МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2008
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2008
Proceedings of the Thirty Seventh Spring Conference of
the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2008

PLANE MODELS OF SMOOTH PROJECTIVE CURVES*

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We study a certain natural relationship between the minimal degree of a plane model of a smooth projective algebraic curve X of genus g and its geometric properties, e.g. the existence of a nontrivial morphism from X onto another curve.

1. Introduction. This article represents the mathematical content of the talk delivered by the author at the 37th Spring Conference of the Union of Bulgarian Mathematicians. There is an original research article which is fairly closely related to what we are going to study [6], jointly with G. Martens on the minimal degree of a plane model of a given algebraic curve.

Even though part of the mathematical contents appearing in this article may also be found in the above mentioned paper, the author tries to make this article as much self-contained as possible so that the readers may obtain a reasonable overview on the topic. For this reason, some parts of this article may appear to be similar to those in [6], which are the outcomes of joint efforts with the author's collaborator. However, the author takes the sole responsibility for all the possible mistakes and inaccuracies in this article, if there are any.

The organization of this paper is as follows. In the next section, we start by observing a couple of examples which may provide a motivation for considering curves without plane models of small degree. The main aim of the section is to persuade the reader that the curves without plane model of small degree can be characterized as curves admitting a degree two morphism onto another curve. In section three, we discuss double coverings of another curve and present a result which is related to the main thesis of this article. In the final section, we raise a question which is related to the theme of the second and third section.

For all the notations and conventions used but not explained, we refer the reader to [2]. Otherwise stated, every curve considered in this paper is smooth irreducible and projective defined over the field of complex numbers. Throughout the text X is always a smooth projective curve of genus g.

2. Motivation and examples. A long time ago (in 1884), Halphen showed that every curve X of genus g can be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$ such that the

* **2000 Mathematics Subject Classification:** 14H50.
Key words: Complex projective algebraic curve, double covering, gonality, Castelnuovo-Severi inequality.
Supported in part by KRF; Grant # 2005-070-C00005.

hyperplane section in $\mathbb{P}^3$ is nonspecial, i.e. every curve of genus g has a nonspecial and very ample linear series of degree $g+3$; cf. [5, page 349; Proposition 6.1].

By projecting from a general point on the embedded curve $\phi(X) \subset \mathbb{P}^3$, one has a plane model X' of $X \subset \mathbb{P}^2$, which is (usually) singular:

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & \phi(X) & \subset & \mathbb{P}^3 \\ & \searrow & & & \downarrow \pi \\ & & X' & \hookrightarrow & \mathbb{P}^2. \end{array}$$

ϕ : Halphen's embedding, $\deg \phi(X) = g+3$
π : projection from a general point $p \in X$, through which there exist only a finitely many trisecant lines, whence $\pi \cdot \phi$ is birational.
$\deg X' = g+2$

Under this circumstances, we may raise the following rather naive but seemingly natural questions.

Questions 2.1.
1. What degree plane models of X might have?
2. Specifically, what is the minimal degree of a plane model X' of a given curve X?
3. Let s_X denotes the minimal degree of a plane model of X. What is the possible range of such s_X's among the curves X having a fixed genus g?

The third question can be answered easily as follows. By the genus formula for plane curves of a given degree s_X, we have

$$g = g(X) \le p_a(X') = \frac{(s_X-1)(s_X-2)}{2}$$

where $p_a(X')$ is the arithmetic genus of the plane model X'. Therefore it follows that

$$m_0 := \frac{3+\sqrt{8g+1}}{2} \le s_X \le g+2,$$

where the second inequality comes from the Halphen's theorem.

Having been able to obtain the interval to which s_X may belong, we further ask:

4. Does every integer in the interval $I := [m_0, g+2]$ occur as a degree s_X for some curve X of genus g?
5. Denoting by $\mathcal{M}_g$ the coarse moduli space of smooth algebraic curves of genus g, we also ask: Is the natural function

$$\mathcal{M}_g \ni X \mapsto s_X \in \mathbb{N}$$

semi-continuous?
6. What are the possible (geometric) descriptions for all those X's with a fixed s_X?

