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ON A CLASS OF BINARY MATRICES

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In the paper is studied the set of all square binary matrices containing an exact number of 1's in each rows and in each column. A connection is established between the cardinality of this set and the cardinality of its subset of matrices containing 1 in the lower right corner. With the help of this result, a new proof of the I. Good and J. Grook theorem is given. In connection with the first result a classification is also made of square binary matrices containing three 1's in each row and column and 1 in the lower right corner.

1. Introduction. A *binary* (or *boolean*, or $(0,1)$)-*matrix* is a matrix whose elements belong to the set $\mathbf{B}=\{0,1\}$. By $\mathbf{B}_n$ we denote the set of all $n\times n$ binary matrices.

Using the notation from [12], we call Λ_n^k-matrices all $n\times n$ binary matrices in each row and each column of which there are exactly k ones.

Let us accept the associative operations "$+$" and "$\cdot$" in the set $\mathbf{B}=\{0,1\}$ to be defined as: $0+0=0$, $1+0=0+1=1+1=1$, $0\cdot 0=1\cdot 0=0\cdot 1=0$, $1\cdot 1=1$. Let the scalar product of two vectors (defined by the n-tuple) with elements from $\mathbf{B}$ be determined by the operations thus introduced. Let us accept that in $\mathbf{B}_n$ the common definition for the operation of matrix product as a scalar product of the corresponding row vector and column vector is determined by the operation of scalar product introduced above. Considering all this, $\mathbf{B}_n$, along with the operation matrix product, is a semigroup that is isomorphic to the semigroup $\mathcal{B}(M)$ comprising all binary relations in a set M, where $|M|=n<\infty$ (see for example [5]).

Let us consider the following combinatorial problem:

Problem 1. *Find the number of all binary relations $\omega\in\mathcal{B}(M)$, $|M|=n<\infty$, such that for all $a\in M$ the equation*

$$|\{x\in M \mid (a,x)\in\omega\}| = |\{x\in M \mid (x,a)\in\omega\}| = k,$$

is correct, where n and k are positive integers.

Using the language of graph theory (see for example [1] or [4]) problem 1 is equivalent to

Problem 2. *Find the number of all bipartite graphs $G=(V_1\cup V_2,E)$, such that $|V_1|=|V_2|=n$ and each vertex is incident with exactly k edges.*

Clearly, problems 1 and 2 can be reduced to the following

Problem 3. *Find the number of all $n\times n$ matrices containing exactly k 1's in each row and each column, i.e. the number of all Λ_n^k-matrices.*

The goal of this paper is to consider certain special cases of problem 3.

Let us denote the number of all Λ_n^k-matrices with $\lambda_{n,k}$.

The following formula is proved in [10]

$$\lambda_{n,2} = \sum_{2x_2+3x_3+\cdots+nx_n=n} \frac{(n!)^2}{\prod_{r=2}^{n} x_r!(2r)^{x_r}}. \tag{1}$$

One of the first recurrence formula for calculation of $\lambda_{n,2}$ appeared in [2]:

$$\begin{aligned} &\lambda_{n,2} = \frac{1}{2}n(n-1)^2\left[(2n-3)\lambda_{(n-2),2} + (n-2)^2\lambda_{(n-3),2}\right] \quad \text{for} \quad n \geq 4 \\ &\lambda_{1,2} = 0, \quad \lambda_{2,2} = 1, \quad \lambda_{3,2} = 6 \end{aligned} \tag{2}$$

Another recurrence formula for the calculation of $\lambda_{n,2}$ occurs in [4]:

$$\begin{aligned} &\lambda_{n,2} = (n-1)n\lambda_{(n-1),2} + \frac{(n-1)^2 n}{2}\lambda_{(n-2),2} \quad \text{for} \quad n \geq 3 \\ &\lambda_{1,2} = 0, \quad \lambda_{2,2} = 1 \end{aligned} \tag{3}$$

The following recurrence system for calculation of $\lambda_{n,2}$ is put forward in [9]:

$$\left| \begin{array}{l} \lambda_{(n+1),2} = n(2n-1)\lambda_{n,2} + n^2\lambda_{(n-1),2} - \pi_{n+1} \quad \text{for} \quad n \geq 2 \\ \pi_{n+1} = \frac{n^2(n-1)^2}{4}[8(n-2)(n-3)\lambda_{(n-2),2} + (n-2)^2\lambda_{(n-3),2} - 4\pi_{n-1}] \quad \text{for} \quad n \geq 4 \\ \lambda_{1,2} = 0, \quad \lambda_{2,2} = 1, \quad \pi_1 = \pi_2 = \pi_3 = 0, \quad \pi_4 = 9 \end{array} \right. \tag{4}$$

where π_n denotes *the number of a special class* of Λ_n^2-matrices.

