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VISCOUS DRAG IN STEADY STATE GAS FLOW BETWEEN ROTATING CYLINDERS WITH DIFFERENT WALL TEMPERATURE*

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The viscous drag (wall shear-stress) exerted at the interface "gas – rotating cylinder wall" is numerically studied by using the Direct Simulation Monte Carlo (DSMC) method and the continuum model based on the Navier-Stokes-Fourier (NSF) equations. The wall drag of the cylinders is studied depending on the temperature of the inner cylinder wall. The temperature of the stationary outer cylinder is constant. Different cases were calculated for a set of velocities and temperatures of rotating inner cylinder. These studies have been accomplished for several fixed Knudsen numbers. The NSF results have been obtained by setting a local value of Knudsen number in the corresponding first order slip boundary conditions. The flow characteristics obtained by both methods are: in a very good agreement at small Knudsen numbers 0.02 and 0.05 and in a satisfactory agreement at 0.1. The work may be of interest to the analysis of various non-planar micro gas flows.

1. Introduction. In microflow device design, the viscous drag "solid surface – gas" is often of critical importance. However, these depend on the characteristics of the flow - the region of local non-equilibrium existing up to one or two molecular mean free paths from the wall in any gas flow near a surface.

The Couette cylindrical flow is a fundamental problem in the rarefied gas dynamics [4, 9, 15, 17]. As such, its modeling and numerical solving is of a great importance for microfluidics, which serves for theoretical background of the analysis of new emerging Micro Electro Mechanical Systems MEMS [5, 7, 19].

The design of adequate mathematical models of gaseous flows in micro devices is one of the most important tasks of the studies. We consider both molecular and continuum models treating the gaseous flow by using different level of mathematical description. Both models take into account the specific microfluidic effects of gas rarefaction and slip-velocity regime at the solid boundaries [10, 11, 13, 14, 15, 16].

In the present paper we compare results for the viscous drag exerted at the interface "gas – rotating cylinder wall" in cylindrical Couette flow obtained by using the molecular Direct Simulation Monte Carlo (DSMC) method with those calculated by a numerical

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solution of the continuum Navier-Stokes-Fourier equations for compressible flow (NSF). Both methods are used to model the cylindrical Couette flow for Knudsen numbers 0.02, 0.05 and 0.1, the inner cylinder dimensionless velocity 0.1, 0.3 and 0.5, the inner cylinder dimensionless temperature 0.1, 0.3, 0.5, 0.7 and 0.9 and the outer cylinder being at rest, with constant dimensionless temperature $T_2 = 1$. The slip boundary conditions are modeled by using local Knudsen number. The aim of the comparison is to illustrate qualitatively the influence of the Knudsen number, the inner cylinder wall velocity and temperature on the viscous drag values.

The comparison illustrates qualitatively the influence of non-equilibrium effects taking place at larger Knudsen numbers Kn ≥ 0.1. Under such conditions the continuum model is no longer valid and a kinetic theory approach must be applied.

2. Formulation of the problem and methods of solution. We study a rarefied gas flow between two coaxial unconfined cylinders (one dimensional, axis-symmetrical problem).

Continuous Model (NSF) and Numerical Simulation. The continuous model is based on the Navier-Stokes-Fourier equations for compressible fluid, completed with the equations of continuity and energy transport. The governing equations are written as follows:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right) = -\frac{\partial P}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{\tau_{\varphi\varphi}}{r}, \tag{2}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{r\varphi}\right), \tag{3}$$

$$\rho c_P \frac{DT}{Dt} = \operatorname{div}(\lambda \operatorname{grad} T) - P\operatorname{div}\mathbf{V} + \mu\Phi, \tag{4}$$

$$P = \rho R T. \tag{5}$$

Where $\mathbf{V}$ is the velocity vector, u and v are the velocity components along axis r and φ. A rather standard notation is used in Eqs. (1)–(5): ρ is density and T is the temperature. ρ, P, T, $u, v = f(r,t)$. The stress tensor components are $\tau_{i,j}$ and Φ is the dissipation function [12]. For a perfect monatomic gas, the viscosity and the coefficient heat transfer read as [1]:

$$\mu = \mu(T) = C_\mu \rho_0 l_0 V_0 \sqrt{T}, \quad C_\mu = \frac{5}{16}\sqrt{\pi}, \tag{6}$$

$$\lambda = \lambda(T) = C_\lambda \rho_0 l_0 V_0 \sqrt{T}, \quad C_\lambda = \frac{15}{32}\sqrt{\pi}. \tag{7}$$