For this series of questions, it is now fairly clear what needs to be studied. Since we are looking for morphisms $\phi : X \to \mathbb{P}^2$ for which ϕ is generically one to one, we are indeed chasing the so-called birationally very ample morphisms into $\mathbb{P}^2$ so that its image curve has minimal degree. This can also be realized as

$$\mathcal{D} := \{\phi^* H | H \in \mathbb{P}^{2^*}\}$$

such that:

(i) for a general $p \in X$, $|\mathcal{D} - p|$ has no base point,

(ii) $\deg D \in \mathcal{D}$ is minimal.

We now begin with a couple of examples which provides a motivation for set fing up the theses in this article.

Example 2.2. We first consider a curve which is most special in the sense of moduli. Let X be an hyperelliptic curve, i.e. $X \in \mathcal{M}_{g,2}$, where

$$\mathcal{M}_{g,2} = \{X \in \mathcal{M}_g |\ \exists\ X \xrightarrow{\pi} \mathbb{P}^1, \deg \pi = 2\}$$

We claim that $s_X = g+2$. For otherwise, take $d = s_X \leq g+1$. Then there exists g^2_d, which is birationally very ample and it follows that there also exists a base point free pencil $g^1_{d-1} = |g^2_d - p|$, where $p \in X$ is a general point, inducing a morphism $\psi : X \overset{g^1_{d-1}}{\to} \mathbb{P}^1$.

$$\begin{array}{ccccc} & & \mathbb{P}^1 & = & \mathbb{P}^1 \\ & \overset{\pi}{\nearrow} & & & \uparrow \text{projection} \\ X & \overset{\pi\times\psi}{\longrightarrow} & X' \subset \mathbb{P}^1 \times \mathbb{P}^1 & \subset & \mathbb{P}^3 \\ & \overset{\psi}{\searrow} & & & \downarrow \text{projection} \\ & & \mathbb{P}^1 & = & \mathbb{P}^1 \end{array}.$$

Since the morphism $\pi \times \psi$ is birationally very ample onto its image, X' is a curve of type $(d-1, 2)$ on a smooth quadric surface in $\mathbb{P}^3$. Hence by the adjunction formula, we have

$$g \leq p_a(X') = ((d-1)-1)(2-1) = d-2$$

which is in compatible with the assumption

$$d = s_X \leq g+1.$$

We next claim that a smooth curve X of genus g having the minimal degree for a plane model $s_X = g+2$ must be hyperelliptic: Suppose $X \notin \mathcal{M}_{g,2}$. Then

$$\exists\ X \overset{|K_X|}{\hookrightarrow} \mathbb{P}^{g-1} \to \cdots \to X' \subset \mathbb{P}^2,$$

where $|K_X|$ induces the canonical embedding, followed by projections from a general point $(g-3)$ times. Therefore we have,

$$\deg X' = \deg K_X - (g-3) = g+1$$

which implies

$$s_X \leq g+1,$$

finishing the proof of the claim.

Example 2.3. At the other extreme, if X is a general curve of genus g, then it was observed already by Severi [8, Anhang G, §10] that

$$s_X = [\frac{2(g+4)}{3}] =: m_1.$$

In fact, this follows essentially from the Brill-Noether theorem, which was believed to be true at that time (and proved later by Griffiths and Harris in late 1970's):
Denoting the variety of special linear systems of degree d and dimension r by $W^r_d(X)$, the so-called "non-existence theorem"asserts that for a general curve of genus g, $W^r_d(X) \neq \emptyset$ if and only if the Brill-Noether number

$$\rho(d, g, r) := g - (r+1)(g-d+r) \geq 0.$$

For $r = 2$, one sees that m_1 is the smallest integer such that $\rho(d, g, 2)$ is non-negative. Moreover, by the fact that a non-degenerate morphism corresponding to a special g^2_d on a general curve of genus g is not composed with an involution [1] , it follows that m_1 is indeed the minimal degree of a plane model of a general curve of genus g.

