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FACTORIZATIONS OF THE GROUPS $P\Omega_8^+(q)$*

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In this paper we consider simple groups G which can be represented as a product of two of their proper non-Abelian simple subgroups A and B. Such an expression $G = AB$ is called a (simple) factorization of G. Here we suppose that G is any one of the simple orthogonal groups with Witt defect 0 in dimension 8 over the finite field $GF(q)$ and determine all factorizations of G.

1. Introduction. Let G be a finite (simple) group. The present paper is concerned with the determination of all factorizations $G = AB$ of G into the product of two non-Abelian simple subgroups A and B. Especially, we treat the case in which G is any member of the series of simple orthogonal groups $P\Omega_8^+(q)$. In [3] we have already considered the factorizations of all series of finite simple groups of Lie type and Lie rank 4 except for the orthogonal groups $P\Omega_8^+(q)$. Thus we complete the determination of all factorizations of all finite simple groups of Lie type of Lie rank 4. The main reason to postpone the investigation of these orthogonal groups is that they possess many classes of isomorphic maximal factorizations (permuted by *triality* automorphisms – graph automorphisms of order 3) and accordingly a lot of possible cases of factorizations (into the product of two simple groups) have to be considered. The following result is proved:

Theorem. *Let $G = P\Omega_8^+(q)$. Suppose that $G = AB$, where A and B are proper non-Abelian simple subgroups of G. Then one of the following holds:*

(1) $q = 2$ and $A \cong A_8$, $B \cong A_9$ or $U_4(2)$, or $A \cong A_9$, $B \cong U_4(2)$, or $A \cong PSp_6(2)$, $B \cong A_6$, A_7, A_8, A_9 or $U_4(2)$;

(2) $q = 3$ and $A \cong \Omega_7(3)$, $B \cong A_9$, $PSp_6(2)$ or $\Omega_8^+(2)$;

(3) $A \cong \Omega_7(q)$, $B = A^\tau$ (τ is a triality automorphism of G), $PSp_4(q)$ ($q > 2$) or $P\Omega_8^-(\sqrt{q})$ (q square);

(4) $q = 2^n > 2$ or $q \equiv -1 \pmod 4$ and $A \cong \Omega_7(q)$, $B \cong L_4(q)$;

(5) $q = 2^n > 2$ or $q \equiv 1 \pmod 4$ and $A \cong \Omega_7(q)$, $B \cong U_4(q)$.

The factorizations of $G = P\Omega_8^+(q)$ into the product of two maximal subgroups (so called maximal factorizations) have been determined in [8]. We make use of this result here.

We shall freely use the notation and basic information on the finite (simple) classical groups given in [6]. Let V be the natural 8-dimensional orthogonal vector space over the

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finite field $\mathbf{F} = GF(q)$ on which G acts. Assume that $(\,,)$ is a nonsingular symmetric bilinear form on V which is associated with the (nondegenerate) quadratic form $Q : V \to \mathbf{F}$ of defect 0 (thus the maximal totally singular subspaces have dimension 4). There is a basis $\{e_i, f_i | i = 1,2,3,4\}$ of V, called a standard basis, such that $Q(e_i) = Q(f_i) = (e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{ij}$ for $i, j = 1,2,3,4$. G acts transitively on totally singular one-dimensional subspaces of V and let P_1 be the stabilizer in G of such a subspace. On the other hand G has two orbits on the set of maximal totally singular subspaces, and let P_3, P_4 be the stabilizers in G of such subspaces in the two different orbits. N_1 means the stabilizer in G of a nonsingular one-dimensional subspace of V and N_2^ε $(\varepsilon = \pm)$ is the stabilizer of a nonsingular two-dimensional subspace of V which has type O_2^ε. From Propositions 4.1.6 and 4.1.20 in [6] we can obtain the structure of P_k $(k = 1,3,4)$, N_1 and N_2^ε. It follows that:

