form as used in (1) for changes in assets and also in equation (2) was employed in the paper [6], among others. We note in advance that the solution approach described below does not exploit any specificity of the utility specification and, consequently, the implications of alternative preference specifications can be investigated with a minimal modification in the computational routine.

In our notation r stands for the interest rate on assets. Throughout the paper we take $\beta(1 + r) < 1$, following [5], Ch.17. This assumption rules out the case of infinitely patient consumers who save forever to finance higher consumption in the indefinite future. The variable x_t denotes the spatial position of the consumer at time t. Our reference to economic space here should be interpreted as the set of different locations where the consumer can work and live. We take $x_t \in [0, x_{\max}]$.

Each period the consumer supplies a fixed amount of labour on the market and in return receives a wage $w(x_t)$ depending on the spatial location. In the computations reported below we use the form $w(x_t) = a_w x_t^2 + b_w x_t + c_w$ for suitably chosen coefficients a_w, b_w and c_w but this functional form assumption is in no way binding.

The term $\xi(x_{t+1} - x_t)^2,\ \xi > 0$, is used to capture the nominal transaction costs borne by the consumer in the process of changing location. These include transportation costs but also a number of other expenses incurred in the process of relocation. Examples of the latter may be asset liquidation costs or costs associated with overcoming cultural or linguistic differences. For this reason relocation costs are not linear in distance.

Finally, the change in location is controlled by the variable z_t as specified in (3). We do not impose any constraints on z_t apart from the obvious one that x_{t+1} should remain in the interval specified. The lack of other constraints on z_t may be interpreted as an absence of physical barriers to relocation from one period to the next. Thus, location dynamics are governed only by expenditure considerations.

The model presented is sufficiently rich to accommodate a number of simulations but in this work we restrict our attention to two questions. First, we study the different relocation decisions of the consumer for different initial points in space x_0. Second, we investigate the effect of the relocation cost parameter ξ on the dynamics of assets and spatial location.

3. Model parameterization and solution. We approach the model as a dynamic programming problem and apply the methods described in [5] and [7]. The control variables c_t and z_t are eliminated by substitution and the optimal trajectories for a_t and x_t are derived by numerically approximating an invariant optimal policy rule that maps a pair (a_t, x_t) to next period's pair (a_{t+1}, x_{t+1}). More precisely, since the state variables a_t and x_t in our problem are constrained, we refrain from employing the traditional Euler equation approach, which requires the optimal state trajectories to lie in the interior of the feasible set. Instead, we compute the value function $V(a_t, x_t)$ for the problem by numerically iterating to convergence on the Bellman equation

$$V_{j+1}(a_t, x_t) = \max_{a_{t+1}, x_{t+1}} \{\ln(c_t) - \eta(a_{t+1} - a_t)^2 + \beta V_j(a_{t+1}, x_{t+1})\}$$

and recording the optimal policy map for each point in the state space.

We work with a discrete state space and force a_t and x_t to live on a finite grid. A programming algorithm to iterate on the value function in this setting is presented in [5], Ch.4 for the univariate case. We modified their algorithm to the bivariate case and coded the routine in C++ to reduce the computational burden arising from the increased dimensionality.

After the policy mapping has been obtained, one can produce a simulation run of suitable length for given parameters and initial conditions (a_0, x_0). Consumption c_t can be computed from (2) when a_t and x_t are known. The optimality of the value function and the associated policy mapping is guaranteed for our model since the state space is a finite set and hence the (continuous) utility function is bounded ([7], Ch.4, p.82-83). This result ensures the existence and uniqueness of the solution. It also guarantees that the sequence of approximations will converge to the optimal value function. The finite state space implicitly imposes an upper bound on assets, an effect that is normally undesirable on economic grounds. In our case, however, the condition $\beta(1+r) < 1$ induces a disinclination to save and the upper limit on assets is not binding.

Throughout the next section we use the following baseline set of parameters for the model:

$$\beta = 0.98,\ r = 0.02,\ \xi = 2.0,\ \eta = 0.003,\ x_{\max} = 21,\ a_w = -0.09,\ b_w = 1.88,\ c_w = 5.0,$$
$$p = 1.5,\ a_0 = 20.0,\ x_0 = 2.$$

In the simulations reported we give explicitly only those parameters whose values have been changed from the set specified above. The (a, x) space was discretized on an integer grid starting at $(0, 0)$ and ending at 101 for a_t and $x_{\max} = 21$ for x_t. The choice of the grid induces a degree of inaccuracy in the results but this is compensated for by substantial economies in computing resources. A typical simulation run we present consists of 30 iterations of the policy mapping, except for cases where more iterations were needed to reach a steady state.