Recall that, by the theorem of Halphen and the genus formula for plane curves, we have

$$m_0 := \frac{3 + \sqrt{8g + 1}}{2} \leq s_X \leq g + 2,$$

and we asked if every integer in the interval

$$I := [m_0, g + 2]$$

occurs as s_X for some curve X of genus g. We now answer this question in the affirmative (at least partially) as follows.

Example 2.4. There exists a curve X of genus g with $s_X = m$ for every $m \in [m_0, m_1]$. Here we provide an outline of the proof of the existence only for the case $m \leq \frac{g+7}{2}$, which will be enough for the next Corollary 2.5, i.e., for the non-semi-continuity of the invariant s_X.

Claim: For $m \leq \frac{g+7}{2}$, $\exists\ X \in \mathcal{M}_g$ with $s_X = m$.

Let X be a smooth model of a general plane nodal curve X' of geometric genus g, $\deg X' = m$. We know that such X or X' always exists since $m \geq m_0$. Denoting by $V_{m,g}$ the Severi variety of plane curves of degree m of genus g, the plane curve X' is indeed a general member of $V_{m,g}$. Note that our numerical assumption $m \leq \frac{g+7}{2}$ is equivalent to the condition $\rho(m - 3, g, 1) < 0$, where $\rho(d, g, r) := g - (r + 1)(g - d + r)$ is the Brill-Noether number. By a result of Coppens [3], we have

$$(*) \qquad \mathrm{gon}(X) = m - 2$$

where the pencil determining the gonality is cut out by lines through a node.

While $s_X \leq m$ is trivially true, the issue here is that we may have smaller degree plane model of X. We now argue that this is not the case.

(i) Suppose $s_X \leq m - 2$. By considering a pencil of lines through a general point of the minimal degree plane model, we see that

$$\mathrm{gon}(X) \leq m - 3,$$

which contradicts the result of Coppens [3].

(ii) Suppose $s_X = m - 1$. If a plane model of minimal degree s_X is singular, then we obtain $\mathrm{gon}(X) \leq m - 3$ by projecting from a singular point. Hence the minimal degree plane model must be a smooth curve of degree $m - 1$, whereas a smooth curve of degree $m - 1$ does not have a base point free and complete g^2_m by a well known theorem of Max Noether.

As a by-product, we obtain the following corollary which answers one of our earlier questions in 2.1.

Corollary 2.5. *The function $\mathcal{M}_g \ni X \mapsto s_X \in \mathbb{N}$ is not semi-continuous.*

Proof. If it were upper or lower semi-continuous, then the generic value m_0 achieved by a general curve of genus g should be the maximal or minimal value among all the possible value of s_X's. However, this is not the case as we have seen in the previous three Examples 2.2, 2.3 and 2.4 □

Recall that Example 2.2 asserts that for $X \in \mathcal{M}_{g,2}$, $s_X = g+2$. Therefore, for a non-hyperelliptic curve

$$X \in \mathcal{M}_g \backslash \mathcal{M}_{g,2}, \quad \text{we have} \quad s_X \leq g+1.$$

We now ask if for a non-hyperelliptic curve X the inequality $s_X \leq g+1$ is indeed sharp. The following theorem, due to Coppens and Martens, provides an answer to the question, cf. [4, Proposition 2.2 and 2.6].

Theorem 2.6 (Coppens-Martens)**.** *Let X be a curve of genus $g \geq 6$. Then*

$$s_X = g+1$$

if and only if

$$X \text{ is bi-elliptic},$$

i.e. $\exists\ \pi : X \to C, \deg \pi = 2$, *$C$ an elliptic curve.*

3. Double coverings of curves of low genus. Theorem 2.6 and Example 2.2 as well indicate that a curve X with big s_X is rather a special curve admitting a morphism of degree two onto a curve of small genus. Therefore, looking for curves with big s_X it seems natural to ask: Does this simple pattern observed for $s_X \geq g+1$ in Example 2.2 and Theorem 2.6 continue to hold, i.e., does $s_X = g+2-h$ imply that C is a double cover of a curve of genus (at most) h - provided that g is not too small with respect to h ? It turns out that the answer to this question is also **YES**; cf. [6].