For the classification of all non defined concepts and notations as well as for common assertion which have not been proved here, we reffer to [1, 6, 8, 11].

2. On a partition of the set Λ_n^k. Let us introduce the notations:

$$\Lambda_n^{k+} = |\{A = (a_{ij}) \in \Lambda_n^k |\ a_{n\,n} = 1\}| \tag{5}$$

$$\Lambda_n^{k-} = |\{A = (a_{ij}) \in \Lambda_n^k |\ a_{n\,n} = 0\}| \tag{6}$$

Obviously,

$$\Lambda_n^{k+} \cap \Lambda_n^{k-} = \emptyset \quad \text{and} \quad \Lambda_n^{k+} \cup \Lambda_n^{k-} = \Lambda_n^k, \tag{7}$$

in other words, $\{\Lambda_n^{k+}, \Lambda_n^{k-}\}$ represents a partition of the set Λ_n^k.

We set:

$$\lambda_{n,k}^{+} = |\Lambda_n^{k+}| \tag{8}$$

$$\lambda_{n,k}^{-} = |\Lambda_n^{k-}| \tag{9}$$

Formula (3) occurs for the first time in [3]. It has been deduced in a manner applicable only to the calculation of the number of the Λ_n^2-matrices. The method for obtaining of the recurrence relation (3) which we offer and which we describe in Section 3.1 is closer to the discovery of the analogical formula for values greater than k. In this case, k represents the number of units in each row and each column of the respective square matrices. The method is based on the following assertion:

Theorem 1. *The equation*

$$\lambda_n^{k-} = \frac{n-k}{k}\lambda_n^{k+}, \tag{10}$$

where $\lambda_{n,k}^{+}$ and $\lambda_{n,k}^{-}$ are given by formulas (8) *and* (9), *respectively, holds true.*

Proof. Let A and B be Λ_n^k-matrices. We say that A and B are ρ-equivalent ($A\rho B$), if the removing of the columns ending in 1 results in equal $n \times (n-k)$ matrices. Obviously, ρ is an equivalence relation. We use ρ_A to denote the set of elements to which A is related by the equivalence relation ρ.

Let $A = (a_{ij})$ be a Λ_n^k-matrix. Let us denote by p^+ the number of all matrices ρ-equivalent to A in which the element in the lower right corner is equal to 1 and by p^- the number of all matrices ρ-equivalent to A in which the element in the lower right corner is equal to 0. Let $K_{j_1}, K_{j_2}, \dots, K_{j_k}$ be the row-vectors of the matrix A with 1 in the final position. The set $J = \{j_1, j_2, \dots, j_k\}$ is partitioned into subsets J_r, $r = 1, 2, \dots s$, such that j_u and j_v are part of the same subset if and only if $K_{j_u} = K_{j_v}$. It is easy to prove that $J = \bigcup\limits_{r=1}^{s} J_r$ and $J_u \cap J_v = \emptyset$ for $u \neq v$. We set $k_r = |J_r|$, $r = 1, 2, \dots, s$. Obviously,

$$k_1 + k_2 + \cdots + k_s = k. \tag{11}$$

Let C be the $n \times (n-k)$ matrix which is obtained from A by removing the columns $K_{j_1}, K_{j_2}, \dots, K_{j_k}$. In this case, with the help of the different ways of adding new columns to C, we obtain all elements of the set ρ_A. Let us first add k_1 columns which equal to those columns of A whose numbers belong to the set J_1. This can be done in $\binom{n-k+k_1}{k_1}$ ways. We can then add k_2 equal columns in $\binom{n-k+k_1+k_2}{k_2}$ possible ways. These equal columns are also equal to the columns in A with number tags from J_2, etc. Therefore,

$$|\rho_A| = \binom{n-k+k_1}{k_1}\binom{n-k+k_1+k_2}{k_2}\cdots\binom{n-k+k_1+k_2+\cdots+k_s}{k_s} =$$

$$= \binom{n}{k_s}\binom{n-k_s}{k_{s-1}}\binom{n-k_s-k_{s-1}}{k_{s-2}}\cdots\binom{n-k_s-k_{s-1}-k_{s-2}+\cdots+k_2}{k_1} =$$

$$= \frac{n!(n-k_s)!(n-k_s-k_{s-1})!\cdots(n-k_s-k_{s-1}-\cdots-k_2)!}{k_s!(n-k_s)!k_{s-1}!(n-k_s-k_{s-1})!\cdots k_1!(n-k_s-k_{s-1}-\cdots-k_2-k_1)!} =$$

$$= \frac{n!}{k_1!k_2!\cdots k_s!(n-k)!}.$$

Analogically, for p^- we get $p^- = \dfrac{(n-1)!}{k_1!k_2!\cdots k_s!(n-1-k)!}$, having in mind the fact that we cannot add new columns after the last column of matrix C.