The above written equations are normalized by using the following scales: for density, $\rho_0 = mn_0$, for velocity $V_0 = \sqrt{2RT_0} - R$ is the gas constant, for length - the distance between the cylinders $L = R_2 - R_1$, for time $t_0 = L/V_0$, for temperature $T_0 = T_{w,1}$ – the wall temperature of both cylinders. The Knudsen number is Kn $= \mathrm{l_0/L}$, where the mean free path is l_o and $\gamma = c_P/c_\mathrm{V} = 5/3$ (c_P and c_V are the heat capacities at constant pressure and constant volume respectively). In this way in the dimensionless model the characteristic number Kn and the constants C_μ and C_λ take part. After the scaling, the

same symbols for the dimensionless ρ, P, T, u, v are used.

For the problem (1)–(4), first-order slip boundary conditions are imposed at both walls, which can be written directly in dimensionless form as follows [6, 22]:

$$v \mp A_\sigma \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right) = V_i, \tag{8}$$

$$u = 0, \tag{9}$$

$$T \pm \zeta_{\mathrm{T}} \mathrm{Kn} \frac{\partial T}{\partial r} = T_i, \tag{10}$$

at $r = R_i$, $i = 1, 2$. In Eqs. (8)–(10) $V_i = v_i/V_0$ and $T_1 = T_{W,1}/T_0$, $T_2 = 1$, are the dimensionless wall velocity and temperature for both cylinders (v_i, $i = 1, 2$ is the dimensional wall velocity). For diffuse scattering we have used the viscous slip and temperature jump coefficients $A_\sigma = 1.1466$ and $\zeta_T = 2.1904$ calculated, respectively in [2, 3], from the kinetic BGK equation by using variation method. The boundary conditions are modeled by using $\mathrm{Kn}_{\mathrm{local}}$.

$$\mathrm{Kn}_{\mathrm{local}} = \frac{l}{L} = \left(L\sqrt{2}\pi\sigma^2 \frac{\rho}{\rho_0}.n_0 \right)^{-1} = \frac{\rho_0.\mathrm{Kn}}{\rho}. \tag{11}$$

In ((11)) with l is denoted the local mean free path.

The wall shear stress or the drag is defined as dimensionless stress tensor component, expressing the viscous interaction between two neighboring thin "shells" of the gas medium, calculated of the driven cylinder wall:

$$\bar{\tau}_{r\varphi} = \tau_{r\varphi} / \left(\rho_0 V_0^2 \right) \tag{12}$$

where $\tau_{r\varphi}$ is the dimensional stress tensor component along the axis φ, according to [11] With a view to axis-symmetric case the stress tensor component reads

$$\tau_{r\varphi} = \tau_{\varphi r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\varphi}{r} \right) \right] . \tag{13}$$

Using the scales introduced in the previously section, the dimensionless stress tensor component $\bar{\tau}_{r\varphi}$ can be expressed through the dimensionless quantities v, T:

$$\bar{\tau}_{r\varphi} = -C_\mu \mathrm{Kn} \sqrt{\bar{T}} \left[r \frac{\partial}{\partial r} \left(\frac{v}{r} \right) \right]. \tag{14}$$

The transfer equations (1)–(4), together with the boundary conditions (8)–(12) and zero initial variations in the clearance between cylinders, written for u, v and T , formulate the unsteady-state initial-boundary value problem. A second order of approximation, central implicit finite difference scheme is used to solve numerically the formulated problem. Starting from the inner cylinder wall M grid nodes are introduced along the coordinate r. The time discretization step is Δt and the step along the axis r is Δr and $r_j = (j-1)\,\Delta r$. j is the grid nod number (more details are given in [6, 23]).

Direct Simulation Monte Carlo (DSMC) Method. The gas considered is simulated as a stochastic system of Nparticles [8, 9]. All quantities used are non-dimensional, so that the mean free path at equilibrium is equal to 1. The basic steps of simulation are as follows:

A. The time interval $\left[0; \hat{t}\right]$ over which the solution is found, is subdivided into subintervals with step Δt.

B. The space domain is subdivided into cells with sides $\Delta z, \Delta r$.

C. Gas molecules are simulated in gap G using a stochastic system of N points (particles) having position $z_i(t), r_i(t)$ and velocities $\overrightarrow{\xi_i(t)}$.

D. N_m particles are located in the m-th cell at any given time. This number varies during the computer simulation by the following two stages:

Stage 1. Binary collisions in each cell are calculated, whereas particles do not move. Collision modelling is realized using Bird's scheme "no time counter".

