МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2013
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2013
*Proceedings of the Forty Second Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2013*

ABOUT SUBALGEBRAS OF PSEUDOCOMPACT AND BOUNDED ALGEBRAS*

Mitrofan M. Choban

This paper deals with the embedding conditions of a universal topological algebra in some pseudocompact or bounded universal topological algebra.

1. Introduction. This paper is a continuation of [1], [6], [7], [4] and [5]. We use the terminology from [2, 8, 5]. Any space is considered to be a Hausdorff space. By $cl_X A$ we denote the closure of a set A in X.

Let $\{E_n : n \in \mathbb{N} = \{0, 1, 2, 3, \dots\}\}$ be a sequence of pairwise disjoint discrete spaces. The discrete sum $E = \oplus\{E_n : n \in \mathbb{N}\}$ is the *signature* of universal E-algebras.

A universal algebra of a signature E or *an E-algebra* is a family $\{G, e_{nG} : n \in \mathbb{N}\}$, where G is a non-empty space and $e_{nG} : E_n \times G^n \to G$ is a mapping for any $n \in \mathbb{N}$. Subalgebras, homomorphisms, isomorphisms and Cartesian product of E-algebras are defined in the traditional way [5].

A topological universal algebra of a signature E or *a topological E-algebra* is a family $\{G, e_{nG} : n \in \mathbb{N}\}$, where G is a non-empty space and $e_{nG} : E_n \times G^n \to G$ is a continuous mapping for any $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ denote by $\exp_o(n)$ the family of all non-empty subsets of the set $\{1, 2, \dots, n\}$. If $n \geq 1$, $L \in \exp_o(n)$ and $a = (a_1, \dots, a_n) \in G^n$, then $G^L_a = \{(x_1, \dots, x_n) \in G^n : x_i = a_i \textit{ for each } i \in \{1, \dots, n\} \setminus L$. If $L = \{1, \dots, n\}$, then $G^L_a = G^n$ for each $a \in G^n$.

Fix a signature E. For each $L \in \exp_o(n)$ denote by E_{nL} some subspace of the space E_n. If $n = 0$ and $L \in \exp_o(n)$, then $E_{0L} = E_0$. We put $\mathcal{E} = \{E_{nL} : n \in \mathbb{N}, L \in \exp_o(n)\}$. A *topological $\mathcal{E}$-algebra* is a family $\{G, e_{nG} : n \in \mathbb{N}\}$, where G is a non-empty space, $e_{nG} : E_n \times G^n \to G$ is a mapping for any $n \in \mathbb{N}$ and the mapping $e_{nLG} = e_{nG}|(E_{nL} \times G^L_a) : E_{nL} \times G^L_a \longrightarrow G$ is continuous for all $n \in \mathbb{N}$ and $L \in \exp_o(n)$.

A subalgebra of a topological $\mathcal{E}$-algebra is a topological $\mathcal{E}$-algebra.

The Cartesian product of topological $\mathcal{E}$-algebras is a topological $\mathcal{E}$-algebra.

Example 1.1. Let $E_0 = \{0\}$, $E_1 = \{-1\}$, $E_2 = \{\cdot\}$ and $E = E_0 \cup E_1 \cup E_2$. The set E is the signature of groups. Consider the following classes of groups with topologies:

1.1.1. Let $E_{1\{1\}} = \varnothing$, $E_{2\{1\}} = E_{2\{2\}} = E_2$ and $E_{2\{1,2\}} = \varnothing$. If a group G with a topology is a topological $\mathcal{E}$-algebra, then G is called a semitopological group.

*2000 Mathematics Subject Classification: 08A05, 22A20, 54A25, 08B05, 08C10, 54C40, 54H15.

Key words: topological universal algebra, totally bounded algebra, bounded algebra, precompact algebra, τ-pseudocompact space, pseudo-τ-compact space, pseudocompact space.

1.1.2. Let $E_{1\{1\}} = \varnothing$ and $E_{2\{1\}} = E_{2\{2\}} = E_{2\{1,2\}} = E_2$. If a group G with a topology is a topological $\mathcal{E}$-algebra, then G is called a paratopological group.

