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ON CHAMLEY'S PROBLEM OF OPTIMAL TAXATION*

Mikhail Ivanov Krastanov, Rossen Alexandrov Rozenov

We provide an example to Chamley's problem of dynamic optimal taxation and show that one of the key assumptions on which previous results are based is not true, in general. We also demonstrate some cases where the problem of the consumer as formulated in the economic literature does not have a solution.

Introduction. In an influential article Chamley [1] introduced the problem of optimal taxation in a dynamic general equilibrium framework. He showed that if the utility function is separable in consumption and labour, then the optimal tax on capital income is zero in the long run. Frankel [3] considered the case of a general utility function and claimed that the capital income tax may be positive or negative depending on whether the sum of the elasticities of marginal utility with respect to consumption and labour is increasing or decreasing over time (Theorem 1). In the process of derivation of their results both Chamley and Frankel used informal arguments about the sign of the one of the co-state variables. Also, Chamley's proof relies essentially on another assumption which, as Xie [5] later proved, is not necessary for optimality. So far, a satisfactory solution to the problem has not been proposed.

In this paper we take the Chamley problem as a specific example of utility function and show that the assumption about the sign of the co-state variable for the private assets equation in the government's problem is not true, in general. Moreover, following the approach in [3], which is based on necessary conditions only, we demonstrate that the problem of the agent may not have a solution.

The setup in the general case as presented in [3] is as follows. There is one representative consumer which maximizes a utility functional subject to a differential equation describing the dynamics of the private assets. Formally, the consumer solves:

$$I = \int_0^\infty e^{-\rho t} u(c(t), l(t)) dt \to \max \tag{1}$$

$$\dot{a}(t) = r^*(t)a(t) + w^*(t)l(t) - c(t) \tag{2}$$

$$a(0) = a_0 \tag{3}$$

$$\lim_{t\to\infty} e^{-\int_0^t r^*(s)ds} a(t) \geq 0 \tag{4}$$

Here the controls are the consumption c and the labour input (hours of work). The functions $r^*(t)$ and $w^*(t)$ represent the after tax returns on assets and labour, respectively,

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and the consumer takes these functions as given by the government. Private assets consist of capital k and government bonds b, i.e. $a = k + b$. The instantaneous utility $u(c, l)$ is assumed to be concave with $u'_c > 0$ and $u'_l < 0$, and ρ is the time preference parameter. Condition (4) is an additional constraint which is typically imposed by economists to ensure that the private assets tend to a non-negative quantity as the time goes to infinity. In other words, the agent is not allowed to end up with debt.

The government solves the following problem:

$$I = \int_0^\infty e^{-\rho t} u(c^*(t), l^*(t))dt \to \max \tag{5}$$

subject to

$$\dot{a}(t) = r(t)a(t) + w(t)l^*(t) - c^*(t) \tag{6}$$

$$\dot{k}(t) = f(k(t), l^*(t)) - c^*(t) - g(t) \tag{7}$$

$$k(0) = k_0 \tag{8}$$

$$a(0) = a_0. \tag{9}$$

The government has control over r and w and takes c^* and l^* as given by the solution of consumer's problem. Since the two optimization problems are interrelated, we are in a situation of a dynamic game and in order to find the solution we have to adopt an appropriate equilibrium concept. Following the literature, we look for a Stackelberg equilibrium [2]. This means that in addition to (6) − (9), we add as an additional constraint the co-state equation for consumer's problem:

$$\dot{\pi}(t) = \pi(t)(\rho - r^*(t)), \tag{10}$$

Proposition 1. *The following assertions hold true:*

(*i*) *Let $\xi > 0$ be an arbitrary real number, c^* and l^* be optimal controls, and a^* be the corresponding optimal trajectory for the problem* (1)–(4)*. Then, c^* and l^* are optimal controls, and $a^*(t) + e^{\int_0^t r^*(s)ds}\xi$ is the corresponding optimal trajectory for the problem* (1)–(4)*, where the equality* (3) *is replaced by $a(0) = a_0 + \xi$;*