Stage 2. Particles move with new initial velocities acquired after collisions, and no external forces act on particles. No collisions are accounted for at this stage.

E. Stage 1 and Stage 2 are repeated until $t = \hat{t}$.

F. Flow macro-characteristics (density, velocity, temperature) are calculated as time-averaged when steady regime is attained. The shear-stress is calculated by

$$\tau_{r\varphi} = \rho \left(\int_S \xi_\varphi \xi_r ds - \overline{\xi_\varphi} . \overline{\xi_r} \right), \tag{15}$$

where S is the cylinder wall surface area.

G. Boundary conditions are diffusive at the cylinder walls and periodic along axis Oy. The number of particles (simulators) used in DSMC calculations is 3200000.

3. Numerical results. The aim of the studies is to determine the influence of the Knudsen number, the cylindrical wall velocity and temperature on the drag between gas and cylinder wall, calculated according Eq. (14) and Eq. (15). In a previous research [12, 13, 14, 15] it has been found that only for Kn = 0.02 and less both method solutions were in an excellent agreement. And here, our studies are for the cases with Kn = 0.02, 0.05 and 0.1 at:

$V_1 = 0.1$, 0.3, 0.5 and $V_2 = 0$ and

$T_1 = 0.1$, 0.3, 0.5, 0.7 and 0.9 and $T_2 = 1$.

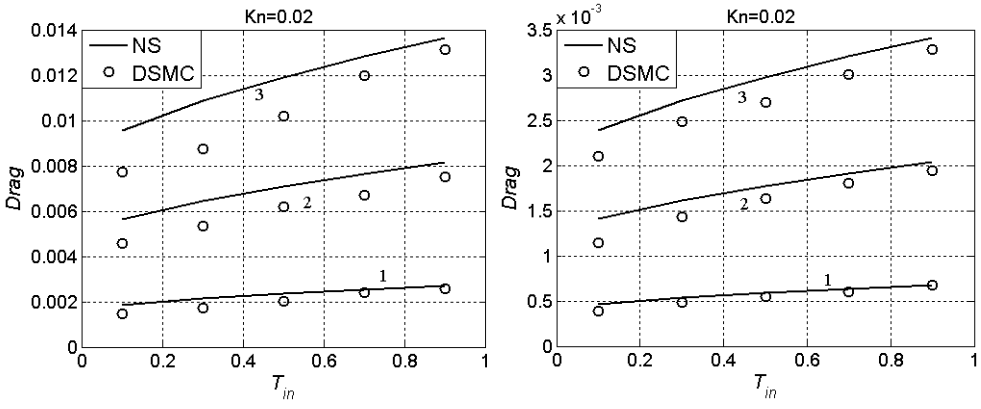

Fig. 1. The viscous drag depending on the temperature of inner cylinder: 1 – $V_{\text{in}} = 0.1$; 2 – $V_{\text{in}} = 0.3$; 1– $V_{\text{in}} = 0.5$ at Kn = 0.02. The left figure the drag on the inner cylinder, the right figure the drag on the outer cylinder

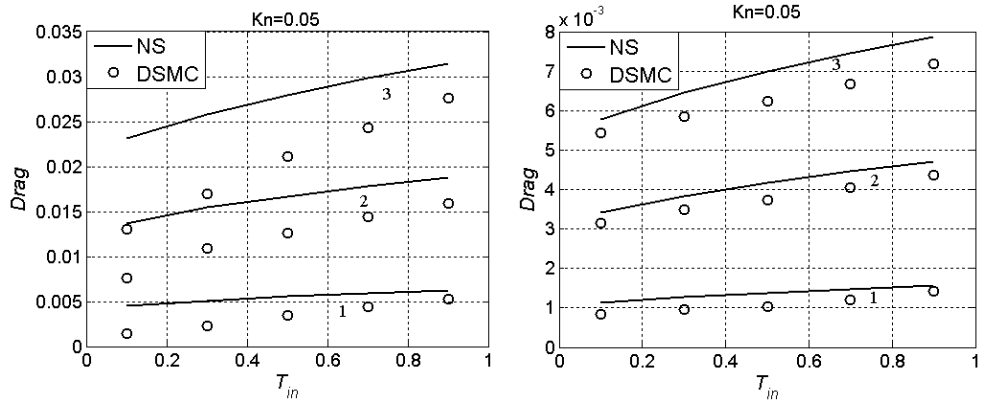

Fig. 2. The viscous drag depending on the temperature of inner cylinder: 1 – $V_{\text{in}} = 0.1$; 2 – $V_{\text{in}} = 0.3$; 1– $V_{\text{in}} = 0.5$ at $\text{Kn} = 0.05$. The left figure the drag on the inner cylinder, the right figure the drag on the outer cylinder