1.1.3. Let $E_{1\{1\}} = \varnothing$, $E_{2\{1\}} = E_2$ and $E_{2\{2\}} = E_{2\{1,2\}} = \varnothing$. If a group G with a topology is a topological $\mathcal{E}$-algebra, then G is called a right topological group.

1.1.4. Let $E_{1\{1\}} = \varnothing$, $E_{2\{2\}} = E_2$ and $E_{2\{1\}} = E_{2\{1,2\}} = \varnothing$. If a group G with a topology is a topological $\mathcal{E}$-algebra, then G is called a leftt topological group.

Fix an infinite cardinal number τ. A space X is called *τ-pseudocompact* if it is completelly regular and for any continuous mapping $f : X \longrightarrow Y$ into a space Y of weight $w(Y) \leq \tau$ the subspace $f(X)$ is compact.

$\aleph_0$-pseudocompact spaces are called pseudocompact spaces. A space X is pseudocompact if and only if any continuous function on X is bounded [8].

A τ-pseudocompact space of weight $\leq \tau$ is a compact space. Hence a space is compact if and only if it is τ-pseudocompact for each cardinal τ.

A subset L of a space X is called a G_τ-set if it is an intersection of τ open subsets of the space X.

A subset Y of a space X is G_τ-dense in X if $Y \cap \neg H \neq \varnothing$ for any non-empty G_τ-set H of the space X.

A space X is τ-pseudocompact if and only if X is G_τ-dense in βX (see [8]) for $\tau = \aleph_0$).

We say that the space X is pseudo-τ-compact if it is completely regular and there exists a dense subspace Y such that the set $cl_X L$ is compact for each subset $L \subseteq Y$ of the cardinality $|L| \leq \tau$.

Any pseudo-τ-compact space is τ-pseudocompact.

Proposition 1.2 ([4], Lemma 5)**.** *A Cartesian product of pseudo-τ-compact spaces is pseudo-τ-compact.*

2. Precompact topological $\mathcal{E}$-algebras. Fix an infinite cardinal τ, a signature $E = \oplus\{E_n : n \in \mathbb{N}\}$ and a family $\mathcal{E} = \{E_{nL} : n \in \mathbb{N}, L \in \exp_o(n)\}$.

A topological $\mathcal{E}$-algebra A is called a *precompact $\mathcal{E}$-algebra* if A is a subalgebra of some Hausdorff compact $\mathcal{E}$-algebra.

Theorem 2.1. *Let A be a precompact topological $\mathcal{E}$-algebra and A is a subalgebra of a Hausdorff compact $\mathcal{E}$-algebra B. For any set L of cardinality $> \tau$ in the compact $\mathcal{E}$-algebra B^L there exists some pseudo-τ-compact subalgebra G which contains A as a closed subalgebra.*

Proof. We put $B_\lambda = B$ for any $\lambda \in L$. For each $x \in A$ we put $\varphi(x) = (x_\lambda \in B_\lambda : \lambda \in L)$, where $x_\lambda = x$ for each $\lambda \in L$. Then φ is an isomorphic embedding of A in B^L. We identify A with $\varphi(A)$. Thus we can assume that A is a topological $\mathcal{E}$-subalgebra of the compact $\mathcal{E}$-algebra B^L.

Fix a point $a \in A$. We put $G = \{(x_\lambda : \lambda \in L) \in B^L : |\{\lambda \in L : x_\lambda \notin A\}| \leq \tau\}$ and $G_1 = \{(x_\lambda : \lambda \in L) \in B^L : |\{\lambda \in L : x_\lambda \neq a\}| \leq \tau\}$.

By construction, we have:

$P1$. $A \subseteq G \subseteq B^L$, A is a closed subspace of the space G and $G_1 \subseteq G \subseteq B^L$.

$P2$. G is a subalgebra of the algebra B^L.

$P3$. G_1 is a dense subspace of the spaces G and B_L.

$P4$. If $Z \subseteq G_1$ and $|Z| \leq \tau$, then $cl_{G_1} L$ is a compact subset of G_1.

