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COMPUTER SUPPORTED TEACHING AND LEARNING OF A DIFFICULT TO MASTER TOPIC IN MULTIVARIABLE CALCULUS*

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The key concept of a limit of two variable functions is considered. Computer Algebra System (CAS) Derive is used as a symbolic, numeric, and graphic tool throughout the teaching-learning process. CAS helps the students to make difference between "iterated limit" and "limit", through experimenting with appropriately selected functions. The experience in giving CAS techniques conceptual dimensions is presented.

I. Introduction.

"A little knowledge is a dangerous thing" [1]

For the last 10 years CAS Derive has been integrated into the teaching and learning of Calculus of one variable at the Technical University of Sofia [2]. Since 1999, a new Bachelor Degree Program in Applied Mathematics has been introduced in which students are taught Multivariable Calculus in the second semester of the first year of study. We decided to extend our experience by implementing computer supported classes.

In this paper, we discuss limits as applied to functions of two variables (FTV). The limit is indispensable tool in mathematics as well as in its applications in Physics, Mechanics, and Engineering. However, this topic has proved to be difficult to be mastered for the majority of students.

Although the definition of limit of FTV is similar to that for a function of one variable, there is an essential difference that makes this concept difficult for the learners. For a function $F(x, y)$, the point (x, y) is allowed to approach the point (a, b) from any "direction" (i.e. in infinitely many ways), not from two directions only (from the left or the right). For a function of two variables, the limit $\lim\limits_{(x,y)\to(a,b)} F(x, y)$ does not exist if its value is not the same for all possible approaches, or paths.

Students, who better understand Calculus, work with CAS Derive [7] (or Maple) one hour a week with a teacher supervising the session. This is an addition to the other classroom activities – lectures and tutorials. The Derive capabilities are used to teach and discuss different types of limits of FTV, important interrelations between them as well as to generate dynamic graphics to give geometrical meaning to definitions, rules and theorems.

*Key words: Calculus, function of two variables, limit, Computer Algebra System (CAS) derive

II. Activities for Teaching and Learning Limits of FTV. Following one of the methodological principles – "From Known to Unknown", the students are told at the beginning that

– Although there is an important difference, the concept of a limit of FTV is much alike the concept of a limit of a function of one variable;
– Analogs of theorems for functions of one variable about the limits of sums, products, quotients, an n-th root, and an absolute value, are also valid for FTV.

The activities aim to enhance student's ability to comprehend and apply the definition of a limit of a FTV and the main theorem about this mathematical object [3].

Definition of Limit. If the values of the function $F(x, y)$ approach a number L for **any and every** mode of approach of (x, y) to (a, b), we say that L is the limit of $F(x, y)$ at (a, b). We denote the limit as $\lim\limits_{(x,y)\to(a,b)} F(x, y)$.

Theorem. *Let $F(x, y)$ be defined in the domain $D = \{(x, y) : x \in X \subset R$ and $y \in Y \subset R\}$. If*
1) *the limit (finite or infinite) $L = \lim\limits_{(x,y)\to(a,b)} F(x, y)$ exists, and*
2) *the simple limit $\lim\limits_{x\to a} F(x, y) = \varphi(y)$ exists for every $y \in Y$,*
then the iterated limit $\lim\limits_{y\to b}(\lim\limits_{x\to a} F(x, y))$ also exists and $\lim\limits_{y\to b}(\lim\limits_{x\to a} F(x, y)) = \lim\limits_{y\to b} \varphi(y) = L$.

If the existing simple limit is with respect to y, then the theorem still holds for the corresponding iterated limit $\lim\limits_{x\to a}(\lim\limits_{y\to b} F(x, y))$.

Activity 1. Application of the definition. The iterated limits exist but the limit does not exist.

Example 1.1 (The *iterated limits* exist but *are not equal*; the limit cannot exist)

Given the function $F(x, y) = \dfrac{x - y}{x + y}$, find the ***iterated limits*** $A = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} F(x, y))$ and $B = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} F(x, y))$. Conclude from the results whether the ***limit*** exists. Verify your assertion using two different directions to approach the point $(0, 0)$.