The values for a_w, b_w and c_w were chosen to preserve the wage function positive over its domain $[0, x_{\max}]$ and produce a maximum in the middle of this interval. In this case the maximum is obtained for $x = 10$. We note, however, that the value for $x = 11$ is very close to this maximum. As pointed out, the model can easily accommodate other assumptions on the wage distribution over space.

4. Simulation results. In this section the results are presented in terms of dynamics of assets and spatial position. We start the presentation of our findings with the outcome for the baseline case (Fig. 1, left). As can be expected, the consumer moves in the direction of the highest wage location and simultaneously draws down on the asset stock to smooth consumption. Once the location of highest wage has been reached, the consumer settles there and holds the asset stock at the optimal (in this case, lower than the initial) level.

Although the focus of the analysis is the change in behaviour with respect to changes in x_0 and ξ, we remind the reader that the background assumptions that are held unchanged can also play an important role. This is illustrated by taking the case of a higher preference parameter η, interpreted as more conservative attitude toward changes in the savings stock (Fig. 1, right).

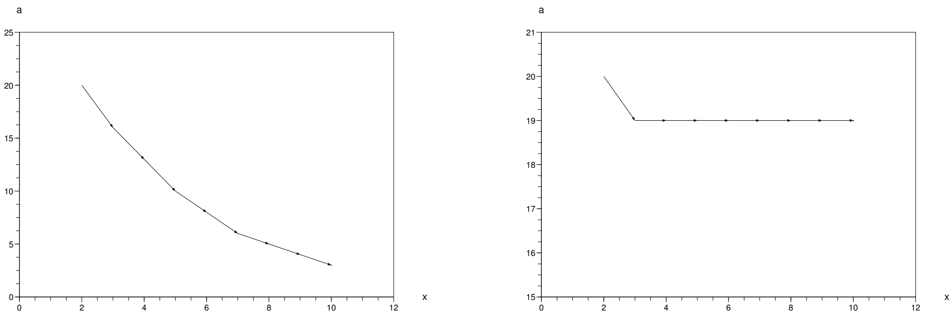

Fig. 1. Asset-location dynamics for the baseline case (left) and for the case $\eta = 0.05$ (right)

Changing the initial point in space also produces reasonable results and the consumer always moves toward the point of highest wage, irrespective of whether the initial location is to the left or to the right of the optimal location (Fig. 2). The left-hand panel in the figure illustrates an interesting situation where the wage at $x = 11$ is so close to the highest wage ($x = 10$) that it makes the cost associated with one more relocation unwarranted. The situation depicted in the right-hand panel shows that consumers starting at more distant (and worse paid) locations may be required to make greater financial efforts to reach the point of highest wage, in this case by reducing their stock of assets to zero. Naturally, a simulation starting at the optimal location does not produce any changes and we do not present the results graphically.

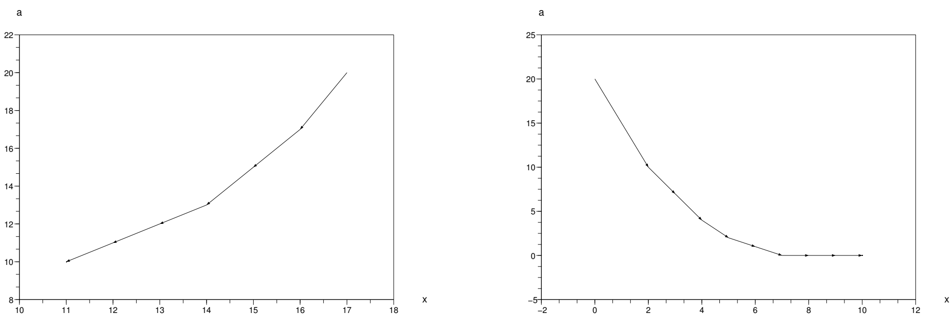

Fig. 2. Asset-location dynamics for the case $x_0 = 17$ (left) and the case $x_0 = 0$ (right)

Finally, a reduction in the relocation cost parameter ξ produces a quicker shift toward the highest wage point with a moderate depletion of assets (Fig. 3, left). The case of higher relocation costs is more interesting. For very high values of the parameter ξ (say, $\xi = 300.0$) the consumer is locked in the initial point in space and does not adjust the initial asset level. If the relocation cost parameter takes an intermediate value ($\xi = 15.0$ in our case), then the adjustment toward the location of highest wage takes place intermittently, with the consumer saving up in the meantime to accumulate enough resources to pay for moving to the next location (Fig. 3, right). As the figure shows, the adjustment process is incomplete in the sense that after several shifts toward the wage peak, wage in the current location becomes sufficiently high to make the outlay on the next relocation plus the psychological discomfort of readjusting the asset level unjustified.

