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APPROXIMATION BY OPERATOR OF CAO-GONSKA TYPE $G^+_{S,N}$. DIRECT AND CONVERSE THEOREM*

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The aim of this paper is to establish the equivalence between the approximating rate of a linear process and the appropriate Peetre K-functional. A general method is applied for proving converse inequalities developed by Z. Ditzian and K. Ivanov.

1. Inroduction. Let $f \in C[-1,1]$, and let n, s be positive integers. The operator $G_{s,n} : C[-1,1] \to \Pi_{sn-s}$ is defined by

$$G_{s,n}(f,x) = \pi^{-1} \int_{-\pi}^{\pi} f(\cos(\arccos x + v)) K_{s,n}(v) dv,$$

where $K_{s,n} = c_{n,s} \left(\dfrac{\sin(nv/2)}{\sin(v/2)} \right)^{2s}$, $\pi^{-1} \int_{-\pi}^{\pi} K_{s,n}(v) dv = 1$.

Let Lf denote the linear function interpolating f at -1 and 1, i.e.

$$L(f,x) := \frac{1}{2} f(1)(x+1) + \frac{1}{2} f(-1)(1-x), \quad -1 \le x \le 1.$$

We consider the sequence of operators

$$G^+_{s,n}(f,x) = G_{s,n}(f,x) + L(f,x) - L(G_{s,n}f,x),$$

where $G_{s,n}$ and L are given as above.

In [C-G] Cao and Gonska proved the following

Theorem A. *Let $n \ge 2$ and $s \ge 3$. Then, for $f \in C[-1,1]$ it holds*

$$\left| G^+_{s,n}(f,x) - f(x) \right| \le c\omega_2(f, \sqrt{1-x^2}.n^{-1}), -1 \le x \le 1,$$

where the second order modulus of continuity $\omega_2(f,\delta)$ is defined for $f \in C[a,b]$ by

$$\omega_2(f,\delta) := \{\sup |f(x+h) + f(x-h) - 2f(x)|, x, x \pm h \in [a,b], 0 \le h \le \delta\}.$$

The aim of this paper is to establish the equivalence between the approximation error of the operator $G^+_{s,n}$ and the appropriate Peetre K-functional.

Let $H_1(g(x)) := \left(1-x^2\right)^{\frac{1}{2}} \dfrac{d}{dx} g(x), \qquad H := H_1^2.$

For $f \in C[-1,1]$, let $\|f\| := \max\{|f(x)| : \ -1 \le x \le 1\}$.

We define for every $f \in C[-1,1]$ and $t > 0$ the $K-$functional

$$K^+(f,t;C[-1,1],C^2,(I-L)H) := \inf\left\{\|f-g\| + t\,\|(I-L)Hg\| \ : g \in C^2\right\}.$$

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The idea for the equivalence of the approximation errors of a given sequence of operators and the values of proper $K-$functionals was studied systematically in [Di-Iv]. Such equivalence (in uniform norm) was established in [Z] for the algebraic version of trigonometric Jackson integrals $G_{s,n}$ and $K-$ functional

$$K\left(f, \frac{1}{n^2}; C[-1,1], C^2, H\right) = \inf\left\{\|f-g\| + \frac{1}{n^2}\|Hg\| : g \in C^2\right\}$$

and for the operator $G^*_{s,n}(f,x) = G_{s,n}(f,x) + L(f,x) - G_{s,n}(Lf,x)$ and $K-$ functional

$$K^*\left(f, \frac{1}{n^2}; C[-1,1], C^2, H(I-L)\right) = \inf\left\{\|f-g\| + \frac{1}{n^2}\|H(I-L)g\| : g \in C^2\right\}$$

(I is the identity operator).

In the present paper we establish the equivalence between the approximating rate of the operator $G^+_{s,n}$ and the value of the $K-$ functional K^+. We prove

Theorem 1. *For $s \geq 3$ and $f \in C[-1,1]$ we have*

$$\left\|G^+_{s,n}f - f\right\| \sim K^+\left(f, \frac{1}{n^2}; C[-1,1], C^2, (I-L)H\right), \quad n \in N.$$

The proof of the equivalence in Theorem 1 could not be carried out by using the technique for $G^*_{s,n}$ in [5]. The theorem follows from several basic inequalties.

