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SOME SIMPLE LINEAR GROUPS AND THEIR FACTORIZATIONS*

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In this paper we consider simple groups G which can be represented as a product of two their proper non-Abelian simple subgroups A and B. Such an expression $G = AB$ is called a (simple) factorization of G. Here we suppose that G is a simple linear group in dimension at most 11 over the finite field $GF(q)$ and determine all the factorizations of G.

1. Introduction. Let G be a finite (simple) group. We are interested in the factorizations of G into the product of two simple subgroups. Our work here concerns the case when G is the simple linear group $L_n(q)$. In [2] the first author determined the factorizations of these series of groups in dimension at most 7. The present paper continues this investigation by considering the factorizations of the finite simple linear groups whose dimensions do not exceed 11. The following result is proved:

Theorem. *Let $G = L_n(q)$ with $8 \leq n \leq 11$. Suppose that $G = AB$, where A, B are proper non-Abelian simple subgroups of G. Then one of the following holds:*
(1) $n = 8,\ q \not\equiv 1\ (mod\ 7)$ *and* $A \cong L_7(q), B \cong PSp_8(q)$;
(2) $n = 10,\ (9, q-1) = 1$ *and* $A \cong L_9(q), B \cong PSp_{10}(q)$.

The factorizations of all the finite simple classical groups into the product of two maximal subgroups (so called maximal factorizations) have been treated in [5]. In particular, an explicit list of the maximal factorizations of the groups $L_n(q)$ has also been given in [5]. We make use of this result here.

In our considerations we shall freely use the notation and basic information about the finite (simple) classical groups given in [4]. Let V be the n-dimensional vector space over the finite field $GF(q)$ on which $G = L_n(q)$ acts naturally, and let P_k be the stabilizer in G of a k-dimensional subspace of V. From Proposition 4.1.17 in [4] we can obtain the structure of P_k. In particular, it follows that $P_1 \cong P_{n-1} \cong \{[q^{n-1}] : GL_{n-1}(q)\}/Z_{(n,q-1)}$. From this it follows immediately that P_1 ($\cong P_{n-1}$) contains a subgroup isomorphic to $L_{n-1}(q)$ if and only if $(n-1, q-1) = 1$.

2. Proof of the Theorem. Let $G = L_n(q)$, where q is a prime power. In our assumptions here $8 \leq n \leq 11$, and $G = AB$ where A, B are proper non-Abelian simple subgroups of G. The list of the maximal factorizations of G is given in [5]. In the case when $G = L_{11}(q)$ only one maximal factorization appears with one factor a (maximal)

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subgroup of G isomorphic to $\{Z_{q^{11}-1/q-1}.11\}/Z_{(11,q-1)}$. Obviously, there is no choice for one of the groups A and B to be a non-Abelian simple subgroup of G. Now, we proceed with the group $G = L_n(q)$ where $n = 8$, $n = 9$, or $n = 10$. Using the list of maximal factorizations in [5] and implicit information about the maximal subgroups there, by order considerations, we come to the following possibilities:

1) $n = 8$ or $n = 10$ and $A \cong L_{n-1}(q)$ (in P_1) with $(n-1, q-1) = 1$, $B \cong PSp_n(q)$;

2) $n = 8$ and $A \cong L_7(q)$ (in P_1) with $q \not\equiv 1$ $(mod\ 7)$, $B \cong L_4(q^2)$ or $B \cong PSp_4(q^2)$ (in a $L_4(q^2)$ subgroup of G);

3) $n = 8$ and $A \cong L_7(q)$ (in P_1) with $q \not\equiv 1$ $(mod\ 7)$, $B \cong \Omega_8^-(q)$ (in a $PSp_8(q)$ subgroup of G) and q even;

4) $n = 8$ and $A \cong L_7(q)$ (in P_1) with $q \not\equiv 1$ $(mod\ 7)$,$B \cong PSp_4(q^2)$ (in a $PSp_8(q)$ subgroup of G);

5) $n = 9$ and $A \cong L_8(q)$ (in P_1) with $(8, q-1) = 1$, $B \cong L_3(q^3)$;

6) $n = 10$ and $A \cong L_9(q)$ (in P_1) with $(9, q-1) = 1$, $B \cong L_5(q^2)$.

Now, we consider these possibilities case by case.

Case 1. These are the factorizations in (1) and (2) of the theorem. It remains to show that these factorizations actually exist. From Proposition 3.3 in [6] we have $SL_n(q) = SL_{n-1}(q).Sp_n(q)$ with natural embeddings of $SL_{n-1}(q)$ and $Sp_n(q)$ in $SL_n(q)$. Factoring out by $Z(SL_n(q))$, we obtain the factorizations in (1) and (2), as $SL_{n-1}(q) \equiv L_{n-1}(q)$ (by the condition $(n-1, q-1) = 1$).