Solution. The teacher can make a choice between paper-and-pencil and computer algebra techniques. CAS Derive can be integrated in the performance of the following steps:

– Calculation of the simple limit when $x \to 0$. The student realizes that this limit does not depend on y (line `#2`), hence, the iterated limit A is actually the limit of the constant sequence $\{-1, -1, \ldots, -1, \ldots\}$, i.e. $A = -1$ (line `#3`).
– The teacher can demonstrate that CAS can serve as a self-assessment tool: the Derive library function `LIM(u, [x, y], [x0, y0])`, where $u(x, y)$ is a FTV, is appropriate for this step. Derive simplifies it by computing the limit first with respect to x, then with respect to y, resulting in the required *iterated limit* A (line `#4`).
– The student repeat the above steps and find the other *iterated limit* B (lines `#5`, `#6` and `#7`).

$$\#1:\quad F(x,\ y) := \frac{x - y}{x + y}$$

$$\#2:\quad \lim_{x \to 0} F(x,\ y) = \frac{0 - y}{0 + y} = -1$$

$$\#3:\quad A = \lim_{y \to 0} \lim_{x \to 0} F(x,\ y) = \lim_{y \to 0} -1 = -1$$

$$\#4:\quad \lim_{[x,\ y] \to [0,\ 0]} F(x,\ y) = -1$$

$$\#5:\quad \lim_{y \to 0} F(x,\ y) = \frac{x - 0}{x + 0} = 1$$

$$\#6:\quad B = \lim_{x \to 0} \lim_{y \to 0} F(x,\ y) = \lim_{x \to 0} 1 = 1$$

$$\#7:\quad \lim_{[y,\ x] \to [0,\ 0]} F(x,\ y) = 1$$

The teacher will be able to notice here that in the Derive function LIM, the order in the list of variables and of their values is important: the result of `LIM(u, [x, y], [x0, y0])` may be different than `LIM(u, [y, x], [y0, x0])` (compare lines #4 and #7).

Comparing the two iterated limits, the student figures out that $A = -1 \neq 1 = B$. The question left is whether the limit L exists. The answer can be approached by the following two ways.

First, let the student assume that the limit L exists and refer to the *Theorem*: as the simple limits exist (#2 and #5), it says that the iterated limits should exist and moreover, $A = B = L$. This equation contradicts the results obtained in lines #4 and #7. So, the limit L does not exist.

Second, the student is recommended to perform the following experiment: find the limit when (x, y) approaches the point $(0, 0)$ along ***two different directions***: the straight lines $y = x$ and $y = x/2$. The results in lines #8 and #9 show ***two different values***, hence, according to the definition the limit L does not exist.

$$\#8:\quad \lim_{x \to 0} F(x,\ x) = 0$$

$$\#9:\quad \lim_{x \to 0} F\left(x,\ \frac{x}{2}\right) = \frac{1}{3}$$

$$\#10:\quad \mathrm{LIM2}(F(x,\ y),\ y,\ x,\ 0,\ 0) = \frac{@1 - 1}{@1 + 1}$$

$$\#11:\quad \mathrm{LIM2}(F(x,\ y),\ x,\ y,\ 0,\ 0) = \frac{1 - @1}{@1 + 1}$$

It is quite understandable that students want to be able to find "all at once", i.e. to find the limit along ***any*** straight line (using only its slope). But there are infinitely many directions! What about the time wasted for that? Can technology help out? This is the right place and the right moment for the teacher to focus on the power of CAS as a time saving tool and to introduce an appropriate Derive function.

The library function `LIM2 (u, x, y, x0, y0)` returns the limit – it simplifies to the limit of $u(x, y)$ as $[x, y]$ approaches $[x_0, y_0]$ along a straight line of a slope denoted by `@1`: if the result is independent of `@1`, then the limit is independent of the direction of approach; if the result depends on the slope `@1`, then the limit depends on the direction of approach. The Derive response in lines #10 and #11 confirms the above conclusion about the non-existence of the limit L.

The student should be told what one can expect from technology: CAS is capable to perform operations for which related knowledge and procedures have been incorporated in it. It is not the CAS, it is the human creativity that produces knowledge and technology. Technology performs operations in a fastest way and (in most cases) without computational mistakes. In such sense, it acts convincingly on the students when the teacher let them verify "by hand" the above response of `LIM2`: it is enough to replace y by kx into the function $F(x, y)$ in order to get the same result as in line `#11` with `@1 = k`.