Thus G is a pseudo-τ-compact subalgebra of the algebra B^L and the algebra A is a closed subalgebra of G. The proof is complete. $\square$

In the case of topological groups and rings and for $\tau = \aleph_0$ the construction of the algebra G in the proof of the above theorem was proposed by M. Urssul [10, 11]. For topological E-algebras this construction was used in [4].

If $E = \varnothing$, then any space is a topological E-algebra. Thus from Theorem 2.1 there follow:

Corollary 2.2. *For any infinite cardinal τ each completely regular space is a closed subspace of some pseudo-τ-compact space.*

Corollary 2.3 (N. Noble [9])**.** *Each completely regular space is a closed subspace of some pseudocompact space.*

3. Bounded topological $\mathcal{E}$-algebras. Fix a signature $E = \oplus\{E_n : n \in \mathbb{N}\}$ and a family $\mathcal{E} = \{E_{nL} : n \in \mathbb{N}, L \in \exp_o(n)\}$.

Let G be a topological $\mathcal{E}$-algebra. Let $n \in \mathbb{N}$, $n \geq 1$, $1 \leq i \leq n$, $a = (a_1, \dots, a_n) \in G^n$ and $\omega \in E_n$. Then the mapping $\omega_{(a,i)} : G \longrightarrow G$, where $\omega_{(a,i)}(x) = e_{nG}(\omega, a_1, \dots, a_{i-1}x, a_{i+1}, \dots, a_n)$ for every $x \in G$ is called a translation on G. This translation is called a continuous translation if for some $L \in \exp_o(n)$ we have $\omega \in E_{nL}$ and $i \in L$. The identical mapping $\iota_G(x) = x$ is also a continuous translation. If $n = 0$ and $\omega \in E_{0G}$, then the constant mapping $\omega_{0G}(x) = e_{0G}(\omega, G^0)$ is also called a continuous translation. The composition of a finite number of continuous translations is called a continuous translation too. Any continuous translation is a continuous mapping of G into G.

A mapping $t : G \longrightarrow G$ is called an admissible translation if there exist $n \in \mathbb{N}$, $n \geq 1$, $L \in \exp_o(n)$, $a \in G^n$ and $\omega \in E_{nL}$ such that:

– $t = \omega_{(a,i)}$ for some $i \in L$;

– for $b \in H^n$ and any $j \in L$ the translation $\omega_{(a,j)}$ is a one-to-one continuous mapping of G onto G.

A topological $\mathcal{E}$-algebra A is called a *bounded $\mathcal{E}$-algebra* if there exists a point $b \in G$ and for each neighbourhood U of the point b in the space G there exist $n, m \geq 1$, $L \in \exp_o(n)$, $i \in L$, $a_1 = (a_{(1,1)}, \dots, a_{(1,n)}), \dots, a_m = (a_{(m,1)}, \dots, a_{(m,n)}) \in G^n$ and $\omega \in E_n$ such that $G = \cup\{t_j(U) : j \leq n\}$ and $t_j = \omega_{(a_j,i)}$ is an admissible translation for each $j \leq m$. The point b is called the fixed point of the topological $\mathcal{E}$-algebra G.

A topological $\mathcal{E}$-algebra A is called a *totally bounded $\mathcal{E}$-algebra* if for all $n \geq 2$, $\omega \in L \in \exp_o(n)$ and $i \in L$ there exists a point $b = b(\omega, i) \in G$ such that for each neighbourhood U of the point b in the space G there exist $m \geq 1$ and $a_1 = (a_{(1,1)}, \dots, a_{(1,n)}, \dots, a_m = (a_{(m,1)}, \dots, a_{(m,n)} \in G^n$ for which $G = \cup\{\omega_{(a_j,i)}(U) : j \leq n\}$ and $t_j = \omega_{(a_j,i)}$ is an admissible translation for each $j \leq m$.

