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FUNCTIONAL TRANSFER THEOREMS FOR MAXIMA OF MOVING AVERAGE*

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In this paper we investigate the asymptotic behavior of sequences of random processes, whose time-intersections are maxima of random number of stationary moving average. The distribution function (df) of the noise components is subexponential and belongs to the domain of attraction of the Gumbel distribution. The time points are almost surely strictly increasing to infinity. The max-increments of these processes are not independent. Here it is proved that such a sequence of random processes converges weakly to a compound extremal process.
In particular, we consider cases in which the counting process is a mixed Poisson, or when the time points constitute a renewal process.

1. Introduction. The class of subexponential distributions (SE) was introduced by Chistyakov [3] in 1964. In 1988 Davis and Resnick [4] use point-processes techniques for studying the extremes of moving average sequence of random variables (*rv's*) from the max-domain of attraction (max-DA) of the Gumbel distribution. The same year Goldie and Resnick [5] obtained necessary and sufficient conditions for $F \in SE$ to belong to the same max-DA.

The asymptotic behavior of sequences of random processes, whose time intersections are random indexed maxima of independent identically distributed (*iid*) *rv's* is investigated in a series of papers of Pancheva and Jordanova (see e.g. [7] and [10]). Independently of them Satheesh et al. [12] investigate the properties of Λ-extremal processes. In [11], Pancheva et al. obtained necessary and sufficient conditions for a compound extremal process to have independent max-increments.

This paper contains analogous results for strictly stationary sequence and, more precisely, for moving average sequence with SE noise in max-DA of the Gumbel distribution.

Throughout this paper let $(\Omega, \mathcal{A}, P)$ be a given complete probability space with filtration $(\mathcal{A}_t)_{t\geq 0}$. We assume that all P – null sets of $\mathcal{A}$ are added to $\mathcal{A}_0$. We denote by $\mathcal{M}([0,\infty))$ the space of non-decreasing, right-continuous functions $y(t) : [0,\infty) \to [0,\infty)$ with finite left limits in $(0,\infty)$, endowed with the Skorokhod topology (see Billingsley [1]). When the sequence of random processes $\{\eta_n\}_{n\in\mathbf{N}}$ converges weakly in the Skorokhod topology to a stochastic process η, we write $\eta_n \Longrightarrow \eta$. All discussed random processes here have sample paths in $\mathcal{M}([0,\infty))$. As these processes have non-decreasing sample

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paths weak convergence of random processes, as random elements in $\mathcal{M}([0,\infty))$ coincide with convergence in J_1-topology of Skorokhod (see e.g [1]).

The next lemma is very useful, when we have to prove weak convergence of random processes. It describes a particular case when composition is a continuous mapping. It is an immediate consequence of Theorem 13.2.4 of [1].

Lemma 1.1. *Let Z_n, Z, θ_n, $n \in \mathbf{N}$ and Λ be random processes, whose path functions lie in $\mathcal{M}([0,\infty))$. We suppose that $Z_n \Longrightarrow Z$ and $\theta_n \Longrightarrow \Lambda$. We assume that for all $n \in \mathbf{N}$, Z_n and θ_n are independent. If the random process Z is stochastically continuous and Λ is such that for all $t > 0$,*

$$P(\forall \epsilon > 0, \Lambda(t-\epsilon) < \Lambda(t+\epsilon)/Z(\Lambda(t)-) < Z(\Lambda(t)) = 1 \quad \text{and} \tag{1}$$

$$P(\Lambda(t-) = \Lambda(t)/Z(\Lambda(t)-) < Z(\Lambda(t)) = 1,$$

then $Z_n \circ \theta_n \Longrightarrow Z \circ \Lambda$, as $n \to \infty$ and the random processes Z and Λ are independent.

If the process Λ has almost surely (a.s.) continuous and strictly increasing sample paths, condition (1) is automatically satisfied.