For p^+ we obtain the equation:

$$p^+ = |\rho_A| - p^- = \frac{n!}{k_1!k_2!\cdots k_s!(n-k)!} - \frac{(n-1)!}{k_1!k_2!\cdots k_s!(n-1-k)!} = \frac{k(n-1)!}{k_1!k_2!\cdots k_s!(n-k)!}.$$

Then, $\dfrac{p^-}{p^+} = \dfrac{n-k}{k}$, i.e. $p^- = \dfrac{n-k}{k}p^+$. Summing over the equivalence classes, we arrive at the equation we had to prove.

By means of equations (5) $\div$ (9) and theorem 1, we get

Corollary 1.

$$\lambda_{n,k} = \lambda_{n,k}^+ + \lambda_{n,k}^- = \lambda_{n,k}^+ + \frac{n-k}{k}\lambda_{n,k}^+ = \frac{n}{k}\lambda_{n,k}^+. \tag{12}$$

3. Some applications. Theorem 1 and Corollary l are useful because they facilitate the calculation Λ_n^{k+} as compared to that of all Λ_n^k-matrices.

3.1. A different proof of the I. Good and J. Grook theorem. Using Corollary 1 in order to obtain a formula for the $n \times n$ binary matrices, it is enough to find a formula for $\lambda_{n,2}^+$. We can do this with the help of the following

Theorem 2. *If $n \geq 3$, then*

$$\lambda_{n,2}^+ = 2(n-1)\lambda_{(n-1),2} + (n-1)^2\lambda_{(n-2),2}.$$

Proof. Let $A = (a_{ij})$ be a Λ_{n-1}^2-matrix. The matrix A gives rise to the Λ_n^2-matrix $B = (b_{ij})$ in the following manner: We choose p, q such that $a_{p\,q} = 1$. This can be accomplished in $2(n-1)$ ways. We set $b_{p\,q} = 0$, $b_{p\,n} = b_{n\,q} = b_{n\,n} = 1$, $b_{i\,n} = b_{n\,j} = 0$ and $b_{ij} = a_{ij}$ for $1 \leq i, j \leq n-1$, $i \neq p$, $j \neq q$. It is easy to see that B is a Λ_n^2-matrix with 1 in the lower right corner. Besides, p and q can be identified uniquely through B and matrix A can be restored. Consequently, $\lambda_{n,2}^+ = 2(n-1)\lambda_{(n-1),2} + t$, where t is the number of all Λ_n^2-matrices containing 1 in the lower right corner, which cannot be generated in the manner described above. These are Λ_n^2-matrices. $B = (b_{ij})$ for which there are p and q such that $b_{p\,q} = b_{n\,q} = b_{p\,n} = b_{n\,n} = 1$ and these are the only 1's (2 in each row and column) in rows with number p and n and in columns with number q and n. In this case, however, removing rows with numbers p and n and columns with numbers q and n, we obtain a Λ_{n-2}^2-matrix. On the contrary, each Λ_{n-2}^2-matrix gives rise to a Λ_n^2-matrix by inserting two new rows, their numbers are p and n, and two new columns, their numbers are q and n, with 0 everywhere except for the places of intersection. Since p and q vary from 1 to $n-1$, $t = (n-1)^2\lambda_{(n-2),2}$. This proves the theorem. $\square$

Applying Theorems 1 and 2 we obtain:

Theorem 3. [3] *The number of all $n \times n$ square binary matrices with exactly two 1's in each row and each column is given by the next formula:*

$$\lambda_{n,2} = (n-1)n\lambda_{(n-1),2} + \frac{(n-1)^2 n}{2}\lambda_{(n-2),2} \quad \textit{for} \quad n \geq 3.$$
$$\lambda_{1,2} = 0, \quad \lambda_{2,2} = 1$$

3.2. On the number of Λ_n^3-matrices. The following formula gives an explicit way for the calculation of $\lambda_3(n)$ offered in [3]:

$$\lambda_3(n) = \frac{n!^2}{6^n}\sum \frac{(-1)^\beta(\beta+3\gamma)!2^\alpha 3^\beta}{\alpha!\beta!\gamma!^2 6^\gamma}, \tag{13}$$

where the sum is wide-spread over all the $\frac{(n+2)(n+1)}{2}$ solutions in nonnegative integers of the equation $\alpha + \beta + \gamma = n$.

As it is noted in [7], the formula (13) does not give us good opportunities to study the behavior of $\lambda_{n,3}$. The aim of the current consideration is to go one step closer to a new recurrence formula for the calculation of $\lambda_{n,3}$, which could help to avoid certain inconveniences results by the use of formula (13).