The animation in Figure 1 is intended to develop students' spatial vision as an essential element of the learning and knowing of mathematics.

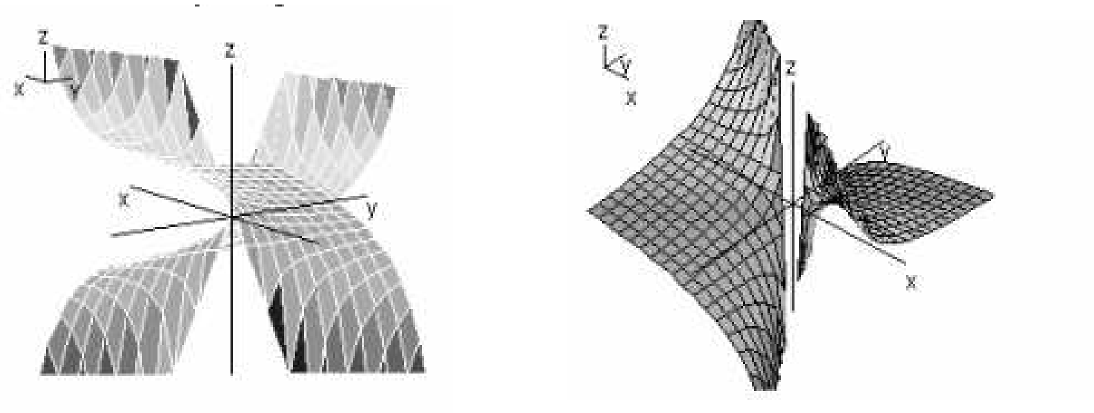

Fig.1 Derive-generated graphs of the surface in Example 1.1

The level curves in Figure 2 are the xy-projections of the traces of the graph in the plane z = constant.

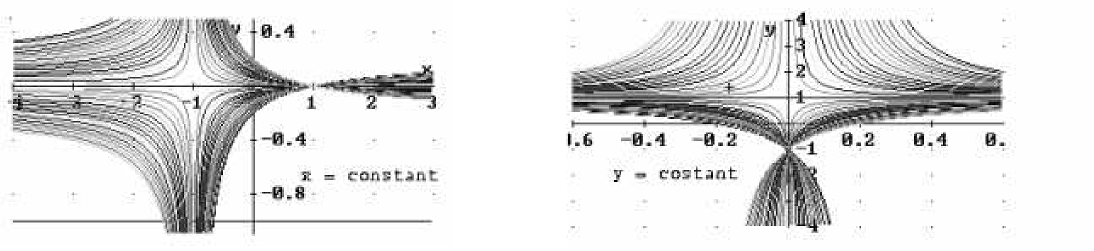

Fig. 2. Level curves of the function in Example 1.1

The next two examples helps students to develop a right feel about the importance of the expression *"any and every mode of approach"* in the definition of a limit of FTV.

Example 1.2. (The *iterated limits* exist and *are equal.* The limit is different along different directions; the limit does not exist)

Consider the function $G(x, y) = \dfrac{x^2y^2}{x^2y^2 + (x - y)^2}$.

(a) Calculate the iterated limits $A = \lim\limits_{y\to 0}(\lim\limits_{x\to 0} G(x, y))$ and $B = \lim\limits_{x\to 0}(\lim\limits_{y\to 0} G(x, y))$.

(b) What can you say about the existence of the limit $L = \lim\limits_{(x,y)\to(0,0)} G(x, y)$? Verify your answer.

Solution. The student is asked to follow the same way of reasoning as in Example 1.1. He/she is free to choose whether to use CAS Derive, or to perform calculations by paper and pencil. Figure 3 contains algebraic and geometric windows the student can obtain with Derive. The graph of the function allows the teacher to illustrate a geometrical meaning/analog of the non-existence of the limit of FTV: in this particular case, there is a hole in the surface.