Case 2. Let $B \cong L_4(q^2)$ and then $|A \cap B| = q^5(q^6-1)(q^4-1).(8, q-1)/(4, q^2-1)$. By the known subgroup structure of $L_4(q^2)$, it follows that $A \cap B$ is contained in a subgroup of B isomorphic to $([q^6] : GL_3(q^2))/Z_{(4,q^2-1)}$. From this fact we derive that an $L_3(q^2)$ subgroup must contain a proper subgroup of order divisible by $(q^6-1)(q^2+1)/6$ which contradicts to the subgroup structure of $L_3(q^2)$ for any q. At last if $B \cong PSp_4(q^2)$ is a subgroup of a $L_4(q^2)$ subgroup of G, then there is no factorization too.

Case 3. Here B is a subgroup of one $B_1(\cong PSp_8(q))$ subgroup of G. From $G = AB$ we have $G = AB_1$ (a covering factorization of $G = AB$). It follows that $|A \cap B| = q^5(q^6-1)(q^2-1)$ and $|A \cap B_1| = |PSp_6(q)|$ (recall that q is even). The obvious fact that $(A \cap B_1) \cap B = A \cap B$ leads, by order considerations, to the factorization $B_1 = (A \cap B_1).B$. Now, looking at the list of the maximal factorizations of $B_1 \cong PSp_8(q)$ (which can be derived from [5]) we see only one possible (maximal) factorization containing ours, namely $B_1 = (\overline{A \cap B_1}).\overline{B}$ where $q = 2$ and $\overline{A \cap B_1} \cong O_8^+(2), \overline{B} \cong O_8^-(2)$. Here $A \cap B_1$ is a subgroup of $\overline{A \cap B_1}(\cong O_8^+(2))$ and, thus, there exist a subgroup of order $|PSp_6(2)|$ in $O_8^+(2)$. The last is possible (by the subgroup structure of $O_8^+(2)$) only if the subgroup $A \cap B_1$ is isomorphic to $PSp_6(2)$, and then we reach the factorization $PSp_8(2) = PSp_6(2).\Omega_8^-(2)$, but $PSp_8(2)$ has no simple factorizations (see [1]).

Case 4. This time (in contrast to case 2) $B \cong PSp_4(q^2)$ is a subgroup of a $B_1 \cong PSp_8(q)$ subgroup of G. As in the previous case we conclude that $B_1 = (A \cap B_1).B$ with $|A \cap B_1| = |PSp_6(q)|.(8, q-1)$. There are four possible maximal factorizations of B_1 which may contain that factorization: $q = 2$ and $B_1 = (S_3 \times PSp_6(2)).(PSp_4(2).2)$ (with $A \cap B_1 < S_3 \times PSp_6(2)$), $B_1 = \widehat{P_1}.(PSp_4(q^2).2)$ (here $\widehat{P_1}$ is the stabilizer in B_1 of one-dimensional totally singular subspace of the 8-dimensional symplectic space on which $B_1 \cong PSp_8(q)$ acts naturally; moreover, $A \cap B_1 < \widehat{P_1}$), and $B_1 = O_8^\varepsilon(q).(PSp_4(q^2).2)$ (with $A \cap B_1 < O_8^\varepsilon(q)$, $\varepsilon = +$ or $\varepsilon = -$, and q is even).

From the first possibility it follows immediately that $A \cap B_1 \cong PSp_6(q)$ and the factorization $PSp_8(2) = PSp_6(2).PSp_4(4)$ arises, which is a contradiction (see [1]).

To discuss the next possibilities we need the following realization of the group $Sp_4(q^2)$ in $Sp_8(q)$. Let V_{2m} be the natural $2m$-dimensional symplectic space over the finite field $k = GF(q)$ on which $G_1 = Sp_{2m}(q)$ acts, and let $(,)$ be a nonsingular symplectic bilinear form on V_{2m}. There is a basis $\{e_i, f_i | i = 1, \ldots, m\}$ of V_{2m}, called a standard basis, such that $(e_i, e_j) = (f_i, f_j) = 0, (e_i, f_j) = \delta_{ij}$ for $i, j = 1, \ldots, m$. The group G_1 has the following (matrix) realization with respect to that basis

$$G_1 = \left\{ X \in SL_8(k) | X^t.L.X = L, L = \left(\begin{array}{c|c} O & E_{m\times m} \\ \hline -E_{m\times m} & O \end{array} \right) \right\}.$$