Example 3.1. Let $E = E_0 \cup E_1 \cup E_2$ and $E_2 = \{\omega_2\}$. Then any E-algebra is a groupoid with 0-ary and 1-ary operators. If $E_{2\{1\}} = E_{2\{2\}} = E_2$ and $E_{2\{1,2\}} = \varnothing$ and a groupoid G with a topology is a topological $\mathcal{E}$-algebra, then G is called a semitopological groupoid. If $E_{2\{1\}} = E_{2\{2\}} = E_{2\{1,2\}} = E_2$ and a groupoid G with a topology is a topological $\mathcal{E}$-algebra, then G is called a paratopological groupoid. If $E_{2\{1\}} = E_2$ and $E_{2\{2\}} = E_{2\{1,2\}} = \varnothing$ and a groupoid G with a topology is a topological $\mathcal{E}$-algebra, then G is called a right topological groupoid. A groupoid G with topology is a left (respectively, right) totally bounded groupoid if G is a left (respectively, right) topological groupoid and a bounded topological $\mathcal{E}$-algebra for respectively $\mathcal{E}$.

In particular, a group G with topology:

– is a left (respectively, right) totally bounded groupoid if G is a left (respectively, right) topological group and for every neighbourhood U of the identity e there exists a finite subset L of G such that $LU = G$ (respectively, $UL = G$);

– totally bounded if G is a semitopological topological group and for every neighbourhood U of the identity e there exists a finite subset L of G such that $LU = G$ and $UL = G$.

For topological groups the conditions of precompactness, boundedness, total boundedness, left total boundedness and right total boundedness are equivalent.

Example 3.2. Let A be a topological multiplicative group and $D = \{0, 1\}$. On $G = A \times D$ consider the following topology and operations $\omega_0, \omega_1, \omega_2$:

– The subspace $A \times \{1\}$ of G is discrete and the neighbourhood of the points $(x, 0)$ in G is of the form $(U \times D) \setminus \{0, 1\}$, where U is a neighbourhood of the point x in A. The space G is Alexandroff duplicate of the space A.

– $\omega_2((x, 0), (y, i)) = (xy, i)$ and $\omega_2((x, 1), (y, i)) = (x, 1)$ for all $x, y \in A$ and $i \in D$.

– $\omega_1(x, i) = (x^{-1}, i)$ for all $x \in A$ and $i \in D$.

– ω_0 is a 0-ary operation and $\omega_0(G^0) = (e, 0)$ where e is the unity of the group A.

The element $b = (e, 0)$ is the unity of the semitopological semigroup G. The semigroup G is left totally bounded if and only if A is a totally bounded topological group. The semigroup G is right totally bounded if and only if A is a finite group.

Example 3.3. Let $\mathbb{K}$ be the group of reals as the Sorgenfrey line ([8], Example 1.2.2) with the binary operation $\omega_1(x, y) = x+y$, $\mathbb{Z}$ be the discrete subgroup of $\mathbb{K}$, $B = \mathbb{K}/\mathbb{Z}$ be the quotient group which is homeomorphic with the subspace $[0, 1) = \{x \in \mathbb{K} : 0 \leq x < 1\}$ of $\mathbb{K}$. The spaces $\mathbb{K}$ and B are homeeomorphic, $\mathbb{K}$ is a non totally bounded paratopological Abelian group and B is a totally bounded paratopological Abelian group. Hence on $\mathbb{K}$ there exists a binary operation $\omega_2(x, y)$ such that $(\mathbb{K}, \omega_2)$ is a totally bounded paratopological Abelian group. Obviously, $\mathbb{K}$ and B are not topological groups. Since any semitopological precompact group and any paratopological pseudocompact group is a topological group ([2], Theorem 2.4.1), B is not a subgroup of some precompact semitopological group or of a pseudocompact paratopological group. If $E = E_2 = \{\omega_1, \omega_2\}$, then $\mathbb{K}$ is a bounded non totally bounded topological E-algebra.

Example 3.4. Let E be a signature and G be a non-empty space. If $n \geq 1$ and $\omega \in E_n$, then we put $\omega(x) = x_1$ for each point $x = (x_1, \ldots, x_n) \in G^n$. If G is a compact space, then G is a precompact non-bounded topological E-algebra. A compact $\mathcal{E}$-algebra without admissible translations is precompact and non-bounded.