(*ii*) *Let $A > 0$ be an arbitrary real number, c and l be admissible controls, and a be the corresponding trajectory for the problem* (1)–(4) *such that*

$$\lim_{t\to\infty} \left(e^{-\int_0^t r^*(s)ds} a(t)\right) = A > 0. \tag{11}$$

If $\int_0^t r^*(s)ds \geq \rho t$ for each $t > 0$, then the controls c and l, and the trajectory a are not optimal for the problem (1)–(4)*.*

Proof. The proof of (i) is evident. In order to prove (ii), we consider the admissible control $(\bar{c}, \bar{l})$, where $\bar{c} = c + \alpha$, $\bar{l} = l$ and $0 < \alpha < \rho A$. Then,

$$I_1 = \int_0^\infty e^{-\rho t} u(\bar{c}(t) + \alpha, \bar{l}(t))dt > \int_0^\infty e^{-\rho t} u(c(t), l(t))dt = I$$

because of the properties of the utility function which is strictly increasing in c. The equality (11) is equivalent to

$$\lim_{t\to\infty}\left(a_0+\int_0^t e^{-\int_0^\tau r^*(s)ds}(w(\tau)l(\tau)-c(\tau))d\tau\right)=A.$$

Apply (4) for the trajectory $\bar{a}$ associated with the control $(\bar{c},\bar{l})$. Then,

$$e^{-\int_0^t r^*(s)ds}\bar{a}(t)=e^{-\int_0^t r^*(s)ds}a(t)-\alpha\int_0^t e^{-\int_0^\tau r^*(s)ds}d\tau$$

$$\lim_{t\to\infty}\left(e^{-\int_0^t r^*(s)ds}\bar{a}(t)\right)=\lim_{t\to\infty}\left(e^{-\int_0^t r^*(s)ds}a(t)-\alpha\int_0^t e^{-\int_0^\tau r^*(s)ds}d\tau\right)\geq$$

$$\geq\lim_{t\to\infty}\left(e^{-\int_0^t r^*(s)ds}a(t)-\alpha\int_0^t e^{-\tau\rho}d\tau\right)=A-\frac{\alpha}{\rho}>0.$$

Thus, we have found control $\bar{c}$ and $\bar{l}$ which give a higher value of the criterion and satisfies all constraints stated in [3]. This completes the proof. ♢

2. An example with separable utility. Next, we investigate Chamley's problem for a concrete example. Let the utility function be $u(c,l)=\ln c-\frac{l^2}{2}$, and the production function be $f(k,l)=\gamma_1 k+\gamma_2 l$ for some $\gamma_1>0$ and $\gamma_2>0$. Then, the Hamiltonian function for the agent's problem is

$$H_1(c,l,a,\pi)=\ln c-\frac{l^2}{2}+\pi(ar^*+w^*l-c)$$

and the necessary conditions are

$$c^*=\frac{1}{\pi}, \tag{12}$$

$$l^*=\pi w^*, \tag{13}$$

$$\dot{\pi}=\pi(\rho-r^*). \tag{14}$$

The private assets equation and the utility function can be rewritten in terms of w and π. The Hamiltonian for the government's problem is

$$H_2(r,w,a,k,\pi,\lambda,\mu,\xi)=-\ln\pi-\frac{\pi^2w^2}{2}+\lambda\left(ar+w^2\pi-\frac{1}{\pi}\right)+\mu\left(\gamma_1k+\gamma_2w\pi-\frac{1}{\pi}-g\right)+$$
$$+\xi\pi(\rho-r)$$

and the necessary conditions are

$$\lambda a-\xi\pi=0, \tag{15}$$

$$-w^*\pi^2+2w^*\lambda\pi+\mu\pi\gamma_2=0, \tag{16}$$

$$\dot{\lambda}=\lambda(\rho-r^*), \tag{17}$$

$$\dot{\mu}=\mu(\rho-\gamma_1), \tag{18}$$

$$\dot{\xi}=r\xi+\frac{1}{\pi}+w^{*2}\pi-\lambda w^{*2}-\frac{\lambda}{\pi^2}-\gamma_2\mu w^*-\frac{\mu}{\pi^2}. \tag{19}$$

