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A NOTE ON THE CLASSES OF STRONGLY STARLIKE AND STRONGLY CLOSE-TO-CONVEX FUNCTIONS*

Donka Pashkouleva

The aim of this paper is to obtain results about Fekete-Szegö problem for the classes of strongly starlike and strongly close-to-convex functions.

1. Introduction and definitions. Let P be the class of functions $h(z)$ given by $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, which are analytic and have positive real part in $E = \{z : |z| < 1\}$. Let Ω be the class of functions analytic in E, such that

$$\omega(0) = 0 \text{ and } |\omega(z)| \le |z| \text{ for } z \in E.$$

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are analytic and univalent in the unit disk E.

A function $f(z)$ analytic in E is said to be starlike in E if $f(0) = f'(0) - 1 = 0$ and

$$\Re \frac{zf'(z)}{f(z)} > 0 \text{ for } z \in E.$$

The class of such functions is denoted by S^*.

A function $f(z)$, analytic in E with $f(0) = f'(0) - 1 = 0$ is said to be convex if and only if, for $z \in E$

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0.$$

The class of such functions is denoted by C.

A function $f(z)$, analytic in E, is said to be close-to-convex in E if there exists a function $g(z) \in S^*$ and a real number γ such that for $z \in E$ and $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$,

$$\Re e^{i\gamma} \frac{zf'(z)}{g(z)} > 0.$$

The class of such functions is denoted by K^γ. Note $K^0 = K$ the class of close-to-convex functions introduced by Kaplan[1].

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The classes S, K^γ, S^* and C are related by proper inclusions

$$C \subset S^* \subset K^\gamma \subset S.$$

A function $g(z)$, analytic in E and normalized so that $g(0) = 0$, $g'(0) = 1$, is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if for $z \in E$

$$\left|\arg \frac{zg'(z)}{g(z)}\right| \leq \alpha\frac{\pi}{2}.$$

We denote such functions by $S^*(\alpha)$. Note that $S^*(1) = S^*$ [7].

A function f(z), analytic in E and normalized so that $f(0) = 0 = f'(0) - 1$ is said to be strongly close-to-convex of order β if and only if there exists a function $g(z) \in S^*$ and a real number γ, $-\frac{\pi}{2} \leq \gamma \leq \frac{\pi}{2}$ satisfying

$$\left|\arg e^{i\gamma}\frac{zf'(z)}{g(z)}\right| \leq \beta\frac{\pi}{2} \text{ for } z \in E \text{ and } \beta \geq 0.$$

The class of such functions will be denoted by $K(\beta)$. Clearly, $K(0) = C$, $K(1) = K$ and for $0 \leq \beta \leq 1$, $f(z)$ is close-to-convex and hence univalent [1].

2. Known results.

Theorem 2.1 ([5], [6]). *If $g(z) \in S^*$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then $|b_n| \leq n$, $n = 2, 3, \ldots$. Equality holds when $g(z) = \frac{z}{(1-\varepsilon z)^2}$, $|\varepsilon| = 1$ and $\left|b_3 - \mu b_2^2\right| \leq \max\{1, |3-4\mu|\}$. The inequality is sharp for the Koebe function $K(z) = \frac{1}{(1-z)^2}$ if $\left|\mu - \frac{3}{4}\right| \geq \frac{1}{4}$ and for function $K_1(z) = K(z^2)^{\frac{1}{2}}$ if $\left|\mu - \frac{3}{4}\right| \leq \frac{1}{4}$.*

Theorem 2.2 ([4]). *If $g(z) \in K$, with $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and if μ is a real number, then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq \frac{1}{3} \\ \frac{1}{3} + \frac{4}{9\mu} & \text{if } \frac{1}{3} \leq \mu \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq \mu \leq 1 \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

For each μ there is a function in K such that equality holds.

Theorem 2.3 ([13]). *Let $f(z) \in K(\beta)$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and let $F(\beta)$ be defined for $z \in E$ by*

$$F_\beta(z) = \frac{1}{2(\beta+1)}\left[\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right] = z + \sum_{n=2}^{\infty} A_n(\beta) z^n,$$

then $|a_n| \leq A_n(\beta)$. The result is sharp for all real β and every integer $n \geq 2$.

Theorem 2.4 ([10]). *Let* $h(z) \in P$*, with* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$*. Then* $|c_n| \leq 2$*,* $n = 1, 2, \ldots,$

$$\left| c_2 - \frac{c_1^2}{2} \right| \leq 2 - \frac{|c_1|^2}{2}.$$

Equality holds when $h(z) = \dfrac{1+z}{1-z}$*.*

In 1933, Fekete and Szegö [2] obtained the sharp bounds for $\left|a_3 - \mu a_2^2\right|$ in S, for each fixed μ in the interval $0 \leq \mu \leq 1$. They showed that for $f(z) \in S$ given by (1.1)

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & \text{if } \ \mu \leq 0 \\ 1 + 2e^{-\frac{2\mu}{1-\mu}} & \text{if } \ 0 \leq \mu \leq 1 \\ 4 - 3\mu & \text{if } \ \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that, for each μ, there exists a function in S such that equality holds. Pfluger [3] considered the problem when μ is complex and $f(z) \in S$.