Theorem 3.1. *Let $0 \leq t \in \mathbb{Z}$ and X be a curve of genus g with $s_X = g+2-h$. Then there is an effective polynomial expression $p(h)$ in h such that $g \geq p(h)$ implies that X is a double cover of a curve of genus at most h.*

Since the proof is somewhat involved using rather conventional (and complicated) techniques, we do not intend to present it here. The reader is advised to look at the paper [6]. Instead we give the proof of the following proposition which may be regarded as the converse part of Theorem 3.1. Since we want to produce a very ample linear series g^r_d on X such that r is large with respect to d or a $g^{r'}_{d'}$ such that d' is large with respect to r' and such that$|K_X - g^{r'}_{d'}|$ is very ample, a reasonable candidate would be $|K_X - \pi^* K_C|$.

Proposition 3.2. *Let $\pi : X \to C$ be a double covering of curves of genus g and h, respectively. If $g \geq 4h$, then $s_X \leq g+2-h$.*

Proof. For any covering $\pi : X \to C$ and any line bundle M on C, it is known ([5, II, Ex.5.1; III, Ex.4.1; IV, Ex.2.6]) that

$$\begin{aligned} H^0(X, \pi^* M) = H^0(C, \pi_* \pi^* M) &= H^0(C, \pi_*(\pi^* M \otimes_{\mathcal{O}_X} \mathcal{O}_X)) \\ &= H^0(C, M \otimes_{\mathcal{O}_C} \pi_* \mathcal{O}_X) \end{aligned}$$

and that $\det \pi_* \mathcal{O}_X \cong \mathcal{O}_C(-D)$ for a divisor D on C such that $2D$ is linearly equivalent to the branch divisor B of π (made up by the points of C over which π ramifies); in particular, the vector bundle $\pi_* \mathcal{O}_X$ on C of rank $\deg \pi$ has degree

$$-\deg D = -\frac{1}{2} \deg B = (\deg \pi) \cdot (\tilde{g} - 1) - (g-1) \leq 0.$$

Moreover ([7, I, 1], if $\deg \pi = 2$, then the rank two vector bundle $\pi_* \mathcal{O}_X$ splits into the line bundles $\mathcal{O}_C$ and $\det \pi_* \mathcal{O}_X$ of degree 0, resp. $2(h-1)-(g-1) = 2h-g-1$. For a double covering $\pi : X \to C$, we thus obtain

$$H^0(X, \pi^* M) = H^0(C, M) \oplus H^0(C, M \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D)),$$

and, if $\deg M < \deg D = g+1-2h$, then $H^0(C, M \otimes_{\mathcal{O}_C} \mathcal{O}_C(-D)) = 0$, i.e.,

$$H^0(X, \pi^* M) = H^0(C, M). \tag{3.2.1}$$

In particular, taking $M = \omega_C$, the canonical sheaf on C, we have

$$\deg M = 2h - 2 \le g - 2h$$

since $g \ge 4h-2$, and by (3.2.1),

$$h^0(X, \pi^* K_C) = h^0(C, K_C) = h.$$

We will show that $|K_X - \pi^* K_C|$ is very ample. Since this series is a complete $g^{g-3h+2}_{2g-4h+2}$ on X, we obtain, by subtracting $g - 3h \ge 0$ general points of X from it, a simple net of degree

$$(2g - 4h + 2) - (g - 3h) = g + 2 - h$$

on X proving that $s_X \le g + 2 - h$.