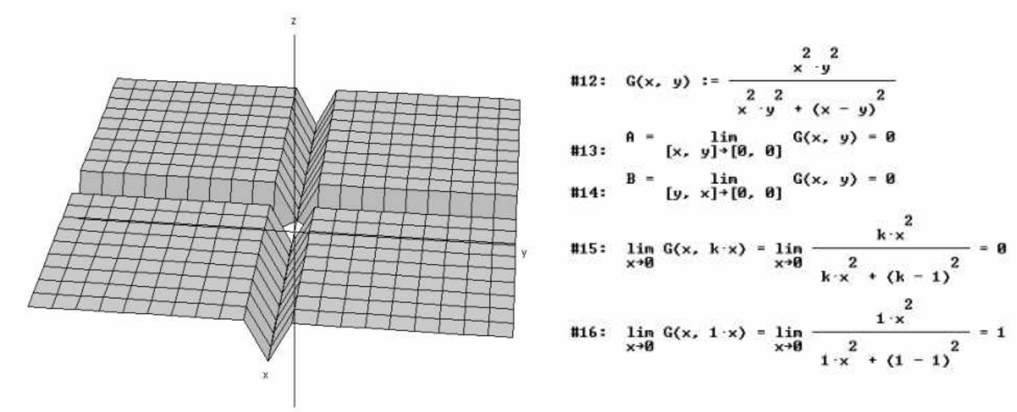

Fig. 3 Example 1.2

(a) Lines #13 and #14 assure the student that $A = B$.

(b) First, the student finds the directional limit (line #15). The result obtained by Derive is misleading. It makes one believe that the limit is zero along *any* straight line through the origin. It is not clear that the bisectrice of the first quadrant is excluded! The developers of Derive have failed to encompass this case and the role of the teacher as a supervisor is important here. He can pose the question about the case with k =1 (line #16) thus, motivating the students to think critically [4]. As the result differs from that with $k \neq 1$, the limit does not exist according to the definition.

This example makes students to assimilate that the existence and equality of the iterated limits is not *sufficient* for the limit to exist. Also, it tells both teachers and students, that: "*Knowledge is Power, Technology is a Powerful Tool*" that implies the *necessity of reflecting on the computer results.*

Example 1.3. (The *iterated limits* exist and *are equal*; the function approaches the same value along any straight line; the limit does not exist)

Show that the iterated limits of the function $H(x,y) = \dfrac{x^2y}{x^4+y^2}$ are equal to zero when $x \to 0$ and $y \to 0$ and the limit along any direction through the origin is also zero.

(a) Does the limit $L = \lim\limits_{(x,y)\to(0,0)} H(x,y)$ exist?

(b) Find the limit of $H(x,y)$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Is your conclusion in (a) correct?

Solution. Supervised by the teacher, students perform the computations shown in the algebraic window in Figure 4. First, the students find out that the iterated limits (lines

#18, #19) and the limit along any arrow $x = \rho\cos\varphi$, $y = \rho\cos\varphi$ through the origin (lines #20, #21) are all zero. They can confirm the latter result using the Derive library function LIM2 (#22, #23). Then, they plot level curves and a segment of the surface in order to get a visual idea of the investigated object (Fig. 4). The Window Shuttle Method makes CAS Derive a powerful instrument for learning through prototypes: numerical, analytical, and graphical. It also supports the teaching-learning process by linking the old (simple limit, level curves) and the new (iterated limit, limit) knowledge.