The notation $\Phi(f,t) \sim \Psi(f,t)$ means that there is a positive constant γ, independent of f and t, such that $\gamma^{-1}\Psi(f,t) \leq \Phi(f,t) \leq \gamma\Psi(f,t)$.

By c we denote positive constants, independent of f and t, that may differ at each occurrence.

For r – natural number we denote
$C^r[a,b] = \left\{f : f, f', ..., f^{(r)} \in C[a,b] (\text{continuous function in } [a,b])\right\}$

2. Proof of Theorem 1.

Definition 2. *Set*

$$Y = \{g \in C[-1,1] : H_1 g \in C[-1,1], Hg \in C[-1,1], H_1 g(\pm 1) = 0\}.$$

$$Z = \{g \in Y : H_1^3 g \in C[-1,1], H^2 g \in C[-1,1], H_1^3 g(\pm 1) = 0\}.$$

Lemma 1. *Let $g \in Y$ and $\widetilde{g}(s) := g(\cos s)$. Then, $\widetilde{g} \in C^2(\mathbb{R})$ and $\widetilde{g}''(s) = Hg(\cos s)$ for $s \in \mathbb{R}$.*

The proof is based on the fact that $\dfrac{d\widetilde{g}(s)}{ds} = 0$ at $s = k\pi, k = 0, \pm 1, \pm 2$. We get that the left and the right second derivative of the function $\widetilde{g}(s)$ at $s = k\pi, k = 0, \pm 1, \pm 2$ are equal.

Lemma 2. *Let Y be the space from Definition 2. Then, for every $f \in C[-1,1]$ and $t > 0$, we have*

$$\begin{aligned} K(f,t;C[-1,1],Y,H) &= K(f,t;C[-1,1],C^2,H), \\ K^+(f,t;C[-1,1],Y,(I-L)H) &= K^+(f,t;C[-1,1],C^2,(I-L)H). \end{aligned}$$

The proof follows arguments as those of Lemma 3 in [3].

Lemma 3. *For a natural number m we have*

$$(G^+_{s,n})^m = L + [(I-L)G_{s,n}]^m = L + (I-L)G^m_{s,n}.$$

Using that for $f \in C[-1,1]$ (as $G_{s,n}$ maps a linear function into a linear function) $(I-L)G_{s,n}Lf = 0$, by induction on m we prove the lemma.

The kernel $K_{s,n}(v)$ is a positive, even trigonometric polynomial and $\int_0^\pi u^k K_{s,n}(v)dv \sim n^{-k}$, $k = 0, \ldots, 2s-2$ (see [4], p.57).

Proof of Theorem 1. The theorem will be proved if we show that $K^+(f, \frac{1}{n^2}; C[-1,1], Y, (I-L)H) \sim \left\|G^+_{s,n}f - f\right\|$ (see Lemma 2). We have to establish the inequalities:

1) $\left\|G^+_{s,n}f - f\right\| \le cK^+\left(f, \frac{1}{n^2}; C[-1,1], Y, (I-L)H\right)$;

2) $K^+\left(f, \frac{1}{n^2}; C[-1,1], Y, (I-L)H\right) \le c\left\|G^+_{s,n}f - f\right\|$.

To prove that $\left\|G^+_{s,n}f - f\right\| \le cK^+ f, \frac{1}{n^2}; C[-1,1], Y, (I-L)H)$, it is sufficient to show that $G^+_{s,n}$ is bounded (which is a well-known fact) and for every $g \in Y$, $\left\|G^+_{s,n}g - g\right\| \le c\frac{1}{n^2}\left\|(I-L)Hg\right\|$ (see [Di-Iv, p.72 , Th. 3.4]). Let $g \in Y$. We set $g(\cos(\arccos x + v)) := g(\cos(t+v)) := \widetilde{g}(t+v), \arccos x = t$. Note that $G^+_{s,n} - I = (I-L)(G_{s,n} - I)$.