$$P_1 \cong [q^6] : (Z_{q-1} \times P\Omega_6^+(q)) \text{ (if } q \text{ is even)},$$
$$P_1 \cong [q^6] : (Z_{(q-1)/2} \times P\Omega_6^+(q)) \text{ (if } q \equiv -1 \pmod 4),$$
$$P_1 \cong [q^6] : 2.(Z_{(q-1)/4} \times P\Omega_6^+(q)).2 \text{ (if } q \equiv 1 \pmod 4);$$
$$P_3 \cong P_4 \cong [q^6] : GL_4(q) \text{ (if } q \text{ is even) or } [q^6] : \left(\frac{1}{2} GL_4(q) / \langle -1 \rangle\right) \text{ (if } q \text{ is odd)};$$
$$N_1 \cong \Omega_7(q);\ N_2^\varepsilon \cong (Z_{q-\varepsilon 1} \times P\Omega_6^\varepsilon(q)).2 \text{ (if } q \text{ is even)},$$
$$N_2^\varepsilon \cong 2.(Z_{(q-\varepsilon 1)/4} \times P\Omega_6^\varepsilon(q)).[4] \text{ (if } q \equiv \varepsilon 1 \pmod 4), \text{ and}$$
$$N_2^\varepsilon \cong (Z_{(q-\varepsilon 1)/2} \times P\Omega_6^\varepsilon(q)).2 \text{ (if } q \equiv -\varepsilon 1 \pmod 4).$$

Recall the well known isomorphisms $P\Omega_6^\varepsilon(q) \cong L_4^\varepsilon(q)$ (here $L_4^\varepsilon(q)$ denotes $L_4(q)$ if $\varepsilon = +$ and $U_4(q)$ if $\varepsilon = -$). From the above it follows immediately that every one of the stabilizers P_1, P_3 and P_4 contains a subgroup isomorphic to $L_4(q)$ if and only if q is even or $q \equiv -1 \pmod 4$; also, in N_1 and N_2^ε there exists a subgroup isomorphic to $L_4^\varepsilon(q)$ only if q is even or $q \equiv -\varepsilon 1 \pmod 4$. The only other sources of subgroups isomorphic to $L_4(q)$ or $U_4(q)$ (for an arbitrary field $GF(q)$) in maximal subgroups of G, taking part in maximal factorizations, are the (maximal) subgroups of Aschbacher's classes C_2 and C_3 (see [1]). In fact, a subgroup isomorphic to $L_4(q)$ in C_2 and a subgroup isomorphic to $U_4(q)$ in C_3 exist only if q is even. In particular, both $L_4(q)$ and $U_4(q)$ simultaneously are subgroups of G only if q is even in any case.

2. Proof of the theorem. Let $G = P\Omega_8^+(q)$ and $G = AB$, where A and B are proper non-Abelian simple subgroups of G. The factorizations of $P\Omega_8^+(2)$ are determined in [2]; this gives *(1)* and *(3)* (with $A \cong B \cong \Omega_7(2) \cong PSp_6(2)$) in the theorem. Thus we can assume that $q \geq 3$. The list of maximal factorizations of G is given in [8]. This leads, by order considerations, to the following possibilities:

1) $A \cong \Omega_7(q)$, $B = A^\tau$ (τ is some triality automorphism of G), $PSp_4(q)$ $(q > 2)$ or $P\Omega_8^-(\sqrt{q})$ (q square);

2) $A \cong \Omega_7(q)$, $B \cong L_4^\varepsilon(q)$, q even or $q \equiv -\varepsilon 1 \pmod 4$;

3) $q = 3$ and $A \cong \Omega_7(3)$, $B \cong A_9$, $PSp_6(2)$ or $\Omega_8^+(2)$;

4) $A \cong L_4(q)$ (in $\Omega_7(q)$ subgroup of G, in N_2^+ or C_2, or in P_1, P_3 or P_4), $B \cong U_4(q)$ (in $\Omega_7(q)$ subgroup of G), q even;

5) $A \cong L_4(q)$ (in $\Omega_7(q)$ subgroup of G, or in P_1, P_3 or P_4), $B \cong U_4(q)$ (in N_2^- or C_3), q even.

All possibilities in case 1) are known factorizations from [8]; the factorization $P\Omega_8^+(3) = \Omega_7(3)\Omega_8^+(2)$, which is the last possibility in case 3), is also proved there. All these factorizations are listed in the theorem. Let us note that in the factorization $G = AB$ with $A \cong \Omega_7(q)$ and $B = A^\tau$ we have $A \cap B \cong G_2(q)$ which is mentioned in [8] (originally in [7], 3.1.1 (vi)).

We consider all possibilities that left case by case.

Case 2). Here $A \cong \Omega_7(q)$ and we choose $B \cong L_4^\varepsilon(q)$ to be a subgroup of $B_1 = A^\tau$ such that $B_1 = (A \cap A^\tau)B$. The last is possible (see [4] and [5]) if q is even or $q \equiv -\varepsilon 1 \pmod 4$. Since $A \cap B = (A \cap B_1) \cap B$ it follows (by order considerations) that $G = AB$. These are the factorizations in *(4)* and *(5)* of the theorem.

