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CALCULATING WITH VECTORS IN PLANE GEOMETRY

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A vector-based method is discussed for solving plane geometry problems. The method is considered as an advantageous approach for presenting of, and educating in, the relevant parts of plane geometry, as well as a practical vehicle for derivation and expression of programs in computational geometry and computer graphics.

Introduction. The way in which geometry is taught in school today, has not changed significantly for quite a long time: almost all results are obtained using century- or millenia-old techniques. This venerable tradition serves remarkably well for exposing a vast amount of the subject of geometry, but not all of it. A number of geometrical problems consist of finding or comparing point locations, distances, angles, and areas and are solved most efficatiously – and aesthetically satisfying – by making use of vector calculus. These kinds of problems appear to be practically important, as witnessed e. g. by their immense presence in computational (algorithmic) geometry and computer graphics programming, but geometry textbooks tend to avoid them, because the classical derivation tools do not handle many of these problems easily, if at all.

The vector apparatus, introduced in different forms in the 19th century by Hamilton, Grassmann, Gibbs, and others, is apparently still a new tool in geometry, and this is reflected by both teaching it and applying it. Exposition of vectors in school is done diffidently, scantly, and almost as an end in itself. In fact, it is usually confined to vector addition and subtraction and multiplication by a scalar.

In our opinion, the limited presence of vectors in school geometry is partly due to not considering, from the outset, a useful enough set of operations on vectors, as well as techniques and methods that would have made use of these operations. The availability of such operations, techniques, and methods in geometry would provide simple solutions to many problems and thus justify a wider recognition of the place vector calculus deserves in the discipline.

In this paper, we are going to show that very modest additions to the way in which vectors are taught in plane geometry, can lead to significantly widening the scope of application of vector calculus for solving planar geometric problems.

Due to lack of space and for the sake of keeping the flow of presentation straight, proofs and derivations are mentioned or sketched rather than given in full. None of them is hard to reconstruct anyway.

Vector operations and equalities. We rely on the notion of a planar vector as usually introduced: a quantity that is determined by its magnitude and direction and also satisfies certain rules of combination, such as admitting addition and subtraction with another vector, and multiplication by a scalar, where addition is commutative and associative and multiplication is distributive over addition as well as associative.

A scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be *the product of the length of* $\mathbf{a}$ *and the oriented projection of* $\mathbf{b}$ *onto* $\mathbf{a}$*'s attitude*. The scalar product is 0 when any of $\mathbf{a}$ and $\mathbf{b}$ is of length 0, or when $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. It is positive when $\mathbf{b}$'s projection has the same orientation as $\mathbf{a}$ and negative when the two orientations are opposite to each other. The well-known properties of the scalar product – commutativity, distrubutivity etc. – follow easily from the definition.

We define a "*perp*" operation on a vector $\mathbf{a} \neq \mathbf{0}$, denoted by $\mathbf{a}^{\perp}$, to mean the vector of the same length as $\mathbf{a}$, rotated at a right angle counterclockwise with respect to $\mathbf{a}$. Thus, for any vector $\mathbf{a} \neq \mathbf{0}$ the ordered pair $(\mathbf{a}, \mathbf{a}^{\perp})$ can be used to represent the positive orientation in the plane. Even more importantly, given a point O, such a vector provides a right Cartesian coordinate system with origin O and axes $\mathbf{a}$, $\mathbf{a}^{\perp}$.

Finally, we define the *area product* $\mathbf{a} \star \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ as

$$\mathbf{a} \star \mathbf{b} = \mathbf{a}^{\perp} \cdot \mathbf{b} \, .$$

There is an obvious relation between the vector (cross) product of spatial vectors and the area product of planar vectors, but the former is a vector and the latter is a scalar.

It is easy to show that $\perp$ and $\star$ satisfy the following relations for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ and scalars k and k':

