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PERTURBATION ANALYSIS FOR A DIFFERENCE MATRIX RICCATI EQUATION*

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Non–local perturbation bounds are obtained for a symmetric difference matrix Riccati equation using the techniques of Lyapunov majorants. Equations of this type arise in the optimal control of linear discrete–time dynamic systems.

Statement of the problem. Consider the symmetric difference matrix Riccati equation

$$\begin{aligned} X_{k-1} &= Q_k + A_k^T (I_n + S_k X_k)^{-1} X_k A_k, \\ X_N &= B, \end{aligned} \tag{1}$$

where $Q_k, A_k, S_k, B \in \mathbb{R}^{n\times n}$ are matrix coefficients such that $Q_k = Q_k^T > 0$, $S_k = S_k^T \geq 0$, $k = 0, 1, \ldots, N$, and $B = B^T \geq 0$. The matrix $X_k = X_k^T \geq 0$ is the solution of equation (1) at the moment k. Here A^T is the transpose of the matrix A, while $A > 0$ (resp. $A \geq 0$) means that the symmetric matrix A is positive (resp. non–negative) definite. The unit $n \times n$ identity matrix is denoted as I_n.

Equations of type (1) arise in the linear–quadratic optimization of discrete–time dynamic systems over a finite time horizon.

Suppose that the matrix coefficients are perturbed as

$$A_k \to A_k + \Delta A_k,\ S_k \to S_k + \Delta S_k,$$

$$B \to B + \Delta B,\ Q_k \to Q_k + \Delta Q_k,\ k = 0, 1, \ldots, N,$$

and let ΔX_k be the perturbation of the solution X_k. Then, the perturbed equation is

$$\begin{aligned} X_{k-1} + \Delta X_{k-1} &= (A_k + \Delta A_k)^T \left(I_n + (S_k + \Delta S_k)(X_k + \Delta X_k)\right)^{-1} \\ &\quad \times (X_k + \Delta X_k)(A_k + \Delta A_k) + Q_k + \Delta Q_k, \\ X_N + \Delta X_N &= B + \Delta B. \end{aligned} \tag{2}$$

The aim of this paper is to derive a non–local perturbation bound for the norm of the perturbation in the solution as a function of the norms of the perturbations of the data.

Main results. As it is well known [3], if M and E are $n \times n$ matrices, such that M is non–singular and E is "small" in the sense that

$$\mathrm{rad}(M^{-1}E) = \mathrm{rad}(EM^{-1}) < 1,$$

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then the following equalities hold:

$$(M+E)^{-1} = M^{-1} - (M+E)^{-1}EM^{-1} = M^{-1} - M^{-1}E(M+E)^{-1}, \tag{3}$$

$$(M+E)^{-1} = M^{-1} - M^{-1}EM^{-1} + (M+E)^{-1}(EM^{-1})^2. \tag{4}$$

Denote

$$G_k = I_n + S_k X_k$$

and

$$F_k = S_k \Delta X_k + \Delta S_k X_k + \Delta S_k \Delta X_k.$$

It can be shown that the matrix G_k is non–singular. Suppose that

$$\mathrm{rad}(G_k^{-1} F_k) = \mathrm{rad}(F_k G_k^{-1}) < 1.$$

Then, using (3) we obtain

$$(G_k + F_k)^{-1} = G_k^{-1} - (G_k + F_k)^{-1} F_k G_k^{-1}.$$

After some calculations, the equation for ΔX_{k-1} takes the form

$$\begin{aligned} \Delta X_{k-1} &= A_k^T G_k^{-1} X_k \Delta A_k + A_k^T G_k^{-1} \Delta X_k (A_k + \Delta A_k) + \Delta A_k^T G_k^{-1} \Delta X_k (A_k + \Delta A_k) \\ &+ \Delta A_k^T G_k^{-1} X_k (A_k + \Delta A_k) + (A_k + \Delta A_k)^T (G_k + F_k)^{-1} F_k G_k^{-1} X_k (A_k + \Delta A_k) \\ &+ (A_k + \Delta A_k)^T (G_k + F_k)^{-1} F_k G_k^{-1} \Delta X_k (A_k + \Delta A_k), \\ \Delta X_N &= \Delta B. \end{aligned} \tag{5}$$

