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SEMILINEAR WAVE EQUATION IN SCHWARZSCHILD METRIC*

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We study the semilinear wave equation in Schwarzschild metric ($3+1$ dimensional space-time). First, we annonce the blow up of the solution for every $p \in]1, 1+\sqrt{2}[$ and non-negative non-trivial initial data. Further, we discuss suitable resolvent estimates for the corresponding Helmholtz equation crucial for the local energy decay of radially symmetric data.

1. Introduction. Consider the manifold

$$\mathbf{M} = \mathbf{R} \times \Omega, \qquad \Omega = \{(r, \omega) : r > 2M,\, \omega \in \mathbf{S}^2\} =]2M, \infty[\times \mathbf{S}^2,$$

equipped with the Schwarzschild metric having the form (see chapter V in [1] or chapter 31 in [4]):

$$g = F(r)\, dt^2 - F(r)^{-1}\, dr^2 - r^2\, d\omega^2. \tag{1.1}$$

Here

$$F(r) = 1 - \frac{2M}{r},$$

the constant $M > 0$ has the interpretation of mass and $d\omega^2$ is the standard metric on the unit sphere $\mathbf{S}^2$.

The D'Alembert operator associated with the metric g is

$$\Box_g = \frac{1}{F}\left(\partial_t^2 - \frac{F}{r^2}\partial_r(r^2 F)\partial_r - \frac{F}{r^2}\Delta_{\mathbf{S}^2}\right),$$

where $\Delta_{\mathbf{S}^2}$ denotes the standard Laplace-Beltrami operator on $\mathbf{S}^2$.

Our goal is to study the existence of global solution to the corresponding Cauchy problem for the semilinear wave equation

$$\Box_g u = |u|^p \qquad \text{in } [0, \infty[\times \Omega. \tag{1.2}$$

This problem can be considered as a natural analogue of the classical semilinear wave equation in the flat Minkowski metric

$$g_0 = dt^2 - dr^2 - r^2\, d\omega^2. \tag{1.3}$$

It is well-known that for any space dimension $n \geqslant 2$, there exists a critical value $p_0 = p_0(n) > 1$ such that the Cauchy problem for (1.2) with the metric g_0 admits a global

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small data solution provided $p > p_0(n)$, while for subcritical values of $p \leqslant p_0(n)$, a blow-up phenomenon is manifested. In the case of space dimension $n = 3$, the critical exponent is $p_0(3) = 1 + \sqrt{2}$.

Our first goal in this work is to prove that the semilinear wave equation in the presence of Schwarzschild metric blows up for $1 < p < 1 + \sqrt{2}$. The main obstacle is related to the fact that (according to the knowledge of the authors) the proof of some concrete dispersive estimates for the wave equation in Schwarzschild metric meets the essential difficulty that there is no simple explicit representation of the corresponding fundamental solution to the D'Alambert operator in Schwarzschild metric.

We introduce the Regge-Wheeler coordinate

$$s(r) = r + 2M \log(r - 2M) \tag{1.4}$$

and restrict (with no loss of generality) our considerations to the case of solutions of the form $u = u(t, s)$. Making further the substitution

$$u(t,s) = \frac{v(t,s)}{r(s)},$$

we obtain the semilinear problem

$$\partial_t^2 v - Pv = Fr^{1-p}|v|^p, \qquad P = \partial_s^2 - \frac{2MF}{r^3}. \tag{1.5}$$

First, we study the local existence of the solution to the Cauchy problem

$$\begin{cases} \partial_t^2 v - Pv = Fr^{1-p}|v|^p, \\ v(0,s) = v_0(s), \qquad \partial_t v(0,s) = v_1(s). \end{cases} \tag{1.6}$$

To study the maximal time interval of existence of solutions to the wave equation in Schwarzschild metric

$$\begin{cases} \Box_g u = |u|^p & \text{in } [0,\infty[\times\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega, \end{cases} \tag{1.7}$$

we suppose that our initial data are radial, i.e.