$$\begin{aligned}
(\mathbf{a}^{\perp})^{\perp} &= -\mathbf{a} & \mathbf{a} \star \mathbf{b} &= -(\mathbf{b} \star \mathbf{a}) \\
(k\,\mathbf{a} + k'\,\mathbf{b})^{\perp} &= k\,\mathbf{a}^{\perp} + k'\,\mathbf{b}^{\perp} & (k\,\mathbf{a} + k'\,\mathbf{b}) \star \mathbf{c} &= k\,(\mathbf{a} \star \mathbf{c}) + k'(\mathbf{b} \star \mathbf{c}) \\
\mathbf{a}^{\perp} \cdot \mathbf{b} &= -\mathbf{a} \cdot \mathbf{b}^{\perp} & \mathbf{a} \star \mathbf{b}^{\perp} &= -(\mathbf{a}^{\perp} \star \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \\
\mathbf{a}^{\perp} \cdot \mathbf{b}^{\perp} &= \mathbf{a} \cdot \mathbf{b} & \mathbf{a}^{\perp} \star \mathbf{b}^{\perp} &= \mathbf{a} \star \mathbf{b} \\
 & & |\mathbf{a} \star \mathbf{b}| &\leq |\mathbf{a}||\mathbf{b}|
\end{aligned}$$

As $\mathbf{a} \star \mathbf{b} = 0$ for any two collinear vectors, the area product is as useful test for collinearity as the scalar product is for perpendicularity. Moreover, while the arithmetic sign of $\mathbf{a} \cdot \mathbf{b}$ indicates an acute or obtuse angle between $\mathbf{a}$ and $\mathbf{b}$, that of $\mathbf{a} \star \mathbf{b}$ determines *orientation*: $\mathbf{a}$ is to the left or to the right of $\mathbf{b}$ for $\mathbf{a} \star \mathbf{b}$ negative or positive, correspondingly. The "symmetricity" of the two kinds of products is further suggested by $(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \star \mathbf{b})^2 = \mathbf{a}^2 \mathbf{b}^2$ (which can be deduced by observing that the area product could have been defined equivalently as *the product of the length of* $\mathbf{a}^{\perp}$ (*or* $\mathbf{a}$) *and the oriented projection of* $\mathbf{b}$ *onto* $\mathbf{a}^{\perp}$*'s attitude*).

The last observation also explains why we call $\star$ "area product": in this form, it is indeed the area of the parallelogram (or twice that of the triangle) spanned by the two vectors involved. Taking into account the sign, it is in fact an oriented area. Thus, unlike the scalar product, the area product has an immediate geometric interpretation!

For any three vectors $\mathbf{p}$, $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} \star \mathbf{v} \neq 0$, we can resolve $\mathbf{p}$ with respect to $\mathbf{u}$ and $\mathbf{v}$ by finding α and β in $\mathbf{p} = \alpha\,\mathbf{u} + \beta\,\mathbf{v}$. Applying $\star\mathbf{v}$ and $\mathbf{u}\star$ to this equation, we obtain $\alpha = (\mathbf{p} \star \mathbf{v})/(\mathbf{u} \star \mathbf{v})$ and $\beta = (\mathbf{u} \star \mathbf{p})/(\mathbf{u} \star \mathbf{v})$, thus:

$$\mathbf{p} = \frac{1}{\mathbf{u} \star \mathbf{v}}((\mathbf{p} \star \mathbf{v})\,\mathbf{u} + (\mathbf{u} \star \mathbf{p})\,\mathbf{v}) \tag{1}$$

Substituting $\mathbf{u}^\perp$ for $\mathbf{v}$, we get:

$$\text{(2)} \qquad \mathbf{p} \;=\; \frac{1}{\mathbf{u}^2}((\mathbf{u}\cdot\mathbf{p})\,\mathbf{u} + (\mathbf{u}\star\mathbf{p})\,\mathbf{u}^\perp)$$

The above general relations make expressing others possible. For example, given points P and M and non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$ the point P', obtained by reflecting P in the direction of $\mathbf{v}$ off the line through M along $\mathbf{u}$ (affine reflection), is given by

$$\mathbf{P'} = \mathbf{M} + \frac{1}{\mathbf{u}\star\mathbf{v}}\,((\overrightarrow{MP}\star\mathbf{v})\mathbf{u} + (\overrightarrow{MP}\star\mathbf{u})\mathbf{v})$$

– P' has, along $\mathbf{u}$ and $\mathbf{v}$, the same as P and the opposite components, respectively. For the usual orthogonal reflection, $\mathbf{v} = \mathbf{u}^\perp$

$$\mathbf{P'} = \mathbf{M} + \frac{1}{\mathbf{u}^2}\,((\overrightarrow{MP}\cdot\mathbf{u})\mathbf{u} + (\overrightarrow{MP}\star\mathbf{u})\mathbf{u}^\perp).$$

Changing the names and doing obvious rearrangements, (1) can be written in the more symmetric form:

$$\text{(3)} \qquad (\mathbf{p}\star\mathbf{q})\,\mathbf{r} + (\mathbf{q}\star\mathbf{r})\,\mathbf{p} + (\mathbf{r}\star\mathbf{p})\,\mathbf{q} \;=\; \mathbf{0}$$

This equality holds true for any three vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, even collinear ones.