2. Description of the Model and Main Results. We suppose that $\{\xi_i\}_{i\in\mathbf{Z}}$ is a sequence of *iid rv's* with df $F \in SE$, in the max-DA of the Gumbel distribution

$$G(x) = \exp\{-e^{-x}\}, \quad x > 0.$$

We assume further, that the tails of F are balanced in the sense that there exists $p \in (0,1]$, such that

$$\lim_{x\to\infty} \frac{P(\xi_1 > x)}{P(|\xi_1| > x)} = p, \quad \lim_{x\to\infty} \frac{P(\xi_1 \le -x)}{P(|\xi_1| > x)} = 1 - p. \tag{2}$$

We denote by $\{X_n\}_{n\in\mathbf{Z}}$ the sequence of linear processes

$$X_n = \sum_{j=-\infty}^{\infty} c_j \xi_{n-j}, \quad n = \dots, -1, 0, 1, \dots. \tag{3}$$

We assume that the real numbers $\{c_i\}_{i\in\mathbf{Z}}$, satisfy the following condition:

There exists $\delta \in (0,1)$ such that

$$\sum_{j=-\infty}^{\infty} |c_j|^\delta < \infty. \tag{4}$$

Without loss of generality we can assume that

$$\max_j |c_j| = 1. \tag{5}$$

Under these conditions, the series in (3) is a.s. convergent and the sequence $\{X_n\}_{n\in\mathbf{Z}}$ is strictly stationary.

Let $q = 1 - p$, $T_0 = 0$ and

$$\tilde{N} = \{(T_k, X_k) : k \in \{0, 1, \dots\}\} \tag{6}$$

be a point process (*pp*) with $T_0 < T_1 < \cdots$, that are a.s. strictly increasing to infinity.

We denote the counting process of the time points by

$$N(t) = max\{n \ge 0 \mid T_n \le t\}. \tag{7}$$

We suppose further that the sequences $\{\xi_i\}_{i\in\mathbf{Z}}$ and $T_1, T_2, \dots$ are independent.

We denote by Y^+ the extremal process, generated by the Poisson *pp* N^+ with mean measure $\mu^+((0,t]\times(x,\infty)) = te^{-x}$ and by Y^- the extremal process, generated by the Poisson *pp* N^- with mean measure $\mu^-((0,t]\times(x,\infty)) = t\frac{q}{p}e^{-x}$. We assume that Y^+ and Y^- are independent.

The weak limit behavior of various quantities related to the extremes of $\{X_n\}_{n\in\mathbf{Z}}$ is discussed for example in [4], [8] and [12].

By the Invariance principle for the maxima of a linear process with SE noise in the max-DA of the Gumbel distribution (see e.g. Theorem 5.5.11 [6]) we have, that there exist sequences $a_n \sim F^{\leftarrow}\left(1-\frac{1}{n}\right)$ and b_n such that

$$Y_n(t) = \begin{cases} \bigvee_{i=1}^{[nt]} \dfrac{X_i - a_n}{b_n}, & t \geq n^{-1} \\ \dfrac{X_1 - a_n}{b_n}, & 0 < t < n^{-1} \end{cases} \Longrightarrow Y(t), \quad \text{in } \mathcal{M}([0,\infty)), \tag{8}$$

where

$$Y(t) = Y^+(t) \vee Y^-(t). \tag{9}$$

is a G_p extremal process, generated by the extreme-value distribution

$$G_p(x) = \exp\{-e^{-x}p^{-1}\}. \tag{10}$$

Now we are ready to state our main result.

Theorem 2.1. *Let $\tilde{N}$ be the pp, defined in* (6) *with counting process N, defined in* (7). *We assume that:*

i) *There exists a random process Λ with a.s. continuous and strictly increasing sample paths, such that*

$$\frac{N(nt)}{n} \Longrightarrow \Lambda(t), \quad n\to\infty \quad \text{in} \quad \mathcal{M}([0,\infty));$$

ii) *The sequence $\{X_n\}_{n\in\mathbf{Z}}$ has representation* (3) *with iid $\{\xi_i\}_{i\in\mathbf{Z}}$, that have SE distribution functions in the max-DA of the Gumbel distribution with centering constants $\{a_n\}_{n\in\mathbb{N}}$ and normalizing constants $\{b_n\}_{n\in\mathbb{N}}$ and that satisfy condition* (2). *The sequence of real numbers $\{c_i\}_{i\in\mathbf{Z}}$ is such that* (4) *and* (5) *are satisfied;*

iii) *The sequences $\{\xi_i\}_{i\in\mathbf{Z}}$ and $T_1, T_2, \ldots$ are independent.*

Then, $\tilde{Y}_n(t) \Longrightarrow Y(\Lambda(t))$, where

$$\tilde{Y}_n(t) = \begin{cases} \bigvee_{i=1}^{N(nt)} \dfrac{X_i - a_n}{b_n}, & t \geq \dfrac{T_1}{n} \\ \dfrac{X_1 - a_n}{b_n}, & 0 < t < \dfrac{T_1}{n} \end{cases}. \tag{11}$$

and Y is defined in (9).