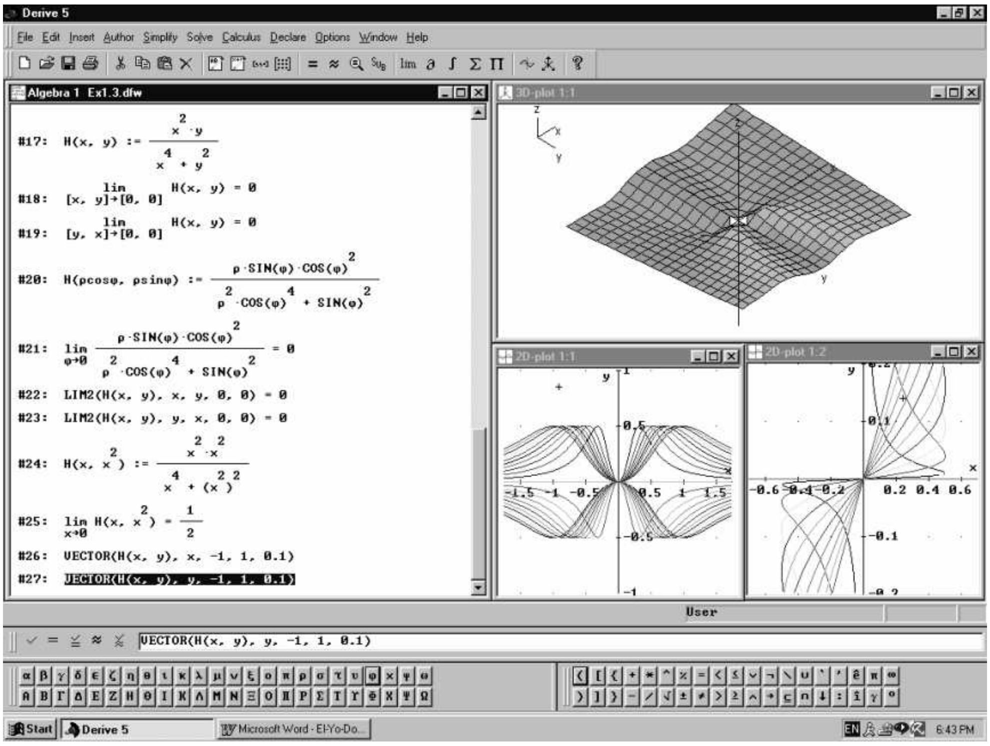

Fig. 4. Example 1.3

Students' answers on Question (a) in Example 1.3 vary: "The limit exists and is zero" or "It could exist" or "I don't know" or "It can't be said". In this situation the theory is silent: the existence and equality of the iterated and directional limits "say" nothing about the existence or non-existence of the function's limit.

For Question (b) the teacher has to let students observe what happens if (x, y) approaches $(0, 0)$ along the parabolic path $y = x^2$ (lines #24, #25) – the limit is not equal to zero! As these different values of the limit belie the definition of limit of FTV, the limit cannot exist.

In addition to Example 1.3, the existence of the limit of $h(x, y) = \dfrac{ye^{-1/x^2}}{e^{-2/x^2} + y^2}$ at $(0, 0)$ could be explored. Students can assure themselves that the function approaches the same zero value along any straight line and any algebraic curve of the type $y = cx^{p/q}$ passing through the origin. But there still exists a mode of approach of (x, y) to $(0, 0)$ for which $h(x,y)$ approaches a non-zero value. The theacher should let students check that its limit, as $(x,y) \to (0,0)$ along the curve $y = e^{-1/x^2}$ is $\frac{1}{2}$, and make the conclusion, based on the definition, that $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ does not exist.

Activity 2. Application of the Theorem. The limit exists but both iterated limits do not exist.

Example 2.1. (The *limit exists*. Both *inner limits do not exist*; both iterated limits can not exist)

Investigate all types of limits of the function $U(x,y) = (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$ at the origin $(0,0)$.

Solution. This example is intended to help students become aware of the importance of the concepts of a necessary condition and a sufficient condition. During tutorial sessions students investigate the function, applying the epsilon-delta definition of a limit and prove that $\lim\limits_{(x,y)\to(0,0)} U(x,y) = 0$ [3]. After that the question arises: Does the existence of the limit ***always yields*** the existence of the iterated limits?

The teacher asks the students first to find the inner (simple) limits and see that neither of them exists! It seems that this fact is very exiting and curious for them: the existence of the limit does not yield the existence even of the simple limits! Then this result can be formulated in terms of sufficiency of a condition, namely: The existence of the limit of FTV ***is not sufficient*** for the existence of the iterated limits?

Further, we let students use CAS Derive to verify that the non-existence of a simple limit indicates the non-existence of the corresponding iterated limit. Then, we let them check whether the results agree with the Theorem and help them come to a deeper understanding of the Theorem: when the limit exists the existence of a simple limit is ***necessary condition*** for the corresponding iterated limit to exist.

To control the student's trajectory of acquiring knowledge of limits of FTV, we ask questions about the directional limits of the function. Some of them need to see the Derive response and then to give the right answers: the limits exist and are all equal to the limit value. Here we tell the students not to "abuse technology" and not to ignore the greatest human advantage - the common sense and recommend them to look back to the roots of the concept of a limit, i.e. to its definition, and to find the answer there.

Through the considered insightful examples the student can gain more comprehension of the concept of a limit. Not every student will be able to investigate algebraically different types of limits of FTV. However, the computer technology could come to the rescue and empower students to carry out their explorations both analytically and geometrically.