Note that, in (1) or (2), if we know $\mathbf{u}$ and $\mathbf{v}$, as well as the products of $\mathbf{p}$ with these two vectors, then $\mathbf{p}$ can be computed. It is often the case that the said products can be found indirectly, as areas or projections. This constitutes a method for solving certain problems: choose a vector $\mathbf{u}$ (or two vectors $\mathbf{u}$ and $\mathbf{v}$), find the corresponding products, then use (1) or (2) to obtain $\mathbf{p}$. The method is illustrated by some of the examples in the following section.

Indirectly computing an area product or a scalar product can bring other benefits, too. Scalar-multiplying (3) by $\mathbf{q}$ and then substituting $-\mathbf{p}^\perp$ for $\mathbf{p}$ in the result, yields:

$$\begin{aligned}(\mathbf{p}\cdot\mathbf{q})\,(\mathbf{q}\star\mathbf{r}) + (\mathbf{p}\star\mathbf{q})\,(\mathbf{q}\cdot\mathbf{r}) &= (\mathbf{p}\star\mathbf{r})\,\mathbf{q}^2\\ (\mathbf{p}\cdot\mathbf{q})\,(\mathbf{q}\cdot\mathbf{r}) - (\mathbf{p}\star\mathbf{q})\,(\mathbf{q}\star\mathbf{r}) &= (\mathbf{p}\cdot\mathbf{r})\,\mathbf{q}^2\end{aligned}$$

which we identify as vector counterparts of the trigonometric equalities for the sine and cosine of a sum of two angles.

If $\mathbf{p}$ and $\mathbf{q}$ are known, then any one of $\mathbf{q}\star\mathbf{r}$, $\mathbf{q}\cdot\mathbf{r}$, and $\mathbf{p}\star\mathbf{r}$ can be obtained by knowing the other two and using the former equality. Similarly, any one of $\mathbf{q}\star\mathbf{r}$, $\mathbf{q}\cdot\mathbf{r}$, and $\mathbf{p}\cdot\mathbf{r}$ can be obtained by knowing the other two and using the latter equality.

That one can do calculations with vectors instead of trigonometric functions, as suggested by the above and other equalities, can be seen as a methodological and practical advantage. Methodological, because introducing angles and angle-specific calculations is avoided until it is really needed; and practical, because purely algebraic expressions are usually simpler to deal width, than those involving trigonometry.

On the other hand, defining cosine and sine as horizontal and vertical projections of a unit-length vector, enables a smooth passage between vectors and angular functions due to the obvious relation of sine and cosine to the area and scalar products. In particular, the equalities $\mathbf{a}\star\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\mathbf{a},\mathbf{b})$ and $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\mathbf{a},\mathbf{b})$ immediately become available.

Rotation is an obvious example of using angle-related functions and vectors together. Indeed, rotating a vector $\mathbf{p}$ to an angle φ about its initial point yields a result $\mathbf{p'}$ whose projections along $\mathbf{p}$ and $\mathbf{p}^\perp$ are proportional to $\cos\varphi$ and $\sin\varphi$, correspondingly:

$$\mathbf{p}' = \cos\varphi\,\mathbf{p} + \sin\varphi\,\mathbf{p}^\perp\,.$$

Finally, wherever needed, we can represent vectors by coordinates in a Cartesian coordinate system and, accordingly, express all vector operations in terms of numeric calculations. Using the fact that if a vector has coordinates (x, y) (as a position vector of a point with the same coordinates), then it equals $\mathbf{x}+\mathbf{y}$ where $\mathbf{x}$ and $\mathbf{y}$ have coordinates $(x, 0)$ and $(0, y)$, respectively, and the properties of the vector arithmetic, it is straightforward to prove that vector addition, negation, subtraction and multiplication by a scalar are coordinate-wise operations, and that if $\mathbf{u} = (x, y)$ and $\mathbf{v} = (x', y')$, then $|\mathbf{u}| = \sqrt{x^2 + y^2}$, $\mathbf{u}\cdot\mathbf{v} = xx' + yy'$, $\mathbf{u}^\perp = (-y, x)$, and $\mathbf{u}\star\mathbf{v} = xy' - x'y$.

