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EXISTENCE OF SOLUTIONS OF ODE'S IN WATER WAVE MODELS*

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We prove existence results for solutions of boundary value problems for fourth-order differential equation arising in water wave models. Variational approach is applied to considered problems.

1. Introduction. In this paper we investigate the existence of travelling wave solutions of fifth-order Korteweg-de-Vries equation of the form

$$w_t + \gamma w_{xxxxx} + \beta w_{xxx} = (\mu(2ww_{xx} + w_x^2) + f(w))_x, \tag{1}$$

which appear in the classical water wave problem with gravity and capillarity (see [3], [4]). In (1) subscripts denote partial differentiation, $\beta, \mu \in \mathbb{R}$, $\gamma > 0$ and $f(w)$ is a polynomial. Looking for travelling waves $w(x,t) = u(x - ct)$, we obtain after appropriate scaling an equation of the form

$$\gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) + g(u), \tag{2}$$

where $g(u) = f(u) + cu$. In [1] existence and symmetry of homoclinic solutions of the equation

$$\gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) + u - u^2, \tag{3}$$

are studied via shooting method.

In this paper we study the existence of periodic solutions of Eq.(2) via variational method. Let $L > 0$.We consider the boundary value problems (P_1) and (P_2) as follows

$$(P_1) \qquad \begin{cases} \gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) + u - u^3, \quad 0 < x < L, \\ u(0) = u(L) = u'(0) = u'(L) = 0. \end{cases}$$

and

$$(P_2) \qquad \begin{cases} \gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) - u - u^2, \quad 0 < x < L, \\ u(0) = u(L) = u'(0) = u'(L) = 0. \end{cases}$$

Note that Eq.(3) turns to

$$\gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) - u - u^2, \tag{4}$$

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after an appropriate change of the variables (see [1]). We obtain $2L$- periodic solutions which are symmetric with respect to $x = 0$ and $x = L$ taking $2L$-periodic extension of the even extension

$$\overline{u}(x) = \begin{cases} u(x), & 0 \le x \le L, \\ u(-x), & -L \le x \le 0. \end{cases}$$

Note that if $u(x)$ is a solution of the equation

$$\gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) + u - u^3, \tag{5}$$

or Eq.(4), then $u(-x)$ is also a solution.

Both problems (P_1) and (P_2) have a variational structure and their weak solutions in the space $X = H_0^2(0, L)$ are critical points of the functionals

$$I(u; L) = \int_0^L \left(\frac{\gamma}{2} u''^2 + \frac{1}{2} u'^2 + \mu u\ u'^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4 \right) dx$$

and

$$J(u; L) = \int_0^L \left(\frac{\gamma}{2} u''^2 + \frac{1}{2} u'^2 + \mu u\ u'^2 + \frac{1}{2} u^2 + \frac{1}{3} u^3 \right) dx.$$

We prove existence of nontrivial solutions using minimization and mountain-pass theorems. Our main results are as follows:

Theorem 1. *Let $0 < \mu < \min(1, 2\gamma)$. Then, problem (P_1) has a solution u which is a minimizer of the functional $I : X \to \mathbb{R}$. If L is sufficiently large, then this solution is nontrivial. Suppose that u is a nonnegative minimizer of $I(., L)$ for sufficiently large L. Then, $u(x) > 0$ for every $x \in (0, L)$. Moreover, for every natural number $n \ge 2$, there exists a solution u_n of Eq. (5) subject to the boundary conditions $u(0) = u(nL) = u'(0) = u'(nL) = 0$ and*

$$\limsup_{n\to\infty} \frac{1}{L} \int_0^L |u_n(nt)| dt \le \left(\frac{8\gamma - 3\mu}{2(1-\mu)(2\gamma - \mu)} \right)^{1/2}. \tag{6}$$

Theorem 2. *Let $0 < \mu < 2\gamma$. Then, problem (P_2) has a nontrivial solution u which is a mountain pass point of the functional $J : X \to \mathbb{R}$.*