$$u_0 = u_0(r), \quad u_1 = u_1(r), \quad (u_0, u_1) \in \mathrm{H}^2(]2M,\infty[) \times \mathrm{H}^1(]2M,\infty[)$$

and that there exists a compact interval $\mathbf{B} \doteq \overline{\mathbf{B}(r_0,R)} \doteq \{|r - r_0| \leqslant R\} \subset]2M,\infty[$ so that

$$\begin{cases} u_0(r),\, u_1(r) \geqslant 0 & \text{almost everywhere,} \\ u_0(r) = u_1(r) = 0 & \text{for } \ |r - r_0| \geqslant R, \\ \int_{2M}^{\infty} u_j(r)\, dr \geqslant \varepsilon & j = 0,1 \end{cases} \tag{1.8}$$

for positive constants ε, $R > 0$ and $r_0 = r_0(\varepsilon, p) \in \Omega$. We also assume that r_0 is near $2M$ for $p \in]2, 1+\sqrt{2}[$, far from it for $p \in]1,2[$ (we make no assumption in the case $p = 2$).

Now we can state the main result.

Theorem 1.1. *For any p, $1 < p < 1 + \sqrt{2}$, there exists a positive number ε_0 such that for any $\varepsilon \in]0, \varepsilon_0[$ there exists $r_0 = r_0(p,\varepsilon)$ and $R = R(p,\varepsilon)$ so that for any initial data*

$$u_0 = u_0(r), \quad u_1 = u_1(r), \quad (u_0,u_1) \in \mathrm{H}^2(]2M,\infty[) \times \mathrm{H}^1(]2M,\infty[),$$

satisfying (1.8) *in $\mathbf{B} \doteq \overline{\mathbf{B}(r_0,R)}$, there exists a positive number $T = T(\varepsilon) < \infty$ and a solution*

$$u \in \cap_{k=0}^2 \mathcal{C}^k([0,T[; \mathrm{H}^{2-k}(]2M,\infty[)$$

of (1.7) *such that*

$$\lim_{t\uparrow T} \|u(t)\|_{\mathrm{L}^2(]2M,\infty[)} = \infty.$$

The above result means that the wave equation in Schwarzschild metric has a similar critical exponent to the free wave equation.

2. Multiplication technique. In this section we assume that $v \in \mathcal{S}(\mathbf{R})$ satisfies the equation

$$zv + Pv = g, \qquad z = \lambda^2 \pm i\varepsilon, \qquad s \in \mathbf{R}, \tag{2.1}$$

where λ is a fixed positive number, $\varepsilon > 0$ is a sufficiently small number and

$$P = \partial_{ss} - W(s). \tag{2.2}$$

We assume further that $W(s)$ is a real-valued $\mathcal{C}^1$ potential satisfying

$$(s - s_0)\partial_s W(s) < -(s - s_0)^2 \,\mathrm{e}^{-\varepsilon_0|s-s_0|} \qquad \forall s \neq s_0 \tag{2.3}$$

for a suitable $s_0 \in \mathbf{R}$, as well as the estimate

$$0 < W(s) \leqslant \frac{C_0}{(1 + |s|)^a}, \qquad a > 2, \tag{2.4}$$

for any $s \in \mathbf{R}$.

Further, we assume that there exists $s_+ = s_+(\lambda) > s_0$ so that

$$W(s) < \frac{\lambda^2}{2},\ \forall s \in (s_+, \infty). \tag{2.5}$$

Similarly, there exists $s_- = s_-(\lambda) < s_0$, so that

$$W(s) < \frac{\lambda^2}{2},\ \forall s \in (-\infty, s_-). \tag{2.6}$$

It is clear that without loss of generality we can assume

$$\Sigma_0 \equiv s_+ - s_0 = s_0 - s_-. \tag{2.7}$$

Consider the quantity

$$E(s) = \frac{1}{2}|v'(s)|^2 + \frac{\lambda^2}{2}|v(s)|^2 - \frac{1}{2}W(s)|v(s)|^2. \tag{2.8}$$

It is clear that

$$\frac{1}{2}|v'(s)|^2 + \frac{\lambda^2}{4}|v(s)|^2 \leqslant E(s) \leqslant \frac{1}{2}|v'(s)|^2 + \frac{\lambda^2}{2}|v(s)|^2 \tag{2.9}$$

for $s > s_+$ and $s < s_-$.