Thus the notions of precompactness, boundedness and totally boundedness are distinct. We should mention that closed subgroups of totally bounded Hausdorff paratopological groups were studied by T. Banakh and S. Ravsky [3].

Theorem 3.5. *Let A be a topological $\mathcal{E}$-algebra and A be subalgebra of a bounded $\mathcal{E}$-algebra B. For any uncountable set S in the bounded topological $\mathcal{E}$-algebra B^S there exists some bounded subalgebra G which contains A as a closed subalgebra. If the topological $\mathcal{E}$-algebra B is totally bounded, then the topological $\mathcal{E}$-algebras B^S and G are totally bounded too.*

Proof. We put $B_\lambda = B$ for any $\lambda \in S$. For each $x \in A$ we put $\varphi(x) = (x_\lambda \in B_\lambda : \lambda \in S)$, where $x_\lambda = x$ for each $\lambda \in S$. Then φ is an isomorphic embedding of A in B^S.

We identify A with $\varphi(A)$. Thus we can assume that A is a topological $\mathcal{E}$-subalgebra of the $\mathcal{E}$-algebra B^S.

Let $b \in A$ be the fixed point of A. We put $G = \{(x_\lambda : \lambda \in S) \in B^S : |\{\lambda \in S : x_\lambda \notin A\}| \leq \tau\}$ and $G_1 = \{(x_\lambda : \lambda \in L) \in B^S : |\{\lambda \in S : x_\lambda \neq a\}| \leq \tau\}$.

By construction, we have:

Claim 1. *$A \subseteq G \subseteq B^S$, A is a closed subspace of the space G and $G_1 \subseteq G \subseteq B^S$.*

Claim 2. *G is a subalgebra of the algebra B^S.*

Claim 3. *G_1 is a dense subspace of the spaces G and B_S.*

Now we mention the next assertion.

Claim 4. *G and B^S are bounded topological $\mathcal{E}$-algebras.*

Let $c = (b_\lambda = b : \lambda \in L) \in G_1$.

Fix a neighbourhood V of the point c in the space B^L. There exists a neighbourhood U of the point b in B and a finite subset $M = \{\lambda_1, \ldots, \lambda_k\} \subseteq M$ such that $\Pi\{U_\lambda : \lambda \in L\} \subseteq V$, where $U_\lambda = U$ for each $\lambda \in M$ and $U_\lambda = B$ for each $\lambda \in S \setminus M$.

Since the topological $\mathcal{E}$-algebra B is bounded, there exist $n, m \geq 1$, $L \in \exp_o(n)$, $i \in L$, $a_1 = (a_{(1,1)}, \ldots, a_{(1,n)}), \ldots, a_m = (a_{(m,1)}, \ldots, a_{(m,n)}) \in G^n$ and $\omega \in E_n$ such that $B = \cup\{t_j(U) : j \leq n\}$ and $t_j = \omega_{(a_j,i)}$ is an admissible translation for each $j \in L$.

Let $H = B^S$ and $C = \{a_{(i,j)} : i \leq m, j \leq n\}$. If $a \in H^n$, $i \in L$, then:

– $\omega_{(a,i)} : H \longrightarrow H$ is a homeomorphism;

– if $a \in G$, then $\omega_{(a,i)}(G) = G$;

– if $a \in A$, then $\omega_{(a,i)}(A) = A$.