From (15) we have that $a\lambda=\pi\xi$ and from (16), after dividing by $\pi>0$, we obtain that $\mu\gamma_2=w^*(\pi-2\lambda)$. Substitute the latter in (19) and multiply both sides of the resulting equation by π. Also, multiply both sides of (14) by ξ and add them. The resulting equation is

$$(20) \qquad (\dot{\xi\pi}) = \rho\xi\pi + 1 + w^{*2}\lambda\pi - \frac{\lambda}{\pi} - \frac{\mu}{\pi}.$$

Follow the same procedure for the equations for a and λ to obtain:

$$(21) \qquad (\dot{a\lambda}) = \rho a\lambda + w^{*2}\lambda\pi - \frac{\lambda}{\pi}.$$

After taking the difference of (20) and (21) and taking into account (15), we find that

$$(22) \qquad \mu(t) = \pi(t)$$

and from (14) and (18) it follows that

$$(23) \qquad r^* = \gamma_1.$$

Using Michel's result [4], which provides a relationship between the Hamiltonian function and the criterion when controls are optimal, we find that

$$(24) \qquad w^2\pi_0^2 = \frac{(\rho\pi_0 a_0 - 1)(\rho - 2\gamma_1)}{\rho}.$$

Since in order the integral (1) to be convergent it is necessary that $\rho < 2\gamma_1$, the last expression implies that $\rho\pi_0 a_0 - 1 < 0$. Finally, noting that the value of the Hamiltonian for agent's problem is equal to the one for government's problem and using (22) and (24), we obtain for $t = 0$ that

$$(25) \qquad \lambda_0 = \frac{\rho\pi_0^2(\gamma_1(a_0 - k_0) + g_0)}{2\gamma_1(\rho\pi_0 a_0 - 1)}.$$

Thus, the sign of λ_0 depends on the sign of $\gamma_1(a_0 - k_0) + g_0$. If the government starts with sufficient savings, then the latter expression could be negative and so λ_0 is positive in contrast with the assertions of Chamley and Frankel.

Since the admissible velocities of the system (2) contain a neighbourhood of the origin, Corollary 1 from [4] implies that $\lim_{t\to+\infty} \pi(t) = 0$. So, $\rho < \gamma_1$, and, hence, $\gamma_1 t > \rho t$ for each $t \geq 0$. Moreover, condition (i) of Proposition 1 implies the existence of a positive number A such that

$$(26) \qquad \lim_{t\to\infty} \left(e^{-t\gamma_1} a^*(t)\right) = A > 0.$$

Then, condition (ii) of Proposition 1 implies that the controls c^* and l^* cannot be optimal. The obtained contradiction shows that consumer's problem has no solution. In fact, this is the fundamental problem with the solutions of Frankel and Chamley. Finally, note that Chamley in [1] does not impose the condition for non-negative assets at infinity which practically allows to increase the value of the functional indefinitely by accumulating more and more debt.

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Mikhail I. Krastanov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str. Bl.8
1113 Sofia, Bulgaria
e-mail: krast@math.bas.bg

Rossen A. Rozenov
Faculty of Mathematics and Informatics
Sofia University
5 James Bourchier Blvd.
1164 Sofia, Bulgaria
e-mail: rossen_rozenov@yahoo.com

ОТНОСНО ЗАДАЧАТА НА ЧАМЛИ ЗА ОПТИМАЛНО ДАНЪЧНО ОБЛАГАНЕ

Михаил Кръстанов, Росен Розенов

Разгледан е моделът на Чамли за оптимално данъчно облагане и е показано, че едно от допусканията, върху които се основават наличните резултати не е изпълнено. Подробно е изледван един конкретен пример, за който са изпълнени всички предположения от икономическата литература, но в който задачата на потребителя няма решение.