Introducing vectors and operations on them without immediate reference to representation is methodologically right, not only because separation of concerns is a good idea in itself, but also because a great many calculations can be carried out using only vectors. The next section is dedicated entirely to illustrating precisely this. However, switching to coordinate representation at some point, thus reducing all calculations to numeric ones, it is necessary, in order to make problem solving, a really practical activity. Without it, problems are at best only solvable symbolically but not in terms of concrete numbers. Unfortunately, to the extent vectors are present in the high-school curriculum, in Bulgaria at least, they are not only lacking essential operations but remain totally unrelated to coordinates. Both of these hinder their use for problem solving.

The next section is a collection of examples of solving problems using the notation and some of the techniques suggested above. The problems are simple but, amazingly, the adequately simple expression of the results and of the derivations that lead to them are not commonly seen in the many relevant books – whether high-school textbooks or application-related texts for computer science students and programming practitioners – or written software, known to the author. In one case, a book discussing computer programming of geometrical problems listed a program of at least 60 lines where there should have been only several. Such is the power of the right choice of concepts and notation – or the lack of them!

Examples.

Example 1. Find the oriented distance d from a point P to a line through B_1 and B_2 – positive when P is to the left of $\overrightarrow{B_1B_2}$. Find the orthogonal projection P' of P on the line.

Answer: $d = \dfrac{\overrightarrow{B_1B_2}\star\overrightarrow{B_1P}}{|\overrightarrow{B_1B_2}|}$, $\quad\mathbf{P}' = \mathbf{B_1} + \dfrac{\overrightarrow{B_1B_2}\cdot\overrightarrow{B_1P}}{B_1B_2^2}\,\overrightarrow{B_1B_2}$.

The oriented distance is the projection of $\overrightarrow{B_1P}$ along $\overrightarrow{B_1B_2}^\perp$. $\overrightarrow{B_1P'}$ is the projection of the same vector along $\overrightarrow{B_1B_2}$.

Example 2. Find an equation for the line through given points B_1 and B_2.

Answer: $\overrightarrow{PB_1}\star\overrightarrow{B_1B_2} = 0$.

Follows immediately from collinearity.

Example 3. Find an equation for the line bisector of given points B_1 and B_2.

Answer: $(\mathbf{P} - (\mathbf{B_1} + \mathbf{B_2})/2)\cdot\overrightarrow{B_1B_2} = 0$.

Follows immediately from perpendicularity.

Parametric equations for the two examples above: $\mathbf{P} = \mathbf{B_1} + s\,\overrightarrow{B_1B_2}$ and $\mathbf{P} = (\mathbf{B_1} + \mathbf{B_2})/2 + s\,\overrightarrow{B_1B_2}^{\perp}$.

Example 4. Given points B_1, B_2, S_1, S_2, and T find the oriented area a of the rectangle with B_1 and B_2 on one of its sides, S_1 and S_2 on the adjacent sides, and T on the opposite side. The area is positive when T is to the left of $\overrightarrow{B_1B_2}$.

Answer: $a = \dfrac{|\overrightarrow{B_1B_2} \cdot \overrightarrow{S_1S_2}|\, \overrightarrow{B_1B_2} \star \overrightarrow{B_1T}}{B_1B_2^2}$.

The area is the product of the lengths of the sides of the rectangle, one of which can be found through $\overrightarrow{B_1B_2} \cdot \overrightarrow{S_1S_2}$ and the other – through $\overrightarrow{B_1B_2} \star \overrightarrow{B_1T}$.

Example 5. Given three non-collinear points A, B, and C, find the circumcentre S of $\triangle ABC$.

Answer: $\mathbf{S} = \dfrac{1}{2}(\mathbf{A} + \mathbf{B}) + \dfrac{1}{2}\dfrac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{\overrightarrow{CA} \star \overrightarrow{CB}}\overrightarrow{AB}^{\perp}$.

S is the crosspoint of the line bisectors of AB and AC. Equating the parametric equations of these lines, and multiplying e. g. by $\cdot\overrightarrow{AC}$, yields an expression for one of the parameters, which can be substituted back in the corresponding equation to obtain the answer.

Example 6. Given three non-collinear points A, B, and C, find the incentre I of $\triangle ABC$.
Answer: $\mathbf{I} = \mathbf{A} + \frac{1}{2p}(AC \cdot \overrightarrow{AB} + AB \cdot \overrightarrow{AC})$, where p is the semiperimeter of $\triangle ABC$.