2. Proofs of the main results. We study the solvability of the problem (P_1). Let $X = H_0^2(0, L)$ be the Sobolev space with the norm $\|u\|^2 = \int_0^L u''^2 dx$, which is equivalent to the usual norm $\|u\|_{H^2}^2 = \int_0^L (u''^2 + u'^2 + u^2) dx$ by Poincare inequalities. We have

Proposition 3. *The functional $\Phi : X \to \mathbb{R}$, $\Phi(u; L) = \int_0^L uu'^2 dx$ is differentiable and*

$$\langle \Phi'(u), v \rangle = \int_0^L (2u\ u'v' + u'^2 v) dx = -\int_0^L (2u\ u'' + u'^2) v dx.$$

Consider now the functional $I : X \to \mathbb{R}$,

$$I(u;L) = \int_0^L \left(\frac{\gamma}{2} u''^2 + \frac{1}{2} u'^2 + \mu u\, u'^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4 \right) dx.$$

By Proposition 3, the functional I is differentiable and

$$\langle I'(u;L), v \rangle = \int_0^L \gamma u'' v'' - u'v' - (\mu(2uu'' + u'^2) + u - u^3) v dx,$$

i.e. critical points of functional I are weak solutions of problem (P_1). Note that, since the embedding $X \subset C^1([0,L])$ is continuous, if u is a critical point of I and u'' has generalized second derivative $u^{(4)} \in L^2(0,L)$, then $u \in H^4(0,L)$ and u''' and u'' are continuous functions. Therefore, $\gamma u^{(4)} = u'' + \mu(2uu'' + u'^2) + u - u^3$ a.e. in $[0,L]$, $u^{(4)}$ is continuous function and u is a classical solution of (P_1).

To obtain critical points of I we use general minimization theorem for weak lower semi-continuous functionals on reflexive Banach spaces (see [2, p. 301]). The functional $I(u)$ is weakly lower semi-continuous on a reflexive Banach space X if $I(u) = I_1(u) + I_2(u)$ where $I_1(u)$ is convex and $I_2(u)$ is sequentially weakly continuous, i.e. if $u_n \to u$ weakly, then $I_2(u_n) \to I_2(u)$ as $n \to \infty$. Let

$$I_1(u;L) = \int_0^L \frac{\gamma}{2} u''^2 dx, \qquad I_2(u;L) = \int_0^L (\frac{1}{2} u'^2 + \mu u u'^2 - \frac{1}{2} u^2 + \frac{1}{4} u^4) dx.$$

Proposition 4. *Let $\mu < \min(1, 2\gamma)$. Then, the functional $I : X \to \mathbb{R}$ is coercive and weakly lower semi-continuous.*

Proof. Since the embedding $X \subset C^1([0,L])$ is continuous, it is clear that $I_2(u;L)$ is sequentially weakly continuous. $I_1(u;L)$ is convex, and then $I(u;L)$ is weakly lower semi-continuous on X. Observe that

$$\int_0^L u u'^2 dx = \int_0^L u u' du = -\int_0^L \left(u u'^2 + u^2 u'' \right) dx$$

and

$$\left| 2 \int_0^L u u'^2 dx \right| = \left| -\int_0^L u^2 u'' dx \right| \le \frac{1}{2} \int_0^L \left(u^4 + u''^2 \right) dx \tag{7}$$

I is coercive functional by $\mu < \min(1, 2\gamma)$, (7) and

$$\begin{aligned} I(u,L) &\ge \frac{2\gamma - \mu}{4} \int_0^L u''^2 dx + \frac{1}{2} \int_0^L u'^2 dx + \int_0^L \left(-\frac{u^2}{2} + \frac{1-\mu}{4} u^4 \right) dx \\ &\ge \frac{2\gamma - \mu}{4} \|u\|^2 - \frac{L}{4(1-\mu)}. \end{aligned}$$