Let $H_i = H$ for each $i \leq n$. The point $a = (a_s : s \in S) \in H$ is (M, C)-marked if $a_s \in C$ for $s \in M$ and $a_s = b$ for $s \in S \setminus M$. A point $x = (x_1, \ldots, x_n) \in H^n$ is (M, C, L)-marked if x_i is (M, C)-marked for each $i \leq n$. The set Z of all (M, C, L)-marked points is finite. By construction, $Z \subseteq G^n$. We affirm that $H = \cup\{\omega_(a,i)(U) : a \in Z\}$ and $G = \cup\{\omega_(a,i)(U \cap G) : a \in Z\}$. Fix $y = (y_s : s \in S) \in H$. If $s \in M$, then there exist $z_s = (z_{1s}, \ldots, z_{ns}) \in B^n$ and $x_s \in U$ such that $\omega_{(z_s,i)}(x_s) = y_s$ and $z_{js} \in C$ for all $j \leq n$. If $s \in S \setminus M$, then $z_s = (b, \ldots, b) \in B^n$ and there exists $x_s \in B$ such that $\omega_{(z_s,i)}(x_s) = y_s$. If $x = (x_s : s \in S) \in H$ and $z = (z_s : s \in S) \in H^n$, then $z \in Z$, $x \in V$ and $\omega_{(z,i)}(x) = y$. If $y \in G$, then $x \in G$ too. Therefore $H = \cup\{\omega_(a,i)(U) : a \in Z\}$ and $G = \cup\{\omega_(a,i)(U \cap G) : a \in Z\}$. Claim 4 is proven. The proof of the following Claim is similar.

Claim 5. *If B is totally bounded, then G and B^S are totally bounded topological $\mathcal{E}$-algebras.*

Thus G is a desired topological $\mathcal{E}$-algebra G. The proof is complete.

Corollary 3.6. *Let A be a paratopological group and A be a subalgebra of a totally bounded paratopological group B. For any uncountable set S in the totally bounded topological paratopological group B^S there exists some totally bounded paratopological group G which contains A as a closed subgroup.*

Corollary 3.7. *Let A be a semitopological group and A be a subalgebra of a totally bounded semitopological group B. For any uncountable set S in the totally bounded topological semitopological group B^S there exists some totally bounded semitopological group G which contains A as a closed subgroup.*

Corollary 3.8. *Let A be a left (right) topopological group and A be a subalgebra of a left (right) totally bounded left (right) topological group B. For any uncountable set S in the left (right) totally bounded topological left (right) topological group B^S there exists some left (right) totally bounded left (right) topological group G which contains A as a closed subgroup.*

REFERENCES

[1] A. V. Arhangel'skii, M. M. Choban. Completeness type properties of semitopological groups, and the theorems of Montgomery and Ellis. *Topology Proceedings*, **37** (2011) 33–60.

[2] A. V. Arhangel'skii, M. Tkachenco. Topological Groups and Related Structures. Amsterdam–Paris, Atlantis Press/World Scientific, 2008.

[3] T. Banakh, S. Ravsky. On subgroups of saturated or totally bounded paratopological groups. *Algebra Discrete Math.*, **4** (2003), 1–20.

[4] M. M. Choban. On precompact and uniformizable algebras. *Serdica Bulg. Math. Publ.* **18** (1992), 129–137.

[5] M. M. Choban. Some topics in topological algebra. *Topology Appl.* **54** (1993), 183–202.

[6] M. M. Choban, I. D. Ciobanu. Compactness and free topological algebras. *ROMAI Journal*, **3** (2007), No 2, 55–85.

[7] M. M. Choban, I. D. Ciobanu. On totally bounded universal algebras. *Creative Mathematics and Informatics*, **21** (2012), No 2, 11–165.

[8] R. Engelking. General Topology. Warszawa, PWN, 1977.

[9] N. Noble. Countably compact and pseudocompact products. *Czechoslov. Math. J.* **19** (1988), No 3, 69–71.

[10] M. I. Urssul. Compact rings and their generalizations. Kishinev, Ştiinţa, 1991 (in Russian).

[11] M. I. Urssul. Topological Groups and Rings. Oradea, Ed. Univ. din Oradea, 1998.

Department of Algebra, Geometry and Topology
Tiraspol State University
5, Gh. Iablocichin
MD2069 Chişinău, Republic of Moldova
e-mail: mmchoban@gmail.com

ВЪРХУ ПОДАЛГЕБРИТЕ НА ПСЕВДОКОМПАКТНИ И ОГРАНИЧЕНИ АЛГЕБРИ

Митрофан М. Чобан

В предлаганата статия са намерени условията за влагане на универсална топологична алгебра в някоя псевдокомпактна или ограничена универсална топологична алгебра