We have

$$(G^+_{s,n}g - g)(x) = (I-L)(G_{s,n} - I)g(x) = (I-L)\pi^{-1}\int_{-\pi}^{\pi}(\widetilde{g}(t+v) - \widetilde{g}(t))K_{s,n}(v)dv.$$

Expanding $\widetilde{g}(t+v)$ by Taylor's formula (as $\widetilde{g} \in C^2(\mathbb{R})$ for $g \in Y$) $\widetilde{g}(t+v) - \widetilde{g}(t) = v\widetilde{g}'(t) + \int_t^{t+v}\widetilde{g}''(\xi)(t+v-\xi)d\xi$ and using that $\int_{-\pi}^{\pi} vK_{s,n}(v)dv = 0$, we obtain

$$(G^+_{s,n}g - g)(x) = (I-L)\pi^{-1}\int_{-\pi}^{\pi}\int_t^{t+v}\widetilde{g}''(\xi)(t+v-\xi)d\xi K_{s,n}(v)dv.$$

We have $\widetilde{g}''(\xi) = (Hg)(\cos\xi)$. Hence,

$$(G^+_{s,n}g - g)(x) = (I-L)\pi^{-1}\int_{-\pi}^{\pi}\int_t^{t+v}[(Hg)(\cos\xi)](t+v-\xi)d\xi K_{s,n}(v)dv.$$

As $(LHg)(\cos\xi) = p\cos\xi + q$, it is easy to obtain

$$\int_{-pi}^{\pi}\int_t^{t+v}[(LHg)(\cos\xi)](t+v-\xi)d\xi K_{s,n}(v)dv = Q + P\cos t = Q + Px,$$

where $P = p\int_{-\pi}^{\pi}(1-\cos v)K_{s,n}(v)dv, \qquad Q = \frac{q}{2}\int_{-\pi}^{\pi}v^2K_{s,n}(v)dv.$

Therefore,

$$(I-L)\int_{-\pi}^{\pi}\int_t^{t+v}[(LHg)(\cos\xi)](t+v-\xi)d\xi K_{s,n}(v)dv = (I-L)(Q+Px) \equiv 0.$$

This implies

$$\begin{aligned}(G^+_{s,n}g-g)(x) &= (I-L)\pi^{-1}\int_{-\pi}^{\pi}\int_{t}^{t+v}[(Hg)(\cos\xi)](t+v-\xi)d\xi K_{s,n}(v)dv\\ &= (I-L)\pi^{-1}\int_{-\pi}^{\pi}\int_{t}^{t+v}[(I-L)(Hg)(\cos\xi)](t+v-\xi)d\xi K_{s,n}(v)dv.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|G^+_{s,n}g-g\right\| &\leq 2\pi^{-1}\int_{-\pi}^{\pi}\int_{t}^{t+v}(t+v-\xi)d\xi K_{s,n}(v)dv\,\|(I-L)Hg\|\\ &= \pi^{-1}\int_{-\pi}^{\pi}v^2K_{s,n}(v)dv\,\|(I-L)Hg\| \leq c\frac{1}{n^2}\|(I-L)Hg\|.\end{aligned}$$

The last estimation completes the proof of the direct implication. The proof of the converse inequality consists of adapting $G^+_{s,n}f$ and $K^+\left(f,\frac{1}{n^2};C[-1,1],Y,(I-L)H\right)$ to the conditions of Theorem 3.1 in ([2], p.69) and Theorem 4.1 in ([2], p.72). We have $G^+_{s,n}f$ for Q_nf, $Df=(I-L)Hf$, $\Phi(f)=\left\|D^2(f)\right\|$, $\lambda(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}v^2K_{s,n}(v)dv\sim n^{-2}$ for $s\geq 2$, $\lambda_1(n)=\frac{1}{12\pi}\int_{-\pi}^{\pi}v^4K_{s,n}(v)dv\sim n^{-4}$ for $s\geq 3$.

The inequality (3.3) from Theorem 3.1 in ([2], p.69) is satisfied as $G^+_{s,n}$ is a bounded operator. To obtain the condition $A<1$ (see [2], p.72, Th. 4.1), it is sufficient to show that $A\frac{\lambda(n)}{\lambda_1(n)}=c\frac{n^2}{m}$ with $A<1$ i.e. $A=c\frac{\lambda_1(n)}{\lambda(n)}\frac{n^2}{m}\leq cm^{-1}<1$, which is true for large m. The results needed for inequalities (3.4), (3.5) and (3.6) from Theorem 3.1 in ([2], p. 69) is given in the following three lemmas from which our theorem follows.