Case 3). Applying same arguments as in the previous case we take $B_1 = A^\tau$ and then $G = AB_1$ with $A \cap B_1 \cong G_2(3)$. In [5] we mentioned that an appropriate choice of a subgroup $B \cong A_9$ or $B \cong PSp_6(2)$ can be made such that it satisfies the factorization $B_1 = (A \cap B_1)B$. Now, as obviously $A \cap B = (A \cap B_1) \cap B$, order considerations imply the factorization $G = AB$; these are the desired cases in *(2)* of the theorem.

Now we proceed to prove that the possibilities in the remaining two cases 4) and 5) do not give rise to any factorizations.

Case 4). First let us suppose that $A < N_1$ and $B < N_1^\tau$ (τ is some triality and $N_1 \cong N_1^\tau \cong \Omega_7(q) \cong PSp_6(q)$; recall that q is even). As we mentioned above $N_1 \cap N_1^\tau \cong G_2(q)$. Any one of the groups A and B is a single conjugacy class of subgroups in N_1 and N_1^τ, respectively. The same holds for $N_1 \cap N_1^\tau$ in both N_1 and N_1^τ. Now, using [4], we can write down the factorizations $N_1 = A(N_1 \cap N_1^\tau)$ and $N_1^\tau = B(N_1 \cap N_1^\tau)$ in which $A_1 = A \cap (N_1 \cap N_1^\tau) \cong SL_3(q)$ and $B_1 = B \cap (N_1 \cap N_1^\tau) \cong SU_3(q)$. Since obviously $A_1 \cap B_1 = A \cap B$, by the orders, we come to the following factorization of $N_1 \cap N_1^\tau \cong G_2(q)$: $N_1 \cap N_1^\tau = A_1B_1$. But the groups $G_2(q)$ have no such a factorization. This eliminates the possibility.

Further, let $A \cong L_4(q)$ be in the single class of subgroups in one of the two conjugacy classes N_2^+ in G. Applying triality, we may take $B \cong U_4(q)$ to be a subgroup of N_1. The two conjugacy classes of N_2^+ are interchangeable by a graph automorphism of order two fixing N_1 (see [7]). Thus we may take A to be the subgroup of index 2 in the group $A_1 \cong O_6^+(q)$ fixing the first two vectors e_1 and f_1 in the standard basis and acting naturally on the subspace generated by the others. There is a second basis (2) $\{e_1+f_1,\ f_1,\ \alpha(e_1+f_1)+e_2,\ e_2+f_2,\ e_3,\ f_3,\ e_4,\ f_4\}$ of V with $\alpha \in \mathbf{F}$ such that $x^2+x+\alpha^2$ is irreducible over $\mathbf{F}$. As $N_1(\cong PSp_6(q))$ has just one conjugacy class of subgroups $U_4(q)$ we may take B to be the subgroup of index 2 in the group $B_1 \cong O_6^-(q)$ which fixes the first two vectors of the last basis and acts naturally on the subspace of type O_6^- generated by the others. It can be read off by ([8], 3.2.4 (e)) that $A \cap B \cong Sp_4(q)$. Order considerations imply $G \neq AB$ in this possibility.

Next we suppose that $A \in C_2$. There are two conjugacy classes of subgroups $A \cong L_4(q)$ in C_2 which are permuted by a graph automorphism of order 2 fixing N_1. Thus (according to the type of subgroups in C_2) we may identify A as the subgroup with the following matrix representation with respect to the basis (3) $\{e_i,\ i = 1 \div 4;\ f_i,\ i = 1 \div 4\}$ (the standard basis with different order of the vectors in it): $A = \left\{ \left(\begin{array}{c|c} S & 0 \\ \hline 0 & (S^{-1})^t \end{array} \right) \mid \det S = 1 \right\}$.
As above (using triality) we may take $B \cong U_4(q)$ to have the same description with respect

to the basis (2). Now, taking into consideration the corresponding matrix realization of B, we can evaluate (in matrix form) the common elements of A and B (with respect to the basis (3)): $A \cap B = \{\mathrm{diag}\,(E_{2\times 2},\, T,\, E_{2\times 2},\ (T^{-1})^t) | \det T = 1\} \cong SL_2(q)$. Hence, by the orders, $G \neq AB$.