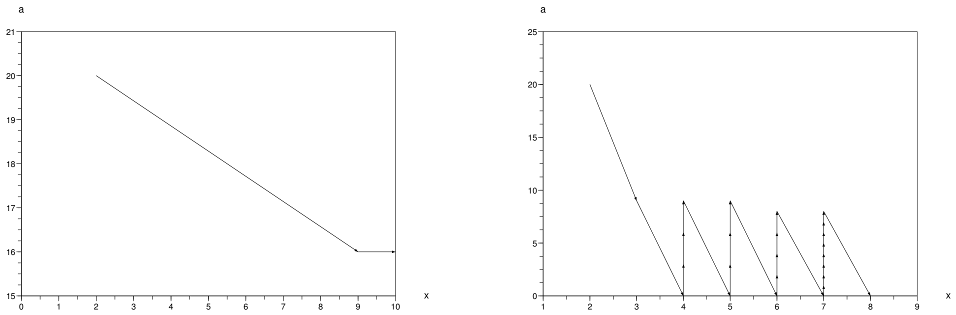

Fig. 3. Asset-location dynamics for the case $\xi = 0.02$ (left) and the case $\xi = 15.0$ (right)

5. Conclusion. The model and the results presented constitute only an initial attempt to study consumption and spatial location decisions in a coherent dynamic optimization framework. A number of extensions to the present setup are possible and, indeed, desirable. First, it would be logical to make prices location dependent. This extension is straightforward in our model but requires a much longer exposition. Second, an analysis of the effects of changes in the specification of the utility function (e.g. preferences over spatial locations, heterogeneity in the parameter η for different consumers, different preferences on the adjustment of asset holdings etc.) would be beneficial. Finally, adding stochastic shocks to the model would bring it closer to the requirements of traditional dynamic general equilibrium modelling.

REFERENCES

[1] T. F. Cooley, E. C. Prescott. Economic Growth and Business Cycles. In T. F. Cooley, ed., Frontiers of Business Cycle Research. Princeton, NJ, Princeton University Press, 1995.

[2] R. G. King, S. T. Rebelo. Resuscitating Real Business Cycles. NBER Working Paper 7534, 2000.

[3] M. Fujita, P. Krugman, A. J. Venables. The Spatial Economy: Cities, Regions and International Trade. Cambridge, MA, MIT Press, 1999.

[4] M. Fujita, J.-F. Thisse. Economics of Agglomeration: Cities, Industrial Location and Regional Growth. Cambridge, UK, Cambridge University Press, 2002.

[5] L. Ljungqvist, T. J. Sargent. Recursive Macroeconomic Theory. Second edition. Cambridge, MA, MIT Press, 2004.

[6] S. Schmitt-Grohé, M. Uribe. Closing small open economy models. *Journal of International Economics*, **61** (2003), 163–185.

[7] N. Stokey, R. E. Lucas Jr. (with E. Prescott). Recursive Methods in Economic Dynamics. Cambridge, MA, Harvard University Press, 1989.

Andrey A. Vassilev
Faculty of Mathematics and Informatics
Sofia University
5p James Bourchier Blvd.
1164 Sofia, Bulgaria
e-mail: avassilev@fmi.uni-sofia.bg

РЕШЕНИЯ ЗА РАЗПОЛОЖЕНИЕ В ПРОСТРАНСТВОТО В ДИНАМИЧНИ ЗАДАЧИ НА ПОТРЕБИТЕЛЯ

Андрей А. Василев

Настоящата работа предлага динамичен модел на решенията за потребление и спестяване, вземани от потребител, който се стреми да намери оптимално положение в икономическото пространство при наличие на разходи по преместването. Разработката разглежда варианти за модифициране на модела и предлага изчислителна реализация на решението ме. Моделът се използва, за да бъде илюстриран типа поведение, който се получава при различни позиции в пространството и разходи по преместване.