One can verify the relation

$$E'(s) = \mathrm{Re}\, v''(s)\,\overline{v'(s)} + \lambda^2 \mathrm{Re}\, v'(s)\,\overline{v(s)} - \frac{1}{2}\,W'(s)\,|v(s)|^2 - W(s)\,\mathrm{Re}\, v'(s)\,\overline{v(s)}. \tag{2.10}$$

Taking into account the fact that u satisfies the equation (2.1), we find

$$E'(s) = \mathrm{Re}\, g(s)\,\overline{v'(s)} - \frac{1}{2}\,W'(s)\,|v(s)|^2 \mp \mathrm{Re}\, i\varepsilon v(s)\,\overline{v'(s)}, \tag{2.11}$$

so we have the relation

$$E'(s) + \frac{1}{2}\,W'(s)\,|v(s)|^2 = \mathrm{Re}\, g(s)\,\overline{v'(s)} \mp \varepsilon \mathrm{Im}\, v'(s)\,\overline{v(s)}, \tag{2.12}$$

In a similar way, we can multiply the equation (2.1) by $\overline{v(s)}$, take the real part and find that

$$\frac{1}{2}\,\partial_{ss}|v(s)|^2 - |\partial_s v(s)|^2 - W(s)\,|v(s)|^2 + \lambda^2|v(s)|^2 = \operatorname{Re} g(s)\,\overline{v(s)}. \tag{2.13}$$

Taking the imaginary part, we have

$$\pm\varepsilon|v(s)|^2 + \Big(\operatorname{Im} v'(s)\,\overline{v(s)}\Big)' = \operatorname{Im} g(s)\,\overline{v(s)}. \tag{2.14}$$

Using the fact that $v(s)$ is rapidly decreasing function, we integrate the last inequality on $(-\infty,\infty)$ and obtain

$$\varepsilon\int_{-\infty}^{\infty}|v(s)|^2ds \leqslant C\int_{-\infty}^{\infty}|g(s)|\,|v(s)|ds. \tag{2.15}$$

Lemma 2.1. *We have the estimate*

$$\varepsilon\int_{-\infty}^{\infty}|v(s)|^2ds + \frac{\varepsilon}{1+\lambda^2}\int_{-\infty}^{\infty}|v'(s)|^2ds \leqslant C\int_{-\infty}^{\infty}|g(s)|\,|v(s)|ds. \tag{2.16}$$

Turning back to the relation (2.12), we take $\sigma > s_0$, integrate from σ to $+\infty$ and derive

$$E(\sigma) - \frac{1}{2}\int_{\sigma}^{\infty} W'(\tau)\,|v(\tau)|^2d\tau = -\operatorname{Re}\int_{\sigma}^{\infty} g(\tau)\,\overline{v'(\tau)}\,d\tau \,\pm\, \varepsilon\int_{\sigma}^{\infty}\operatorname{Im} v'(\tau)\,\overline{v(\tau)}d\tau,$$

Setting

$$K = \sqrt{1+\lambda^2}\int_{-\infty}^{\infty}|g(s)|\,|v(s)|ds + \int_{-\infty}^{\infty}|g(s)|\,|v'(s)|ds, \tag{2.17}$$

we apply Lemma 2.1 and get

$$E(\sigma) \leqslant CK \quad \forall\sigma > s_0. \tag{2.18}$$

Taking $\sigma > s_+$, we can use the inequality (2.9) and find

$$\frac{1}{2}|v'(\sigma)|^2 + \frac{\lambda^2}{4}|v(\sigma)|^2 \leqslant CK \quad \forall\sigma > s_+. \tag{2.19}$$

In a similar way, we get

$$\frac{1}{2}|v'(\sigma)|^2 + \frac{\lambda^2}{4}|v(\sigma)|^2 \leqslant CK \quad \forall\sigma < s_-. \tag{2.20}$$