Here the student can go on his own with the next

Example 2.2. (The limit exists. Only one of the inner limits exists; only the corresponding iterated limit exists). What is common and what is different in the behavior of the function $V(x,y) = x\sin\frac{1}{y}$ in the vicinity of the point (0, 0), compared to the function in Example 2.1?

Activity 3. Continuity of functions of two variables. CAS can be also used to experiment with different functions by "What if...?" questions, e.g.: What if the function is defined at the point (a, b)? Does the value of $F(a, b)$ have an effect on the limit of F at (a, b)? What if they are equal? Will the graph still have holes or gaps?

The next example illustrates the concept of continuity of FTV, and further laboratory experiments can be recommended to the students.

Example 3. Discuss the continuity of the "dog saddle" function: $Z(x, y) = 4x^3y - 4xy^3$ at the origin.

5. Conclusion.

"Challenge is energy of life"

The motto "Knowledge is Power, Technology is a Powerful Tool", quoted in Section II, has to be interpreted strategically. Technology is a tool with great power and potential that are "addressed" to teacher's and student's creativity in order to convert it into an effective instrument in the educational process. It allows them to explore the wisdom "Nobody has a monopoly on truth" and find their own way to an adequate application of technology. The freedom to choose, encourages the students to investigate and the teachers to innovate (not imitate). The impact of technology on the educational process can be spread in many aspects. It empowers and enriches traditional approaches; makes possible the development of new approaches; promotes links between mathematics conceptual and procedural knowledge; makes teaching and learning more interactive, dynamic, flexible and exiting. Also, some unattractive for students and teachers topics, can become more attractive thanks to the opportunity for experimental observations with technology on a variety of insightful examples.

It is very important to use properly appropriate educational technology, defined as in [1]:

$$\text{Educational Technology} = \text{Technology OF Education} + \text{Technology IN Education.}$$

Tradition and Technology have to go hand by hand. Technology does can "replace" hundreds of teachers but a powerful tradition and teachers can give thousands of technologies vitality.

We mainly use CAS Derive because it has capabilities suitable to the best Bulgarian tradition in Mathematics education. It is a great responsibility for the teachers to use efficiently the potentials and limitations of CAS in order to discover the power and beauty of our tradition.

REFERENCES

[1] K. L. Kumar. Educational Technology. New Age international, Publishers, New Delhi, 1996.

[2] B. Kutzler, V. Kokol-Voljc. Introduction to Derive 5, The Mathematical Assistant for Your PC. Texas Instruments, 2000.

[3] G. M. Fihtengoltz. Differential and Integral Calculus - I. Nauka, Moscaw, 1966.

[4] E. De Bono. Teaching Thinking. Penguin Books, London, 1984.

[5] M. Meagher. Curriculum and Assessment in the Age of Computer Algebra Systems. *Int. Journal of Computer Algebra in Mathematics Education*, **8**, No. 1 (2002).

[6] W. Peschek, E. Schneider. How to Identify Basic Knowledge and Basic Skills? Features of Modern Education in Mathematics. *Int. Journal of Computer Algebra in Mathematics Education*, **8**, No.1 (2002).
[7] E. Varbanova. A CAS supported environment for learning and teaching Calculus, Proceedings of the First KAIST Int. Symposium on Enhancing University Mathematics Teaching, Korean Advanced Institute of Science and Technology, (2005),
`http://math.kaist.ac.kr/2005/proceedings/`

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КОМПЮТЪРНО-ПОДПОМОГНАТО ИЗУЧАВАНЕ НА ЕДНА ТРУДНА ЗА УСВОЯВАНЕ ТЕМА ОТ ДИФЕРЕНЦИАЛНОТО СМЯТАНЕ НА ФУНКЦИИ НА ДВЕ ПРОМЕНЛИВИ

Елена Апостолова Върбанова, Йорданка Добрева Панева-Коновска

Разглежда се основното понятие граница на функция на две променливи. Показано е приложение на системата за компютърна алгебра (СКА) Derive като помощно средство в учебния процес за символни преобразования, числени пресмятания и получаване на графики. СКА подпомагат студентите в усвояване и различаване на понятията граница и последователна граница, посредством експериментиране с подходящо избрани функции. Представен е един опит за разкриване на концептуалните измерения на възможностите на СКА.