Lemma 4. *For $f\in Z$ we have*

$$\left\|f-G^+_{s,n}f+\lambda(n)Df\right\|\leq\lambda_1(n)\Phi(f). \tag{1}$$

Lemma 5. *For $f\in C[-1,1]$ we have*

$$\left\|D^2(G^+_{s,n})^{2m}f\right\|\leq A\frac{\lambda(n)}{\lambda_1(n)}\left\|D(G^+_{s,n})^mf\right\|\ . \tag{2}$$

Lemma 6. *For $f\in C[-1,1]$ we have*

$$\left\|D(G^+_{s,n})^mf\right\|\leq\frac{B}{\lambda(n)}\|f\|. \tag{3}$$

We prove inequalities (2.2) and (2.3) for a power m of $G^+_{s,n}$ such that $A<1$.

Proof of Lemma 4. Let $f\in Z$. From $Z\subset Y$ and Lemma 1 we get $\widetilde{f}\in C^2(\mathbb{R})$. Applying Lemma 1 for Hf, we get $\widetilde{f}\in C^{(4)}(\mathbb{R})$.

We put $G_{s,n}f(x)=\pi^{-1}\int_{-\pi}^{\pi}f(\cos(\arccos x+v))c_{n,s}\left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^{2s}dv=\pi^{-1}\int_{-\pi}^{\pi}\widetilde{f}(t+v)$

$K_{s,n}(v)dv = \pi^{-1}\int_{-\pi}^{\pi}\widetilde{f}(t-v)K_{s,n}(v)dv \equiv \widetilde{G}_{s,n}\widetilde{f}(t)$, where $\widetilde{f}(t+v) := f(\cos(t+v) := f(\cos(\arccos x + v))$.

Using that $\frac{1}{2\pi}\int_{-\pi}^{\pi} v^2\widetilde{f}''(t)K_{s,n}(v)dv = \lambda(n)\widetilde{f}''(t)$, we get

$$\begin{aligned}
& f(x) - G^{+}_{s,n}f(x) + \lambda(n)Df(x) \\
= \ & f(x) - (L + (I-L)G_{s,n})f(x) + \lambda(n)(I-L)Hf(x) \\
= \ & (I-L)(f(x) - G_{s,n}f(x) + \lambda(n)Hf(x)) \\
= \ & (I-L)(\widetilde{f}(t) - \widetilde{G_{s,n}}\widetilde{f}(t) + \lambda(n)\widetilde{f}''(t)) \\
= \ & (I-L)\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\widetilde{f}(t) - \widetilde{f}(t+v) + \frac{1}{2}v^2\widetilde{f}''(t)\right]K_{s,n}(v)dv.
\end{aligned}$$

Expanding $\widetilde{f}(t+v)$ by Taylor's formula and using that $\int_{-\pi}^{\pi} v\widetilde{f}'(t)K_{s,n}(v)dv = 0$, $\int_{-\pi}^{\pi} v^3\widetilde{f}'''(t)K_{s,n}(v)dv = 0$, we obtain

$$f(x) - G^{+}_{s,n}f(x) + \lambda(n)Df(x) = -(I-L)\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{t}^{t+v}\frac{1}{3!}\widetilde{f}^{(4)}(\xi)(t+v-\xi)^3 d\xi K_{s,n}(v)dv.$$

We have $\widetilde{f}^{(4)}(\xi) = (H^2 f)(\cos\xi)$. Now we will prove that

$$(I-L)\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{t}^{t+v}\frac{1}{3!}[(H^2f)(\cos\xi)](t+v-\xi)^3 d\xi K_{s,n}(v)dv$$

$$= (I-L)\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{t}^{t+v}\frac{1}{3!}[(I-L)(H^2f)(\cos\xi)](t+v-\xi)^3 d\xi K_{s,n}(v)dv.$$

The last equality will be proved if we show that

$$\int_{-\pi}^{\pi}\int_{t}^{t+v}[(LH^2f)(\cos\xi)](t+v-\xi)^3 d\xi K_{s,n}(v)dv = P_1\cos t + Q_1 = P_1 x + Q_1.$$

Since $(LH^2f)(\cos\xi) = p\cos\xi + q$, we have to show that

$$\int_{-\pi}^{\pi}\int_{t}^{t+v}(p\cos\xi + q)(t+v-\xi)^3 d\xi K_{s,n}(v)dv = P_1\cos t + Q_1. \tag{4}$$

Integrating by parts 3 times the inner integral, we get (4) with

$$P_1 = \int_{-\pi}^{\pi}\left[3v^2 - 6\,(1-\cos v)\right]K_{s,n}(v)dv, \qquad Q_1 = \frac{1}{4}\int_{-\pi}^{\pi} v^4 K_{s,n}(v)dv.$$