Let K be a field extension of $k = GF(q)$ of degree 2. There is an element ω of K such that $1, \omega$ form a basis of K over k, and $\omega^2 = 1 + t.\omega$ where $t \in k$. Further, let $S = (s_{ij}^0 + s_{ij}^1.\omega)_1^4 = S_0 + S_1.\omega$ with $S_0 = (s_{ij}^0)_1^4, S_1 = (s_{ij}^1)_1^4, s_{ij}^l \in k$ $(l = 0, 1; i, j = 1, 2, 3, 4)$ be any unimodular matrix such that $S^t L S = L$, and $m = 2$. Let us denote by $\tilde{B}$ the set of all the matrices S which satisfy these properties. So $\tilde{B} \cong Sp_4(q^2)$ has a standard symplectic realization over the field K. Then, the following matrices form a subgroup of G_1 (with $m = 4$) isomorphic to $Sp_4(q^2)$: $W = P. \left(\begin{array}{c|c} S_0 & S_1 \\ \hline S_1 & S_0 + S_1 \end{array} \right) .P^{-1}$,where $P = (p_{ij})_1^8$ and $p_{ij} = \begin{cases} 0, & \text{if } j \neq \pi(i) \\ 1, & \text{if } j = \pi(i) \end{cases}$ for the permutation $\pi = (1)(2,3,5)(4,7,6)(8)$. The isomorphism is given by the map $\sigma : S \mapsto W$.

Now, let $A \cap B_1 < \widehat{P_1}$. According to [4] there is only one conjugacy class of each of the subgroups $\widehat{P_1}$ and $Sp_4(q^2)$ in $Sp_8(q)$. Direct computations (using the above representation of $Sp_4(q^2)$ as a subgroup in $Sp_8(q)$) show that all the elements of this $Sp_4(q^2)$ subgroup stabilizing one-dimensional totally singular subspace have the following form (in terms of the matrices S above):

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ 0 & s_{22} & s_{23} & s_{24} \\ 0 & 0 & s_{11}^{-1} & 0 \\ 0 & s_{42} & s_{43} & s_{44} \end{pmatrix},$$

where $s_{11} \in k^*$;$s_{12} = (s_{23}s_{42} - s_{22}s_{43}).s_{11}, s_{14} = (s_{23}s_{44} - s_{43}s_{24}).s_{11}$, and $s_{22}s_{44} - s_{24}s_{42} = 1$. Thus, we can obtain the structure of the common subgroup of these $\widehat{P_1}$ and $Sp_4(q^2)$ subgroups in $Sp_8(q)$: $E_{q^6} : (Z_{q-1} \times SL_2(q^2))$. The corresponding group in $PSp_8(q)$ is isomorphic to $E_{q^6} : (Z_{q-1} \circ SL_2(q^2))$ and $A \cap B$ (of order $q(q^4 - 1).(8, q - 1)/(2, q^2 - 1)$) should be a subgroup of that group. It means that an $L_2(q^2)$ group contains proper subgroup of order divisible by $(q + 1)(q^2 + 1)$, which is impossible for any q.

Finally, let $A \cap B_1 < D \cong O_8^\varepsilon(q)$ and q be even. From the list of the maximal subgroups of $O_8^\varepsilon(q)$ (given in [4] and [5]) it follows $A \cap B_1 \cong PSp_6(q)$ and $A \cap B_1$ is a subgroup of the subgroup $D_1 \cong \Omega_8^\varepsilon(q)$ in D. Then, the factorization $B_1 = D_1.B$ is valid. We claim that this factorization does not really exist. Indeed, from the relevant maximal factorization (see 3.2.1 (d) p. 48 in [5]), in particular, it follows the factorization $B_1 = D.B$ with $H = D \cap B \cong O_4^\varepsilon(q^2)$. Using the information there and the above realization of $Sp_4(q^2) \cong PSp_4(q^2)$ (q is even), we can construct (in matrix form) the group H. For example, if $\varepsilon = +$ (so $D \cong O_8^+(q)$ and $D_1 \cong \Omega_8^+(q)$), then we choose B_1 to be the group G_1