Consider the operator

$$L = \beta(s)\partial_s + \frac{\partial_s\beta(s)}{2} = \beta(s)\partial_s + \gamma(s), \tag{2.21}$$

where $\beta(s)$ is a suitable real-valued $\mathcal{C}^3$ function that will be defined later on and $\gamma = \beta_s/2$. The operator L is formally anti self-adjoint so satisfies the relation

$$\langle Lh_1, h_2\rangle = -\langle h_1, Lh_2\rangle, \tag{2.22}$$

where $h_1, h_2 \in \mathcal{S}(\mathbf{R})$ and here and below $\langle\cdot,\cdot\rangle$ is the standard L^2-product in $\mathbf{R}$. Note also that the operator P in (2.2) is formally self-adjoint operator so

$$\langle Ph_1, h_2\rangle = \langle h_1, Ph_2\rangle, \tag{2.23}$$

It is easy to combine the equation (2.1) and the relations (2.22), (2.23), deducing the relation

$$2\ \operatorname{Re}\langle zLv, v\rangle = 2\ \operatorname{Re}\langle Lg, v\rangle +\ \operatorname{Re}\langle [P,L]v, v\rangle, \tag{2.24}$$

where $[A,B]$ denotes the commutator of the operators A and B, i.e. $[A,B] = AB - BA$.

For the commutator in the right side of the above relation, we have the representation

$$[L, \partial_{ss} - W] = -2\beta_s \partial_{ss} - 2\beta_{ss}\partial_s - \left(\frac{1}{2}\beta_{sss} + \beta W_s\right). \tag{2.25}$$

This operator is formally symmetric, since can be represented in the form

$$[L, \partial_{ss} - W] = -2\partial_s \beta_s \partial_s - \left(\frac{1}{2}\beta_{sss} + \beta W_s\right). \tag{2.26}$$

Since $z = \lambda^2 \pm i\varepsilon$, we have also the relation

$$2 \text{ Re}\,\langle zLv, v\rangle = \mp 2\varepsilon \text{ Im}\,\langle \beta\partial_s v, v\rangle. \tag{2.27}$$

Hence, the relation (2.24) can be rewritten as follows:

$$\mp 2\varepsilon \text{ Im}\,\langle \beta\partial_s v, v\rangle + 2\langle \beta_s \partial_s v, \partial_s v\rangle - \frac{1}{2}\langle (\beta_{sss} + 2\beta W_s)\, v, v\rangle = 2 \text{ Re}\,\langle Lg, v\rangle. \tag{2.28}$$

Taking into account the symmetry assumption (2.7), we can choose

$$\beta(s) = (s - s_0)\varphi_+(s - \Sigma_0)\varphi_-(s + \Sigma_0). \tag{2.29}$$

Here and below $\varphi_+(s)$ is a smooth decreasing function, such that

$$\varphi_+(s) = \begin{cases} 1, & \text{if } s \leqslant 0; \\ 0, & \text{if } s \geqslant 1. \end{cases} \tag{2.30}$$

Similarly, $\varphi_-(s)$ is a smooth increasing function, such that

$$\varphi_-(s) = \begin{cases} 1, & \text{if } s \geqslant 0; \\ 0, & \text{if } s \leqslant -1. \end{cases} \tag{2.31}$$

The function $\beta(s)$ satisfies the properties

$$\beta(s) = s - s_0,\ \beta'(s) = 1,\ \beta'''(s) = 0,\ \forall s \in (s_-, s_+), \tag{2.32}$$

$$\beta(s) = \beta'(s) = \beta'''(s) = 0,\ \forall s \text{ outside } (s_- - 1, s_+ + 1), \tag{2.33}$$

$$|\beta(s)| + |\beta'(s)| + |\beta'''(s)| \leqslant C\,\langle \Sigma_0\rangle, \quad \forall s \in\ (s_- - 1, s_+ + 1), \tag{2.34}$$

where $\langle x\rangle = \sqrt{1 + |x|^2}$.