Further, we assume that

$$\|G_k^{-1}\| \, \|F_k\| \le \frac{1}{2}.$$

Then, using (4), we see that the inequality

$$\left\|(G_k + F_k)^{-1}\right\| \le \frac{\left\|G_k^{-1}\right\|}{1 - \|G_k\| \, \|F_k\|} \le 2 \left\|G_k^{-1}\right\| \tag{6}$$

holds.

Next we introduce the perturbation vector

$$\delta \;=\; (\delta_B, \delta_{A_1}, \dots, \delta_{A_N}, \delta_{Q_1}, \dots, \delta_{Q_N}, \delta_{S_1}, \dots, \delta_{S_N})^T \in \mathbb{R}_+^{3N+1}$$

with elements δ_Z equal to the norms $\|\Delta Z\|$ of the perturbations ΔZ in the data matrices $Z = B, A_k, Q_k, S_k$. Consider the following quantities:

$$\begin{aligned} a_0(\delta,k) &= \delta_{Q_k} + (2\|A_k\| + \delta_{A_k}) \left\|G_k^{-1}\right\| \|X_k\| \delta_{A_k} \\ &\quad + 2\left(\left\|G_k^{-1}\right\| \|X_k\| (\|A_k + \delta_{A_k})\right)^2 \delta_{S_k}, \\ a_1(\delta,k) &= \left\|G_k^{-1}\right\| (\|A_k + \delta_{A_k})^2 \left(1 + 2\|X_k\|(\|S_k\| + 2\delta_{S_k})\right), \\ a_2(\delta,k) &= 2\left\|G_k^{-1}\right\| (\|A_k + \delta_{A_k})^2 (\|S_k\| + \delta_{S_k}), \end{aligned} \tag{7}$$

depending on the perturbation vector δ and the current time k.

As a corollary from (5) and (6), the following bound for the perturbation $\|\Delta X_{k-1}\|$ is obtained

$$\begin{aligned} \|\Delta X_{k-1}\| &\le a_0(\delta,k) + a_1(\delta,k)\|\Delta X_k\| + a_2(\delta,k)\|\Delta X_k\|^2, \\ \|\Delta X_N\| &= \|\Delta B\| \le \delta_B, \end{aligned} \tag{8}$$

where the coefficients $a_i(\delta,k)$, $i = 0,1,2$, are defined by the equalities (7).

Let

$$h(\delta,\rho) = a_0(\delta,k) + a_1(\delta,k)\rho + a_2(\delta,k)\rho^2$$

be the Lyapunov majorant for equation (8), see [3]. Hence, supposing that $\|\Delta X_k\| \le \rho$ for some $\rho > 0$, the corresponding majorant equation is

$$a_0(\delta,k) - (1 - a_1(\delta,k))\rho + a_2(\delta,k)\rho^2 = 0.$$

If the inequalities

$$a_1(\delta,k) + 2\sqrt{a_0(\delta,k)a_2(\delta,k)} \le 1, \tag{9}$$

hold for each $k = N, N-1, \dots, 1$, then the estimation

$$\|\Delta X_{k-1}\| \le \frac{2a_0(\delta,k)}{1 - a_1(\delta,k) + \sqrt{(1 - a_1(\delta,k))^2 - 4a_0(\delta,k)a_2(\delta,k)}} \tag{10}$$

is true.