Using (1), $\overrightarrow{AI}$ can be expressed as a linear combination of $\overrightarrow{AB}$ and $\overrightarrow{AC}$, then expressions for the doubled oriented areas of $\triangle ABI$, $\triangle AIC$, and $\triangle ABC$ can be substituted for $\overrightarrow{AB} \star \overrightarrow{AI}$, $\overrightarrow{AI} \star \overrightarrow{AC}$, and $\overrightarrow{AB} \star \overrightarrow{AC}$ in the result. Those expressions make use of AB, AC, p, and the inradius.

Example 7. Given a point P, non-collinear vectors $\mathbf{u}$ and $\mathbf{v}$, and $r > 0$ find the centre C of a circle with a radius r, inscribed between the rays with a common initial point P and oriented towards $\mathbf{u}$ and $\mathbf{v}$. Find the points of tangency A and B of the circle with the rays.

Answer: If $\mathbf{u}\star\mathbf{v} > 0$, then $\mathbf{C} = \mathbf{P} + \frac{r}{\mathbf{u} \star \mathbf{v}}(|\mathbf{v}|\mathbf{u} + |\mathbf{u}|\mathbf{v})$, $\mathbf{A} = \mathbf{C} - r\,\dfrac{\mathbf{u}^{\perp}}{|\mathbf{u}|}$, and $\mathbf{B} = \mathbf{C} + r\,\dfrac{\mathbf{v}^{\perp}}{|\mathbf{v}|}$. If $\mathbf{u} \star \mathbf{v} < 0$, then $\mathbf{u}$ and $\mathbf{v}$ change places.

Using (1), $\overrightarrow{PC}$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, in which expressions for the doubled areas of $\triangle PAC$ and $\triangle PCB$ can be substituted for $\mathbf{u} \star \overrightarrow{PC}$ and $\overrightarrow{PC} \star \mathbf{v}$. The expressions make use of $|\mathbf{u}|$, $|\mathbf{v}|$ and r. Then, we find $\overrightarrow{CA}$ and $\overrightarrow{CB}$ as collinear to $\mathbf{u}^{\perp}$ and $\mathbf{v}^{\perp}$, respectively, and being of length r.

Example 8. Find the point of tangency of the tangent line through given point P to a circle with centre C and radius r.
Answer: $\mathbf{T} = \mathbf{C} + \frac{r}{\mathbf{d}^2}(r\,\mathbf{d} + s\sqrt{\mathbf{d}^2 - r^2}\,\mathbf{d}^{\perp})$, where $\mathbf{d} = \overrightarrow{CP}$. If $s = 1$, T is a point on the line that is right-hand side tangent to the circle, and if $s = -1$ it is on the left-hand side tangent.

If R is the right-hand side tangent point, then resolving $\overrightarrow{CR}$ with respect to $\mathbf{d}$ and $\mathbf{d}^{\perp}$ (using (2)) we take into account $\mathbf{d} \cdot \overrightarrow{CR} = r^2$ and $\mathbf{d} \star \overrightarrow{CR} = r\sqrt{\mathbf{d}^2 - r^2}$, and similarly for the left-hand side tangent point.

Example 9. Find the points of tangency of the common tangent lines of two circles.
Answer: Let the two circles have centres C_1 and C_2 and radii r_1 and r_2, correspondingly, and let $\mathbf{d} = \overrightarrow{C_1C_2}$ and $r = r_1 + s_1 r_2$. The points in search are given by the formula $\mathbf{T}_i = \mathbf{C}_i + (-1)^{i+s_1=3}\frac{r_i}{\mathbf{d}^2}(r\,\mathbf{d} + s_2\sqrt{\mathbf{d}^2 - r^2}\,\mathbf{d}^\perp)$, where the subscript $i = 1, 2$ denotes a point on the first and the second circle, respectively. s_1 and s_2 each take values 1 or -1. For $s_1 = 1$ we consider the two inner tangents, and for $s_1 = -1$ – the outer ones. For $s_2 = 1$ the two lines tangent from the right to the first circle are considered, and for $s_2 = -1$ – the ones from the left.

Similar to the above problem, we resolve with respect to $\mathbf{d}$ and $\mathbf{d}^\perp$ and substitute for the thus emerging scalar and area products, although there are more cases to consider here which sum up in the result.