Now, using Lemma 2.1, we find

$$|\varepsilon \text{ Im}\,\langle \beta\partial_s v, v\rangle| + |\text{Re}\,\langle Lg, v\rangle| \leqslant C\,\langle \Sigma_0\rangle\, K, \tag{2.35}$$

where K is defined in (2.17).

From these estimates we find

$$\begin{aligned} &\int_{\Sigma_0^\delta < |s - s_0| < \Sigma_0} \langle s - s_0\rangle^{-1/\delta}\, |v'(s)|^2 ds + \\ &\quad + \int_{\Sigma_0^\delta < |s - s_0| < \Sigma_0} \langle s - s_0\rangle^{-1/\delta} (s - s_0)^2\, \mathrm{e}^{-\varepsilon_0 |s - s_0|}\, |v(s)|^2 ds \leqslant CK, \end{aligned} \tag{2.36}$$

where $\langle s - s_0\rangle = \sqrt{1 + |s - s_0|^2}$.

Moreover, for any positive $\Sigma_1 < \Sigma_0$ it follows that

$$\frac{1}{\langle \Sigma_1\rangle} \int_{-\Sigma_1}^{\Sigma_1} |v'(s)|^2 ds + \frac{1}{\langle \Sigma_1\rangle} \int_{-\Sigma_1}^{\Sigma_1} (s - s_0)^2\, \mathrm{e}^{-\varepsilon_0 |s - s_0|}\, |v(s)|^2 ds \leqslant CK. \tag{2.37}$$

Now we can use the estimate (2.19), (2.6) together with the last two estimates (2.36) and (2.37), choose $\Sigma_1 = \Sigma_0^\delta$, $\delta > 0$, small enough and derive

$$\int_{-\infty}^{\infty} \langle s - s_0\rangle^{-1/\delta} \, |v'(s)|^2 ds + \int_{-\infty}^{\infty} (s - s_0)^2 \, \mathrm{e}^{-2\varepsilon_0|s-s_0|} \, |v(s)|^2 ds \leqslant C \; \langle \Sigma_0^\delta \rangle \, K. \tag{2.38}$$

One can see that the weight $(s - s_0)^2 \, \mathrm{e}^{-2\varepsilon_0|s-s_0|}$ in the above inequality can be replaced by $\mathrm{e}^{-3\varepsilon_0|s-s_0|}$. Indeed, the estimate

$$v(x) = v(x+10) + \int_0^{10} v'(x+\tau)d\tau \leqslant v(x+10) + \sqrt{10}\left(\int_0^{10} |v'(x+\tau)|^2 d\tau\right)^{1/2}$$

implies

$$\int_0^1 |v(x)|^2 dx \leqslant C \int_{10}^{11} |v(x)|^2 dx + C \int_0^{11} |v'(x)|^2 dx.$$

Thus, we modify (2.38) as follows

$$\int_{-\infty}^{\infty} \langle s - s_0\rangle^{-1/\delta} \, |v'(s)|^2 ds + \int_{-\infty}^{\infty} \mathrm{e}^{-3\varepsilon_0|s-s_0|} \, |v(s)|^2 ds \leqslant C \; \langle \Sigma_0^\delta \rangle \, K. \tag{2.39}$$

Using (2.39), we obtain

$$\| \, \mathrm{e}^{-\delta s^2} \, v\|_{L^2}^2 + \| \, \mathrm{e}^{-\delta s^2} \, v_s\|_{L^2}^2 \leqslant C \left(|Re\langle Lg, v\rangle| + \varepsilon |Im\langle \beta v_s, v\rangle|\right). \tag{2.40}$$

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ПОЛУЛИНЕЙНО ВЪЛНОВО УРАВНЕНИЕ В МЕТРИКА НА ШВАРЦШИЛД

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Изучаваме полулинейно вълново уравнение в метрика на Шварцшилд ($3+1$ мерно пространство – време). Първо анонсираме резултат за избухване на решението за всяко $p \in]1, 1+\sqrt{2}[$ и неотрицателни нетривиални начални данни. По-нататък обсъждаме подходяща резолвентна оценка на съответното уравнение на Хелмхолц, което е решаващо за доказване локалното намаляване на енергията за радиални данни.