Let us denote

$$\begin{aligned}
\alpha &= \max\{\|A_1\|, \|A_2\|, \dots, \|A_N\|\},\\
\gamma &= \max\left\{\left\|G_1^{-1}\right\|, \left\|G_2^{-1}\right\|, \dots, \left\|G_N^{-1}\right\|\right\},\\
\sigma &= \max\{\|S_1\|, \|S_2\|, \dots, \|S_N\|\},\\
\chi &= \max\{\|X_1\|, \|X_2\|, \dots, \|X_N\|\},\\
\delta_A &= \max\{\delta_{A_1}, \delta_{A_2}, \dots, \delta_{A_N}\},\\
\delta_S &= \max\{\delta_{S_1}, \delta_{S_2}, \dots, \delta_{S_N}\},\\
\delta_Q &= \max\{\delta_{Q_1}, \delta_{Q_2}, \dots, \delta_{Q_N}\},\\
\widehat{\delta} &= (\delta_B, \delta_A, \delta_S, \delta_Q)^T \in \mathbb{R}_+^4.
\end{aligned}$$

Using (7), we define a new quadratic majorant function

$$\widehat{h}(\widehat{\delta},\rho) = \widehat{a}_0(\widehat{\delta}) + \widehat{a}_1(\widehat{\delta})\rho + \widehat{a}_2(\widehat{\delta})\rho^2$$

with coefficients

$$\begin{aligned}
\widehat{a}_0(\widehat{\delta}) &= \delta_Q + 2\alpha\gamma\chi\delta_A + \gamma\chi\delta_A^2 + 2\gamma^2\chi^2(\alpha+\delta_A)^2\delta_S,\\
\widehat{a}_1(\widehat{\delta}) &= \gamma(\alpha+\delta_A)^2(1 + 2\chi(\sigma+\delta_S) + 2\gamma\chi\delta_S),\\
\widehat{a}_2(\widehat{\delta}) &= 2\gamma(\alpha+\delta_A)^2(\sigma+\delta_S).
\end{aligned} \tag{11}$$

The new majorant equation $\widehat{h}(\widehat{\delta},\rho) = \rho$ now is

$$\widehat{a}_0(\widehat{\delta}) - (1 - \widehat{a}_1(\widehat{\delta}))\rho + \widehat{a}_2(\widehat{\delta})\rho^2 = 0. \tag{12}$$

Let us define the set

$$\Omega = \left\{\widehat{\delta} \in \mathbb{R}_+^4 : \widehat{a}_1(\widehat{\delta}) + 2\sqrt{\widehat{a}_0(\widehat{\delta})\widehat{a}_2(\widehat{\delta})} \le 1\right\}.$$

If $\widehat{\delta} \in \Omega$, then the equation (12) has real roots. We denote by $f(\widehat{\delta})$ the smallest positive root of this equation.

Applying the techniques of Lyapunov majorants, we state the following theorem.

Theorem 1. *Suppose that $\widehat{\delta} \in \Omega$. Then, the following estimates are valid*

$$\|\Delta X_{k-1}\| \le f(\widehat{\delta}) = \frac{2\widehat{a}_0(\widehat{\delta})}{1 - \widehat{a}_1(\widehat{\delta}) + \sqrt{(1 - \widehat{a}_1(\widehat{\delta}))^2 - 4\widehat{a}_0(\widehat{\delta})\widehat{a}_2(\widehat{\delta})}},\ k = 1, 2, \dots, N,$$

where the quantities $\widehat{a}_i(\widehat{\delta})$, $i = 0, 1, 2$, *are defined in* (11).

We point out that perturbation bounds for a periodic discrete–time matrix Riccati equation are obtained in [5]. Perturbation analysis for the algebraic discrete–time Riccati equation is done in [2] and [6], see also [3].

A similar perturbation problem for the differential matrix Riccati equation is formulated and solved in [4]. An alternative approach to this problem is presented in [1].