Sketch of the proof: By Theorem 5.5.11 [6], under conditions in ii) we get (8) with Y, defined in (9). Because of i) and iii) we can apply Lemma 1.1. □

Corollary 2.1. *Let $\tilde{N}$ be the pp, defined in* (6) *with counting process N, which is a* *mixed Poisson process with random intensity* Λ *and* $E\Lambda < \infty$. *We assume that condition* ii) *is satisfied and the sequence* $\{\xi_i\}_{i\in\mathbf{Z}}$ *and the random process* N *are independent.*

Then, $\widetilde{Y}_n(t) \Longrightarrow \widetilde{Y}(t) = Y(\Lambda t)$ *in* $\mathcal{M}([0,\infty))$, *where* Y *is defined in* (9).

$\widetilde{Y}(t)$ *is a self-similar with respect to (w.r.t.) continuous one-parameter group (c.o.g.)* $\Gamma = \{(st, x + \ln s) \mid s > 0\}$.

Proof. Since the counting process N is mixed Poisson,

$$\frac{N(nt)}{n} \Longrightarrow \Lambda.t, \quad n \to \infty \text{ in } \mathcal{M}([0,\infty)).$$

The limiting process has continuous and strictly increasing sample paths, therefore we can use Theorem 2.1.

It is easy to check that $\Gamma = \{(st, x + \ln s) \mid s > 0\}$ is a c.o.g. w.r.t. the composition. Let $t > 0$. By the Law of total probability we have, that for all $s > 0$

$$P(\tilde{Y}(st) < x) = P(Y(\Lambda st) < x) = E\exp\{-\Lambda ste^{-x}p^{-1}\} =$$

$$= E\exp\{-\Lambda te^{-(x-\ln\ s)}p^{-1}\} = P(Y(\Lambda t) + \ln\ s < x) = P(\tilde{Y}(t) + \ln\ s < x). \qquad \square$$

If the time points $T_1, T_2, \ldots$ of the pp $\tilde{N}$, defined in (6) constitute a renewal process, with time between renewals $J_1, J_2, \ldots$, with $EJ_1 < \infty$, then the counting process N could be interpreted as a mixed Poisson with constant intensity $\Lambda = \dfrac{1}{EJ_1}$. An immediate consequence of Renewal theory and this corollary is that the limiting process is $Y\left(\dfrac{t}{EJ_1}\right)$. The last process is self-similar and max-stable.

When EJ_1 is not finite, we cannot apply the above theorem.

Theorem 2.2. *Let* $J_1, J_2, \ldots$ *be a.s. positive, iid random variables with df* J, *such that* $1 - J \in RV_{-\beta}$, $\beta \in (0,1)$. *Let* $\tilde{N}$ *be the pp, defined in* (6) *with time points* $T_1, T_2, \ldots$, *that constitute a renewal process with times between renewals* $J_1, J_2, \ldots$ *and counting process* N, *defined in* (7). *We suppose that conditions* ii) *and* iii) *are satisfied. Then,*

$$\bigvee_{k=1}^{N(nt)} \frac{X_k - \widetilde{a}_n}{\widetilde{b}_n} \Longrightarrow \tilde{Y}(t) = Y(E_\beta(t)), \quad \text{in } \mathcal{M}([0,\infty)),$$

where Y *is the* G_p *extremal process, defined in* (9), $E_\beta(t) = inf\{x \geq 0 \mid S_\beta(x) > t\}$ *is the hitting time process of the strictly stable Levy motion* $\{S_\beta(t)\}_{t\geq 0}$, *with*

$$S_\beta(1) \sim S_\beta\left(\sqrt[\beta]{(-\beta)\Gamma(-\beta)\cos\frac{\pi\beta}{2}}, 1, 0\right)$$

and $\widetilde{a}_n \sim F^{\leftarrow}(J(n))$.