Example 10. Find a circle of radius r, tangent to a given line and passing through a point P.
Answer: Let the line be defined by a point M on it and a direction vector $\mathbf{u}$. There are at most two circles of the wanted kind, and their centres are given by $\mathbf{C}_{1,2} = \overrightarrow{MP'} + \frac{1}{|\mathbf{u}|}(\pm d'\,\mathbf{u} + s\,r\,\mathbf{u}^\perp)$, where $d' = \sqrt{d(2r-d)}$, d is the distance from P to the line, P' is the projection of P on the line, and $s = \pm 1$ for P on the left/right of $\mathbf{u}$.

We use (2) for resolving $\overrightarrow{P'C_{1,2}}$ with respect to $\mathbf{u}$ and $\mathbf{u}^\perp$ and substitute for the area and scalar products. P' and d are found as in example 1.

Concluding remarks. Adding $\perp$ and the area product to the scalar product, and using the resolution of a vector into a linear combination of two vectors helps to solve a number of problems in plane geometry that cannot be approached using the scalar product alone. Fortunately, this addition is "for free", essentially not requiring any new knowledge and not calling for a paradigm shift of any scale. In fact, it does the opposite: the amount of trigonometry that might be otherwise needed to solve certain problems, can be significantly reduced using the proposed method, thus leading to a purer form of geometry. Moreover, with this method it is easier to stay in the coordinate-free, vector-only language of expressing locations and lengths.

Of course, the area product is not a new concept. It is essentially Grassmann's *exterior product*. Russian mathematics has been making use of it for at least half a century [4], under a name that can be translated as "skew product". Much later, the product was reinvented independently in [2], thereby becoming popular in the English speaking world as "perp product". The article [2] actually went further by also recognizing the utility of $\perp$ as an operation in its own right, apart from the perp/area product. The three-vector equality (3) was also derived there.

We see the contribution of the present article in the following directions:

- providing, by means of examples, further evidence of the utility of the scalar and area products and the $\perp$ operation,
- extending the applicability of calculation with products by showing how the equality expressing a vector decomposition with respect to two vectors can be used "backwards",
- showing that some widely used trigonometric equalities have natural analogues in vector calculus, and more generally, that geometric calculations involving trigonometric functions often can be replaced by calculating with vectors,

- similarly, showing that coordinate transformations, such as rotations and reflections, whose algebraic representation traditionally involves matrices and determinants and coordinate-wise formulae, or alternatively complex numbers, do have natural enough expression in a pure vectoral form.

In view of the above, we hope to have shown that vector calculus can be more usefully employed both in teaching and in applying geometry.

Clifford algebra [3, 1], or as it is more specifically called recently – geometric algebra – is a powerful algebraic and calculational tool which can handle spaces of arbitrary dimension by dealing with multi-dimensional directed volumes generalizing the concept of vector. The geometric algebra approach subsumes the one presented here by offering a more general calculational framework. However, perhaps due to that same generality, it seems somewhat less practical for solving planar geometry problems – both conceptually and as an actual computational vehicle, including the implications of implementing a tool of this sort programmatically. Therefore, for the teaching and practical uses of plane geometry, we consider the approach started by [4, 2] and further developed here more directly applicable than geometric algebra. The more so that it does not require the conceptual and notational shift needed in the case of geometric algebra.

The approach presented here is used on a regular basis by the author in his teaching computational geometry to university students, as well as in teaching algorithm design to high-school students training in programming competitions.

REFERENCES

[1] R. G. Calvet. Treatise of plane geometry through geometric algebra, 2001. http://www.xtec.es/~rgonzal1/treatise.htm.

[2] F. S. Hill jr. The pleasures of "perp dot" products, *Graphics Gems IV*, Academic Press, 1994, pp. 138–148.

[3] P. Lounesto. Clifford algebras and spinors, 2nd ed., Cambridge University Press, 2001.

[4] В. Г. Болтянский, И. М. Яглом (ред.) Энциклопедия элементарной математики, кн. 4 Геометрия, Гос. изд. физикомат. лит-ры, 1963, 338–350.

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СМЯТАНЕ С ВЕКТОРИ В ПЛАНИМЕТРИЯТА

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Разглежда се метод за решаване на планиметрични задачи посредством смятане с вектори. Считаме, че методът носи предимства както в обучението по съответните части на планиметрията, така и като практичен инструмент при извеждане и съставяне на програми в областта на алгоритмичната геометрия и компютърната графика.