In order to show that $|K_X - \pi^* K_C|$ is very ample, we need to show that

$$h^0(X, (\pi^* K_C) + P + Q) \le h^0(\pi^* K_C)$$

for any two points P, Q on X. Let $p := \pi(P)$, $q = \pi(Q)$, $P + P' := \pi^*(p)$ and $Q + Q' := \pi^*(q)$. Then

$$(\pi^* K_C) + P + Q = \pi^*(K_C + p + q) - P' - Q'.$$

By (3.2.1)

$$h^0(X, \pi^*(K_C + p + q)) = h^0(C, K_C + p + q) = 2h - h + 1 = h + 1$$

because we still have $\deg(K_C + p + q) = 2h \le g - 2h$. Since $|K_C + p + q|$ is base point free ([5, IV, 3.2]), so is $|\pi^*(K_C + p + q)|$, and it follows that

$$\begin{aligned} h^0((\pi^* K_C) + P + Q) &= h^0(\pi^*(K_C + p + q) - P' - Q') \\ &\le h^0(\pi^*(K_C + p + q)) - 1 = h = h^0(\pi^* K_C). \end{aligned}$$

□

4. Epilogue. We saw in Section 1 that for a smooth curve X of genus g, the minimal degree s_X of a plane model of X lies in the interval

$$\left[\frac{3 + \sqrt{8g+1}}{2}, g + 2\right],$$

and that every integer in the sub-interval actually occurs as s_X for some curve X of genus g. However, we still don't know if the integers in the other part of the interval really occur as s_X for some curve X of genus g. Of course, reasonable candidates are double coverings of curves of genus h, as we saw for the cases $h = 0, 1$.

The following result due to Dongsoo Shin may provide some clue toward a satisfactory answer to this question, at least partially [9].

Theorem 4.1. *Let X be a smooth irreducible curve of genus g. If X is a double cover of a smooth irreducible curve Y of genus $h \ge 2$, $g \ge 4h - 2$, then*

$$s_X \ge g - 2h + 1 + \mathrm{gon}(Y),$$

where $\mathrm{gon}(Y)$ *denotes the gonality of* Y.

Recall that a curve of genus g can be embedded in $\mathbb{P}^3$ as a curve of degree $g+3$ by a theorem of Halphen. Naturally, one may want to have a more precise version of Halphen's statement. For example, it would be nice to have a description of those curves which may be embedded in $\mathbb{P}^3$ as a curve of degree smaller than $g+3$. A first step in this direction was obtained by Harris [2, Exercise B, p.221], who showed the following using a theorem of Mumford [2, Theorem 5.2, p.193].

Theorem 4.2 (Harris)**.** *Let* X *be a curve of genus* g. *If* X *is not hyperelliptic, trigonal or bi-elliptic, then* X *embeds into* $\mathbb{P}^3$ *as a curve of degree* $g+2$.

Let

$$\mathcal{M} := \{X \in \mathcal{M}_g |\ X \text{ is not hyperelliptic, trigonal or bi-elliptic}\},$$

be the classes of curves admitting embeddings of degree $g+2$ into $\mathbb{P}^3$. Note that if $\mathrm{Cliff}\, X \geq 3$, then $X \in \mathcal{M}$ and hence X carries a very ample g^3_{g+2} by Theorem 4.2. However one has

$$\mathcal{M}_3 := \{X \in \mathcal{M}_g |\, \mathrm{Cliff}(X) \geq 3\} \subsetneq \mathcal{M}.$$

Therefore one can expect: $X \in \mathcal{M}_3$ **may satisfy a stronger condition, say, the existence of a very ample** g^3_{g+1} **on** X**.**

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РАВНИННИ МОДЕЛИ НА ГЛАДКИ АЛГЕБРИЧНИ КРИВИ

Чангхо Ким

В тази работа ние изучаваме някои естествени зависимости между минималната степен на равнинния модел на гладка алгебрична крива X от род g и нейните геометрични свойства, в частност съществуването на нетривиални морфизми от X към друга крива.