Moreover, $\tilde{Y}$ *is a self-similar process w.r.t. c.o.g.* $\Gamma = \{(st, x + \beta.\ln s) : s > 0\}$ *and*

$$P(\tilde{Y}(t) < x) = \sum_{n=0}^{\infty} \frac{(-e^{-x}p^{-1}t^\beta)^n}{\Gamma(1 + n\beta)}, \quad x > 0.$$

Proof. By Theorem 5.5.11 [6] we have (8) with Y, defined in (9) and $a_n \sim \left(\dfrac{1}{1-F}\right)^{\leftarrow}(n)$.

By Theorem 3.6 [9], for $n \to \infty$,

$$\frac{N(nt)}{d_n} \Longrightarrow E_\beta(t), \ in\ \mathcal{M}([0,\infty))$$

where $d_n \sim \dfrac{1}{1-J(n)} \in RV_\beta$.

The sample paths of the processes $\dfrac{N(nt)}{d_n}$ and E_β are in $\mathcal{M}[0,\infty)$, but the sample paths of $E_\beta(t)$ are not a.s. strictly increasing. That is why, we have to check condition (1). This means that E_β should be a.s. continuous and strictly increasing at every point $t_0 > 0$, such that $E_\beta(t_0)$ is a point of discontinuity of Y. When we interpret this for S_β and Y, they a.s. should not have simultaneous jumps. This is obviously true, because these processes are independent and stochastically continuous. So condition (1) is satisfied. Now we use condition *iii)*, apply Lemma 1.1 and obtain

$$Y_{d_n}\left(\frac{N(nt)}{d_n}\right) \Longrightarrow Y(E_\beta(t))$$

and $\widetilde{a_n} \sim a_{d_n} \sim F^{\leftarrow}(J(n))$.

Let us now obtain self-similarity of the limiting process. Let $t > 0$. As the mixing process E_β is self-similar w.r.t. $\mathcal{L} = \{(st, xs^\beta) \mid s > 0\}$, we have that for all $s > 0$

$$P(\tilde{Y}(st) < x) = P(Y(E_\beta(st)) < x) = P(Y(s^\beta E_\beta(t)) < x).$$

By the Law of total probability we get

$$P(Y(s^\beta E_\beta(t)) < x) = E\exp\{-E_\beta(t)s^\beta e^{-x}p^{-1}\} =$$

$$= E\exp\{-E_\beta(t)e^{-(x-\beta \ln\ s)}p^{-1}\} = P(Y(E_\beta(t)) + \beta \ln\ s < x) = P(\tilde{Y}(t) + \beta \ln\ s < x).$$

To derive the *df* of the limiting process $\widetilde{Y}$ we use Corollary 3.2,(a) in [9]:

$$\begin{aligned} P(\tilde{Y}(t) < x) &= P(Y(E_\beta(t)) < x) = E\exp\{-t^\beta E_\beta(1)e^{-x}p^{-1}\} \\ &= E\exp\{-t^\beta e^{-x}p^{-1}(S_\beta)^{-\beta}\}. \end{aligned}$$

According to Bondesson, Kristiansen and Steutel [2], $(S_\beta)^{-\beta}$ is Mittag-Leffler distributed. So, we complete the proof. □

Note. The limiting processes in the above theorems are max-stable and stochastically continuous, but their max-increments are not independent in general.

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ФУНКЦИОНАЛНИ ГРАНИЧНИ ТЕОРЕМИ ЗА ПРЕНОСА ПРИ МАКСИМУМИ НА ПЛЪЗГАЩИ СЕ СРЕДНИ

Павлина Калчева Йорданова

В тази статия изследваме асимптотичното поведение на редици от случайни процеси, чиито сечения във фиксирани моменти от време са максимуми от случаен брой стационарни плъзгащи се средни. Функцията на разпределение на шумовите компоненти е субекспоненциална и принадлежи на областта на макс-привличане на разпределението на Гумбел. Моментите от време образуват почти сигурно строго растяща към безкрайност редица. Макс-нарастванията на тези процеси не са независими. Тук е доказано, че такава една редица от случайни процеси клони слабо към съставен екстремален процес.
В частност разглеждаме случаите, когато броящият процес е смесен Поасонов, или когато моментите от време образуват процес на възстановяване.