Thus, $\left\|f - G^{+}_{s,n}f + \lambda(n)Df\right\|$

$$\begin{aligned}
&= \left\|(I-L)\frac{1}{\pi}\int_{-\pi}^{\pi}\int_{t}^{t+v}\frac{1}{3!}[(I-L)(H^2f)(\cos\xi)](t+v-\xi)^3 d\xi K_{s,n}(v)dv\right\| \\
&\le \frac{2}{\pi 3!}\int_{-\pi}^{\pi}\int_{t}^{t+v}(t+v-\xi)^3 d\xi K_{s,n}(v)dv\,\left\|(I-L)H^2 f\right\| \\
&= \frac{1}{12\pi}\int_{-\pi}^{\pi} v^4 K_{s,n}(v)dv\,\left\|(I-L)H^2 f\right\| = \lambda_1(n)\Phi(f).
\end{aligned}$$

In the last equality we used that the operator H maps a linear function into a linear function ($H(ax+b)=-ax$) and hence, $D^2f=(I-L)H(I-L)Hf=(I-L)H^2f$.

Proof of Lemma 5. We consider both sides of (2.2). Using Lemma 3 we have

$$\begin{aligned}
D^2(G^+_{s,n})^{2m}f &= (I-L)H(I-L)H(G^+_{s,n})^{2m}f=(I-L)H^2(G^+_{s,n})^{2m}f\\
&= (I-L)H^2(L+(I-L)G^{2m}_{s,n})f\\
&= (I-L)H^2Lf+(I-L)H^2(I-L)G^{2m}_{s,n}f\\
&= (I-L)H^2(I-L)G^{2m}_{s,n}f=(I-L)H^2G^{2m}_{s,n}f,\\
D(G^+_{s,n})^m f &= (I-L)H(G^+_{s,n})^m f=(I-L)H(L+(I-L)G^m_{s,n})f\\
&= (I-L)HLf+(I-L)H(I-L)G^m_{s,n}f\\
&= (I-L)H(I-L)G^m_{s,n}f=(I-L)HG^m_{s,n}f.
\end{aligned}$$

Thus, (2.2) is equivalent to

$$\left\|(I-L)H^2G^{2m}_{s,n}f\right\|\le A\frac{\lambda(n)}{\lambda_1(n)}\left\|(I-L)HG^m_{s,n}f\right\|. \tag{5}$$

We recall the inequality

$$\left\|HG^m_{s,n}f\right\|\le c\frac{n^2}{m}\left\|f\right\|, \tag{6}$$

which is valid for every $s\ge 2$ (see [Z, p.191]). Using the above inequality and that operators $G_{s,n}$ commute with H, we have

$$\begin{aligned}
\left\|(I-L)H^2G^{2m}_{s,n}f\right\| &= \left\|(I-L)(HG^m_{s,n})(HG^m_{s,n})f\right\|\\
&= \left\|(I-L)(HG^m_{s,n})(I-L)(HG^m_{s,n})f\right\|\\
&\le 2\left\|HG^m_{s,n}(I-L)HG^m_{s,n}f\right\|\\
&\le c\frac{n^2}{m}\left\|(I-L)HG^m_{s,n}f\right\|=A\frac{\lambda(n)}{\lambda_1(n)}\left\|(I-L)HG^m_{s,n}f\right\|.
\end{aligned}$$

This establishes (2.5) and, hence, completes the proof of (2.2).

Proof of Lemma 6. We have

$$\begin{aligned}
\left\|D(G^+_{s,n})^m f\right\| &= \left\|(I-L)HG^m_{s,n}f\right\|\\
&\le 2\left\|HG^m_{s,n}f\right\|\le c\frac{n^2}{m}\left\|f\right\|=\frac{B}{\lambda(n)}\left\|f\right\|.
\end{aligned}$$

In the last inequality we used the estimation (2.6). This completes the proof of (3).

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ПРИБЛИЖАВАНЕ С ОПЕРАТОРИ ОТ ТИП НА КАО-ГОНСКА $G^+_{S,N}$. ПРАВА И ОБРАТНА ТЕОРЕМА

Теодора Димова Запрянова

В работата се доказва еквивалентност на ръста на приближаване с линейния оператор от тип Као-Гонска и подходящо дефиниран K-функционал. Приложен е обща метод (развит от З. Дитциан и К. Иванов) за доказване на обратни неравенства.