Let $X = (x_{ij}) \in \Lambda_n^{3+}$ and all 1's in the last columns and in the last row be respectively the elements $x_{s\,n}$, $x_{t\,n}$, $x_{n\,p}$, $x_{n\,q}$, $x_{n\,n}$, where $s, t, p, q \in \{1, 2, \ldots, n-1\}$, $s \neq t$, $p \neq q$. $\widetilde{X}$

denotes the 2×2 submatrix

$$\widetilde{X} = \begin{pmatrix} x_{sp} & x_{sq} \\ x_{tp} & x_{tq} \end{pmatrix}. \tag{14}$$

The set Λ_n^{3+} is partitioned into the following nonintersecting subsets:

$$\mathrm{A}_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \tag{15}$$

$$\mathrm{B}_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\} \right\} \tag{16}$$

$$\Gamma_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} \in \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \right\} \tag{17}$$

$$\Delta_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} \right\} \tag{18}$$

$$\mathrm{E}_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \right\} \tag{19}$$

$$\mathrm{Z}_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \right\} \tag{20}$$

$$\mathrm{H}_n = \left\{ X \in \Lambda_n^{3+} \middle| \widetilde{X} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \tag{21}$$

We set

$$\begin{gathered} \alpha_n = |\mathrm{A}_n|, \quad \beta_n = |\mathrm{B}_n|, \quad \gamma_n = |\Gamma_n|, \quad \delta_n = |\Delta_n|, \\ \epsilon_n = |\mathrm{E}_n|, \quad \zeta_n = |\mathrm{Z}_n|, \quad \eta_n = |\mathrm{H}_n|. \end{gathered} \tag{22}$$

Obviously,

$$\gamma_n = \delta_n \tag{23}$$

and

$$\lambda_{n,3}^{+} = \alpha_n + \beta_n + \gamma_n + \delta_n + \epsilon_n + \zeta_n + \eta_n. \tag{24}$$

Theorem 4.

$$\lambda_{n,3}^{+} = \frac{3(n-1)(3n-8)}{2} \lambda_{(n-1),3} + \alpha_n + \beta_n + 2\gamma_n - \eta_n.$$

Proof. Let $Y = (y_{ij}) \in \Lambda_{n-1}^3$. In Y we choose two 1's not belonging to the same row or column. Let then these be the elements y_{sp} and y_{tq}, $s, t, p, q \in \{1, 2, \ldots, n-1\}$, $s \neq t$, $p \neq q$. This can happen in $\dfrac{3(n-1)[3(n-1)-5]}{2} = \dfrac{3(n-1)(3n-8)}{2}$ ways. The 1's, thus selected, are turned into 0's and in Y in the last place one more column (number n) and one more row (number n) is added, so that $y_{sn} = y_{tn} = y_{np} = y_{nq} = y_{nn} = 1$ and $y_{in} = y_{nj} = 0$ for $i \notin \{s, t, n\}$, $j \notin \{p, q, n\}$. Obviously, the matrix, thus formed, belongs to one of the sets E_n, Z_n or H_n.

On the contrary, let $X = (x_{ij})$ be a matrix from E_n or Z_n. Then $\widetilde{X}$ has unique zero diagonal whose elements we turn into 1's and remove the last row (number n) and the last column (number n). In this way a Λ_{n-1}^3-matrix is generated.

Let $X = (x_{ij}) \in \mathrm{H}_n$ and $\widetilde{X} = \begin{pmatrix} x_{sp} & x_{sq} \\ x_{tp} & x_{tq} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. We select one diagonal of

$\widetilde{X}$ and turn 0's of this diagonal into 1's. Then we remove the last row (with number n) and the last column (with number n). We get a Λ_n^3-matrix. Consequently, two H_n-matrices correspond to each Λ_n^3-matrix. Then,

$$\lambda_{n,3}^{+} = \frac{3(n-1)(3n-8)}{2}\lambda_{(n-1),3} - \eta_n + t,$$

where $t = \alpha_n + \beta_n + \gamma_n + \delta_n$. Considering (23) and (24), we prove the theorem. □

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ВЪРХУ ЕДИН КЛАС БИНАРНИ МАТРИЦИ

Красимир Янков Йорджев

В работата е разгледано множеството от всички квадратни бинарни матрици с точно k единици на всеки ред и всеки стълб. Получена е връзка между мощността на това множество и мощността на неговото подмножеството от матрици с единица в долния десен ъгъл. С помощта на получения резултат е предложено ново доказателство на теорема на I. Good и J. Grook. Във връзка с първия резултат е направена и една класификация на квадратните бинарни матрици с точно три единици на всеки ред и всеки стълб и единица в долния десен ъгъл.