□

Proposition 5. *Let $\mu < \min(1, 2\gamma)$. Then, for sufficiently large L*

$$\inf\{ I(u,L) : u \in X \} < 0.$$

Proof. Let us take a test function $v(x) = \varepsilon \sin^2\left(\frac{\pi x}{L}\right)$ which satisfies the boundary conditions. A direct calculation shows that

$$I(v, L) = \varepsilon^2 L\left(\frac{\pi^3\gamma}{4L} + \frac{\pi^2}{4L^2} - \frac{\mu\pi\varepsilon}{4L} + \frac{35\varepsilon^2}{512} - \frac{3}{16}\right)$$

Then, taking $\varepsilon : \varepsilon^2 < \frac{96}{35}$ and L sufficiently large such that

$$(96 - 35\varepsilon^2)L^3 + 128\pi^2(\varepsilon\mu - 1)L > 128\pi^3\gamma$$

we obtain that $I(v, L) < 0$. Then, $\inf\{I(u, L) : u \in X\} < 0$. $\square$

Proof of Theorem 1. Existence part directly follows from a general minimization theorem and Propositions 4 and 5. The maximum principle and (6) follow by tedious computations. $\square$

Consider now the problem (P_2) and corresponding functional

$$J(u; L) = \int_0^L \left(\frac{\gamma}{2}u''^2 + \frac{1}{2}u'^2 + \mu u\ u'^2 + \frac{1}{2}u^2 + \frac{1}{3}u^3\right) dx,$$

on the space X. As before the critical points of J are classical solutions of (P_2). We prove that J satisfies the assumptions of mountain-pass theorem (see [5, p. 7]).

Proposition 6. *The functional $J : X \to \mathbb{R}$ satisfies (PS) condition, i.e. if $(u_n)_n$ is a sequence in X such that $J(u_n)$ is bounded and $J'(u_n) \to 0$, then $(u_n)_n$ has a convergent subsequence.*

Proof of Theorem 2. It follows from Proposition 6 and geometric assumptions of mountain-pass theorem as: (i) There exists $r > 0$ such that $J(u, L) \geq 0$ for $u : \|u\| \leq r$. (ii) There exists $v \in X$ such that $J(v, L) < 0$. We have

$$J(u, L) \geq \frac{2\gamma - \mu}{4}\int_0^L u''^2 dx + \int_0^L \left(\frac{u^2}{2} - \frac{|u|^3}{3} - \frac{\mu}{4}u^4\right) dx.$$

Let k be the embedding constant $X \subset C([0, L]) : |u|_C \leq k\|u\|$. Taking $\|u\|$ sufficiently small we have $\frac{u^2}{2} - \frac{|u|^3}{3} - \frac{\mu}{4}u^4 \geq 0$ and $J(u, L) \geq 0$. To show (ii) let us take $u_0 \in X$ such that $3\mu u_0 u_0'^2 + u_0^3 < 0$ in $(0, L)$. It may be $u_0 = -\sin^2\left(\frac{\pi x}{L}\right)$. Then, from

$$J(tu_0, L) = t^2\int_0^L (\gamma u_0''^2 + u_0'^2 + u_0^2)dx + t^3\int_0^L (3\mu u_0\ u_0'^2 + u_0^3)dx,$$

taking t sufficiently large negative, we obtain that for $v = tu_0$, $J(tu_0, L) < 0$. $\square$

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СЪЩЕСТВУВАНЕ НА РЕШЕНИЯ НА ОБИКНОВЕНИ ДИФЕРЕНЦИАЛНИ УРАВНЕНИЯ В МОДЕЛИ ЗА ВОДНИ ВЪЛНИ

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Доказани са две теореми за съществмуване на решения на гранични задачи за диференциални уравнения от четвърти ред в теорията на водните вълни. Приложени са вариационни методи за доказване на резултатите.