Now we deal with the subcase in which A stabilizes one-dimensional totally singular subspace. Such a possible subgroup A (let us call it "standard") has already been described as a subgroup also in N_2^+ of the second considered subcase above. B is the same discussed in this subcase subgroup of N_1. As it has been seen the intersection $A \cap B$ is isomorphic to $Sp_4(q)$. Let us denote by $\widehat{A}$ any subgroup of P_1 which is isomorphic to $L_4(q)$. Then $|A \cap \widehat{A}| \geq \dfrac{|A|^2}{|P_1|} = \dfrac{|A|^2}{q^6.(q-1).|A|} = (q^4-1).(q^3-1).(q+1)$. Thus in A there exist two subgroups: $A \cap B\,(\cong Sp_4(q))$ and $A \cap \widehat{A}$ (of order not less than $(q^4-1).(q^3-1).(q+1)$). Hence $|\widehat{A} \cap B| \geq |(A \cap B) \cap (\widehat{A} \cap A)| \geq \dfrac{|A \cap B|.|\widehat{A} \cap A|}{|A|} = \dfrac{|\widehat{A} \cap A|.|Sp_4(q)|}{|L_4(q)|} = \dfrac{|\widehat{A} \cap A|}{q^2.(q^3-1)} \geq \dfrac{(q^4-1).(q+1)}{q^2} > q^2 - 1$ and again it follows (by order considerations) that $G \neq AB$ in this subcase.

Lastly, if A stabilizes a maximal totally singular subspace we can choose this subspace to be generated by the vectors $\{e_i,\ i = 1 \div 4\}$; B is the same subgroup of N_1 discussed throughout the hole case so far (this choice can be made in view of the fact that P_3 and P_4 are permuted by a graph automorphism of order 2 which fixes N_1). For one possible subgroup A we have already proved (see the subcase $A \in C_2$, A obviously stabilizes this subspace) that no factorization gives rise to. To prove this for an arbitrary group $A \cong L_4(q)$, taking into account the structure of $P_3 \cong P_4$, we consider all possible type $L_4(q)$ Sylow 2-subgroups L in the stabilizer of the mentioned subspace. Routine but exhausting calculations show that such a group L always contains involution of B. This eliminates the possibility.

Case 5). In this case if A is a subgroup in an $\Omega_7(q)$ subgroup of G, applying triality τ, we may take A to be a subgroup of N_1 as τ permutes all isomorphic to $\Omega_7(q)$ subgroups of G; τ also permutes the stabilizers P_1, P_3 and P_4 between themselves so we can restrict the other subcases of A to the inclusion $A \in P_1$ only. On the other hand G has uniquely determined subgroups $B(\cong U_4(q))$ in any one of the two conjugacy classes of either N_2^- or in C_3. These subgroups are interchangeable by a graph automorphism of order two which fixes N_1 and P_1; so only one possible subgroup B in one N_2^- and in C_3 is enough to be considered. According to [1] the groups in C_3 correspond to appropriate extensions of the basic field $\mathbf{F}$.

First we deal with the subcase $A < N_1$ and $B \in N_2^-$. In order to proceed this possibility we need to introduce into consideration two more bases of V with respect to which the subgroups A and B will be presented in matrix form. These are the basis (4) $\{e_1 + f_1, f_1, e_i, f_i;\ i = 2, 3, 4\}$ (in which $e_1 + f_1$ is a nonsingular vector) and the basis (5) $\{e_1 + f_1, e_1 + e_2 + \mu f_2, e_1 + f_1 + f_2, e_2 + \mu f_2, e_i, f_i;\ i = 2, 3, 4\}$ with $\mu \in \mathbf{F}$ such that $x^2 + x + \mu$ is irreducible over $\mathbf{F}$. The subspace generated by the first two vectors of (5) has type O_2^- and the other six vectors generate a subspace of type O_6^-; V is an orthogonal decomposition of these its subspaces. Let us consider the subgroup A_1 of the group $G_1 = SO_8^+(q) = O_8^+(q) = G.2$ which fixes each of the vectors $e_1 + f_1$ and f_1 of the basis (4) and acts naturally on the subspace generated by the other vectors of that basis: $A_1 = A.2 \cong O_6^+(q)$. A subgroup $B_1 = B.2 \cong O_6^-(q)$ can be taken to be the subgroup of G_1 fixing vectors $e_1 + f_1$ and $e_1 + e_2 + \mu f_2$ and stabilizing the subspace generated by the other vectors in the basis (5). Now these copies of A_1 and B_1 have clearly described matrix representations with respect to the bases (4) and (5), respectively. Taking into account the necessary "orthogonal" conditions onto the entries of the matrices in these representations we calculate the common elements (in the representation related to the basis (4)) of A_1 and B_1. It follows that $A_1 \cap B_1$ contains a subgroup $O_4^+(q)$ acting naturally on the subspace generated by the vectors e_3, f_3, e_4, f_4 and fixing $e_1 + f_1$, f_1, e_2, f_2. As $O_4^+(q) \cong (SL_2(q) \times SL_2(q)).2$, the intersection $A \cap B$ contains a subgroup isomorphic to $SL_2(q) \times SL_2(q)$ which (by order considerations) eliminates this possibility. Obviously, the treated copy of $L_4(q)$ is also in P_1. So, in this particular case, we have eliminated the possibility $A \in P_1$, $B \in N_2^-$ too.