The decrease of the inner cylinder temperature value leads to decrease the drag between the gas and the cylinders. The drag on the inner cylinder is greater than the drag on the outer cylinder in all studied cases (no matter whether the rotating cylinder is inner or outer). The Figures 1, 2 and 3 show that the drag is linearly dependent on the cylinder temperature. This indicates that the effects caused by the increasing drag "gas-wall" at the Knudsen number increasing, can be of interest at the MEMS development.

In previous research was found that the differences between the two methods occur in temperature values, which naturally leads to increasing differences with increasing temperature gradient [15, 16, 24].

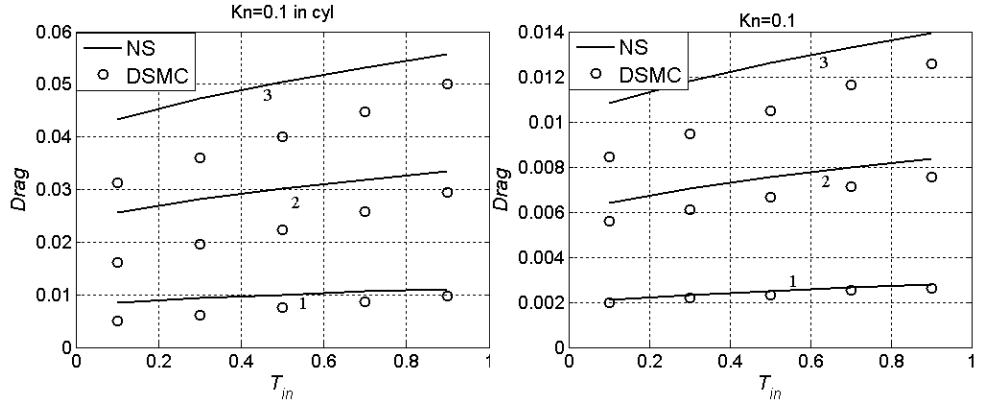

Fig. 3. The viscous drag depending on the temperature of inner cylinder: 1– $V_{\text{in}} = 0.1$; 2 – $V_{\text{in}} = 0.3$; 1– $V_{\text{in}} = 0.5$ at $\text{Kn} = 0.1$. The left figure the drag on the inner cylinder, the right figure the drag on the outer cylinder

4. Conclusions. The results obtained by both methods are: in a very good agreement at small Knudsen numbers Kn = 0.02 and Kn = 0.05 and in a satisfactory agreement at Kn = 0.1. The flow character is maintained at increasing the Knudsen number while the differences are in the macro-characteristics value. With decreasing the wall temperature the drag "gas-wall" decreases. This is due to the fact that with increasing of Knudsen number the NSF model can't adequately describe the gas flow.

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ВИСКОЗНО ТРИЕНЕ НА СТЕНИТЕ, ПРИ СТАЦИОНАРНО ТЕЧЕНИЕ НА ГАЗ МЕЖДУ ВЪРТЯЩИ СЕ ЦИЛИНДРИ С РАЗЛИЧНА ТЕМПЕРАТУРА

Добри Данков, Владимир Русинов, Петър Господинов

Числено е изследвано вискозното триене върху стената, възникващо на фазовата граница „газ – въртящ се цилиндър“, посредством метода на прякото статистическо моделиране (DSMC) и решението на континуален модел, с използване на уравненията на Навие-Стокс и Фурие (NSF). Разгледана е зависимостта на триенето върху стената на въртящия се вътрешен цилиндър от неговата температура. Външният цилиндър е неподвижен и с постоянна температура на стената. Числените резултати са получени за няколко характерни стойности на числото на Кнудсен. В граничните условия на плъзгане, от първи ред в континуалния модел, е използвано локално число на Кнудсен. Сравнението на резултатите, получени по двата метода, показва много добро съвпадение за числа на Кнудсен 0.02 и 0.05 и задоволително съответствие за число на Кнудсен равно на 0.1. Разработката представлява интерес при изследването на микротечения на вискозен газ.
Ключови думи: Механика на флуидите, кинетична теория, разреден газ, пряко статистическо моделиране (DSMC).