Example 1. Let us consider equation (1) in the case $n = 2$ with matrix coefficients

$$A_k = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; S_k = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; Q_k = \frac{1}{2k}\begin{pmatrix} \frac{3k+4}{4k+4} & 0 \\ 0 & 1 \end{pmatrix}$$

for each $k = 1, 2, \ldots, N$, and

$$B = \frac{1}{2(N+1)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The solution here is

$$X_{k-1} = \frac{1}{2k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for $k = 1, 2, \ldots, N$. Let the matrix coefficients be perturbed as

$$\Delta A_k = \frac{1}{2}\begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}, \; \Delta S_k = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix}$$

and

$$\Delta Q_k = \frac{\varepsilon}{2k}\begin{pmatrix} \frac{16+4\varepsilon+29k+5k\varepsilon+4(k+1)\varepsilon^2+12k^2}{(4k+4)(4k+4+\varepsilon+\varepsilon^2)} & 0 \\ 0 & 1 \end{pmatrix}$$

for $k = 1, 2, \ldots, N$, where $\varepsilon > 0$ is a small parameter. Then, we obtain

$$\Delta X_{k-1} = \frac{\varepsilon}{2k}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for each $k = 1, 2, \ldots, N$. Suppose that the Frobenius norm is used. Then, we have

$$\|A_k\| \le \frac{1}{2}, \; \|S_k\| \le \frac{1}{2}, \; \|X_k\| \le \frac{\sqrt{2}}{2}, \; \|G_k^{-1}\| \le \frac{\sqrt{2}}{2},$$

$$\|\Delta A_k\| \le \frac{\varepsilon}{2}, \; \|\Delta S_k\| \le \frac{\varepsilon}{2}, \; \|\Delta Q_k\| \le \frac{\varepsilon\sqrt{2}}{2}$$

for each $k = 1, 2, \ldots, N$. Thus, the perturbation vector is

$$\widehat{\delta} = (\delta_B, \widehat{\delta}_A, \widehat{\delta}_S, \widehat{\delta}_Q)^T = \left(0, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\sqrt{2}\varepsilon}{2}\right)^T \in \mathbb{R}^4_+.$$

The coefficients of the majorant function $\widehat{h}$ are calculated as

$$\begin{aligned} \widehat{a}_0(\widehat{\delta}) &= \frac{(3 + 2\sqrt{2} + 3\varepsilon + \varepsilon^2)\varepsilon}{4}, \\ \widehat{a}_1(\widehat{\delta}) &= \frac{(1+\sqrt{2})(1+\varepsilon)^3}{4}, \\ \widehat{a}_2(\widehat{\delta}) &= \frac{\sqrt{2}(1+\varepsilon)^3}{4}. \end{aligned}$$

In the Table below the perturbation bound $f(\widehat{\delta})$ is given according to Theorem 1 for different values of ε. This bound is compared with the greatest perturbation in the solution $\|\Delta X_0\|$. The ratio of both quantities shows that Theorem 1 gives relatively tight perturbation bounds in this particular case.

ε	$f(\widehat{\delta})$	$\|\Delta X_0\|$	$\frac{f(\widehat{\delta})}{\|\Delta X_0\|}$
0.01	0.0402946623	0.0070710678	5.6985257974
0.001	0.0037065442	0.0007071067	5.2418451673
0.0001	0.0003678493	0.0000707106	5.2021760124
0.00001	0.0000367572	0.0000070710	5.1982597657
0.000001	0.0000036754	0.0000007071	5.1978686400

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ПЕРТУРБАЦИОНЕН АНАЛИЗ НА ДИФЕРЕНЧНОТО МАТРИЧНО РИКАТИЕВО УРАВНЕНИЕ

Галина Божилова Пелова

Получени са нелокални пертурбационни граници за симетричното диференчно матрично Рикатиево уравнение в обратно дискретно време, с използване техниката на мажорантите на Ляпунов. Уравнения от този тип възникват при оптималното управление на линейни дискретни динамични системи върху краен времеви интервал.