Next we suppose that A is already considered subgroup of N_1 (in matrix form with respect to (4)) and let B be in C_3. Here we shall need a (matrix) realization of B in $G_1 = SO_8^+(q) = G.2$. The group G_1 has the following (matrix) realization with respect to the basis (3)

$$G_1 = \left\{X \in SL_8(q) | X^t.L.X = L,\ Q(e_jX) = Q(e_j),\ Q(f_jX) = Q(f_j)\right\},$$

where $L = \left(\begin{array}{c|c} O & E_{4\times4} \\ \hline E_{4\times4} & O \end{array}\right)$. Let $\mathbf{K}$ be a field extension of $\mathbf{F} = GF(q)$ of degree 2. There is an element ω of $\mathbf{K}$ such that $\omega + \omega^q = 1$ and $\omega^2 = p_0 + p_1.\omega$ where $p_0, p_1 \in \mathbf{F}$. Further, let $S = (s_{ij}^0 + s_{ij}^1.\omega)_1^4 = S_0 + S_1.\omega$ with $S_0 = (s_{ij}^0)_1^4$, $S_1 = (s_{ij}^1)_1^4$, $s_{ij}^l \in \mathbf{F}$ $(l = 0, 1;\ i, j = 1, 2, 3, 4)$ be any unimodular matrix such that $\overline{S}^t TS = T$, where $T = \left(\begin{array}{c|c} O & E_{2\times2} \\ \hline E_{2\times2} & O \end{array}\right)$. Let us denote by $\tilde{B}$ the set of all matrices S which satisfy these properties. So $\tilde{B} \cong U_4(q)$ has a standard unitary realization over the field $\mathbf{K}$. Then the following matrices form a subgroup B of G_1 isomorphic to $U_4(q)$: $W = P.\left(\begin{array}{c|c} S_0 & p_0S_1 \\ \hline S_1 & S_0 + p_1S_1 \end{array}\right).P^{-1}$, where $P = (p_{ij})_1^8$ and $p_{ij} = \begin{cases} 0, & \text{if } j \neq \pi(i) \\ 1, & \text{if } j = \pi(i) \end{cases}$ for the permutation $\pi = (1)(2,3,5)(4,7,6)(8)$. The isomorphism is given by the map $\sigma : S \mapsto W$. Now calculations show that $A \cap B \cong SL_2(q)$ and again there is no factorization in this subcase of *Case* 5). Let us note that for the considered copy of A we have also eliminated the case $A \in P_1$ and B in C_3.

To finish our considerations we have to discuss in full the possibility $A \in P_1$ and B is one of the subgroups in N_2^- or in C_3 already described above. As P_1 is a single conjugacy class of subgroups in G we may take A to be in the stabilizer in G of the first vector from the standard basis. Applying matrix realizations of these groups (with respect to the corresponding bases, making necessary transitions between them), according to the structure of P_1, we consider all subgroups R in P_1 that can be isomorphic to a Sylow 2-subgroup of possible $A \cong L_4(q)$ in P_1. Calculations show that any such subgroup R always has common elements with the subgroup B. Thus $2 | |A \cap B|$ and then, by the orders, there are no factorizations in these cases.

We considered all the possible cases. The theorem is proved.

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ФАКТОРИЗАЦИИТЕ НА ОРТОГОНАЛНИТЕ ГРУПИ $P\Omega_8^+(q)$

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Разглеждат се прости групи G, които могат да се представят като произведение на две свои собствени неабелови прости подгрупи A и B. Всяко такова представяне $G = AB$ се нарича (проста) факторизация на G. В настоящата работа предполагаме, че G е проста ортогонална група с нулев дефект на Вит от размерност 8 над крайно поле $GF(q)$ и определяме всички факторизации на G.