in the above realization (with $m = 4$) and let $B = \sigma(\tilde{B})$. Further, let D be the subgroup of B_1 of all the matrices preserving the quadratic form $x_1x_5 + x_2x_6 + x_3x_7 + x_4x_8$ where $x_1, \dots, x_8$ are the coordinates of a vector in V_8 over the field k with respect to the standard symplectic basis. Now, $H = \sigma(\tilde{H})$ where $\tilde{H}$ is the subgroup of $\tilde{B}$ of all the matrices S preserving the quadratic form $x_1x_3 + x_2x_4$ with x_1, x_2, x_3, x_4 – coordinates of a vector in V_4 over the field K with respect to the standard symplectic basis. The isomorphism $O_4^+(q^2) \cong (L_2(q^2) \times L_2(q^2)).Z_2$ is well-known. Thus, in terms of the described groups we have $H = F.\langle r\rangle$ with $F \cong L_2(q^2) \times L_2(q^2)$ and

$$r = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = r_{e_1+f_1}.r_{e_2+f_2},$$

where $r_{e_1+f_1}$ and $r_{e_2+f_2}$ are the matrices in the basis $\{e_i, f_i | i = 1, 2, 3, 4\}$ of the reflections in the (nonsingular) vectors $e_1 + f_1$ and $e_2 + f_2$ respectively. It is obvious that F is a subgroup of D_1 and the same is true for the group $\langle r\rangle \cong Z_2$ (basic facts concerning the properties of the orthogonal groups, see [4]). Hence, H is a subgroup of D_1 and then $H = D_1 \cap B$. With similar considerations we can see that H is the common group of the groups D_1 and B also in the case when $\varepsilon = -$. Thus, by the orders, the last two groups do not give rise to any factorization.

Case 5. This case is similar to one of those considered in [3]. In [3] we have eliminated the case $G = AB$ with $G \cong U_6(q)$, $A \cong U_5(q)$, $B \cong L_3(q^2)$, and $q \geq 3$. Here, repeating the same considerations for the corresponding groups (mainly for the group $L_3(q^3)$ replacing the group $L_3(q^2)$), we reduce our case to only one possible factorization: $L_9(2) = L_8(2).L_3(8)$; the interception group has to be of order $2.3^2.7$. $L_9(2)$ has a single conjugacy class of subgroups isomorphic to $L_3(8)$. We construct (in matrix form) a representative of this class directly. Let $GF(8) = GF(2)(\omega)$ where $\omega^3+\omega+1 = 0$. Further, let $S \in SL_3(8) \equiv L_3(8)$ and $S = S_0 + S_1.\omega + S_2.\omega^2$, where the matrices $S_i (i = 0, 1, 2)$ have their entries in the field $GF(2)$. Then, the following matrices form a subgroup of $L_9(2)$ isomorphic to $L_3(8)$:

$$W = \begin{pmatrix} S_0 & S_2 & S_1 \\ S_1 & S_0 + S_2 & S_1 + S_2 \\ S_2 & S_1 & S_0 + S_2 \end{pmatrix}.$$

The isomorphism is given by the map $S \mapsto W$. $L_8(2)$ subgroup of $L_9(2)$ can be contained only in the maximal classes represented by P_1 and P_8. Each of them forms a single conjugacy class of subgroups in $L_9(2)$. Now, direct computations show that the common subgroup of the constructed $L_3(8)$ subgroup and P_1 or P_8 is isomorphic to $E_{2^6} : L_2(8)$. The fact that such a group has to contain a subgroup of order $2.3^2.7$ implies the existence in $L_2(8)$ of a proper subgroup of order divisible by 63, but this is impossible by the subgroup structure of $L_2(8)$.

Case 6. If such a factorization exists the order of $A \cap B$ should be $q^{11}(q^8-1)(q^6-1)(q^4-1).(10, q-1)/(5, q^2-1)$. A subgroup of $B \cong L_5(q^2)$ of that order has to be contained in a maximal subgroup isomorphic to $([q^8] : GL_4(q^2))/Z_{(5,q^2-1)}$. Consequently, $L_4(q^2)$ must have a proper subgroup of order divisible by $q^3(q^8-1)(q^6-1)(q^2+1).(10,q-1)/((5,q^2-1).(4,q^2-1))$. Now, looking at the list of the maximal subgroups of $L_4(q^2)$ we see that there is no such a possibility.

We considered all the posible cases. The theorem is proved.

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НЯКОИ ПРОСТИ ЛИНЕЙНИ ГРУПИ И ТЕХНИТЕ ФАКТОРИЗАЦИИ

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В тази статия разглеждаме прости групи G, които могат да се представят като произведение на две свои собствени неабелеви прости подгрупи A и B. Всяко такова представяне $G = AB$ се нарича (проста) факторизация на G. В настоящата работа предполагаме, че G е проста линейна група от размерност ненадминаваща 11 над крайното поле $GF(q)$ и определяме всички нейни факторизации.