In the case of C, S^* and K^γ the subclasses of S consisting of convex, starlike and close-to-convex functions respectively, the above inequalities can be improved [4], [6]. In particular for $f(z) \in K^\gamma$ and given by (1.1) Keogh and Marks [4] showed that when μ is complex

$$\left|a_3 - \mu a_2^2\right| \leq \max\left\{1, 3|\mu - 1|, |4\mu - 3|\right\}.$$

In 1987 Koepf [6] showed that for $f(z) \in K^\gamma$ and μ real

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 3 - 4\mu & \text{if } \ 0 \leq \mu \leq \dfrac{1}{3} \\ \dfrac{1}{3} + \dfrac{4}{9\mu} & \text{if } \ \dfrac{1}{3} \leq \mu \leq \dfrac{2}{3} \\ 1 & \text{if } \ \dfrac{2}{3} \leq \mu \leq 1. \end{cases}$$

3. Strongly Starlike and Strongly Close-to-convex Functions In this section we extend Theorem 2.1 to the class $S^*(\alpha)$ of strongly starlike functions of order α, $0 < \alpha \leq 1$. This class has been investigated in [7], [8], [9].

Theorem 3.1. *Let* $g(z) \in S^*(\alpha)$ *and be given by* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$*, then for* μ *real*

$$\left|b_3 - \mu b_2^2\right| \leq \max\left\{\alpha, \alpha^2|3 - 4\mu|\right\}.$$

The result is sharp for $g_0(z) = z \exp\left\{\int_0^z \left[\frac{1}{t}\left(\frac{1+t}{1-t}\right)^\alpha - 1\right] dt\right\}$ *if* $\left|\mu - \frac{3}{4}\right| \geq \frac{1}{4\alpha}$ *and for* $g_1(z) = z \exp\left\{\int_0^z \left[\frac{1}{t}\left(\frac{1+t^2}{1-t^2}\right)^\alpha - 1\right] dt\right\}$ *if* $\left|\mu - \frac{3}{4}\right| \leq \frac{1}{4\alpha}$*.*

Proof. Since $g(z) \in S^*(\alpha)$ it follows that we can write

$$zg'(z) = g(z)h(z)^\alpha \tag{3.1}$$

for $h(z) \in P$ and $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Comparing the coefficients in (3.1), we obtain $b_2 = \alpha c_1$ and $2b_3 = \frac{3}{2}\alpha^2 c_1^2 - \frac{\alpha}{2} c_1^2 + \alpha c_2$ so that

$$b_3 - \mu b_2^2 = \frac{\alpha}{2}\left(c_2 - \frac{1}{2}c_1^2\right) + \frac{\alpha^2 c_1^2}{4}(3 - 4\mu) = \frac{\alpha}{2}\left[\left(c_2 - \frac{1}{2}c_1^2\right) + \alpha c_1^2\left(\frac{3}{2} - 2\mu\right)\right].$$

Thus

$$\left|b_3 - \mu b_2^2\right| \le \frac{\alpha}{2}\left[\left|c_2 - \frac{1}{2}c_1^2\right| + \alpha\left|\frac{3}{2} - 2\mu\right| |c_1|^2\right] \le \frac{\alpha}{2}\left[2 - \frac{1}{2}|c_1|^2 + \alpha\left|\frac{3}{2} - 2\mu\right| |c_1|^2\right]$$

where we have used Theorem 2.4.

If $\left|\mu - \frac{3}{4}\right| \le \frac{1}{4\alpha}$, then we have

$$|b_3 - \mu b_2| \le \frac{\alpha}{2}\left[2 - \frac{1}{2}|c_1|^2 + \frac{1}{2}|c_1|^2\right] = \alpha.$$

On the other hand, if $\left|\mu - \frac{3}{4}\right| \ge \frac{1}{4\alpha}$, Theorem 2.4 and the fact that $|c_2| \le 2$ we have

$$\left|b_3 - \mu b_2^2\right| \le \frac{\alpha}{2}\left[2 + \left(\alpha\left|\frac{3}{2} - 2\mu\right| - \frac{1}{2}\right)|c_1|^2\right] \le \frac{\alpha}{2}\left[2 + 4\alpha\left|\frac{3}{2} - 2\mu\right| - 2\right] = \alpha^2|3 - 4\mu|,$$

which completes the proof. □

In 1987 Koepf [11] considered the Fekete-Szegö problem for the class $K(\beta)$ and obtained sharp results when $\mu = \frac{2\beta}{3(\beta+1)}$. Koepf showed that

$$\left|a_3 - a_2^2\right| \le \begin{cases} \frac{1+2\beta}{3} & \text{for } 0 \le \beta \le 1 \\ \frac{\beta^2+\beta}{3} & \text{for } \beta > 1. \end{cases}$$

He also established this result [12] using different method.

We now extend the result of Keogh and Marks in Theorem 2.2 to the class $K(\beta)$ of strongly close-to-convex functions of order β. All the results of Koepf [11], with the exception of the case $\mu = 1$ and $\beta \ge 1$ are contained in the following Theorem.

Theorem 3.2. *Let $f(z) \in K(\beta)$ and be given by (1.1), then for $0 \le \beta \le 1$*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} 1 - \mu + \frac{\beta(2-3\mu)(\beta+2)}{3} & \text{if } \mu \le \frac{2\beta}{3(\beta+1)} \\ 1 - \mu + \frac{2\beta}{3} + \frac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]} & \text{if } \frac{2\beta}{3(\beta+1)} \le \mu \le \frac{2}{3} \\ \frac{2\beta+1}{3} & \text{if } \frac{2}{3} \le \mu \le \frac{2(\beta+2)}{3(\beta+1)} \\ \mu - 1 + \frac{\beta(3\mu-2)(\beta+2)}{3} & \text{if } \mu \ge \frac{2(\beta+2)}{3(\beta+1)}. \end{cases}$$

For each μ there is a function in $K(\beta)$ such that equality holds.

For the case when $\mu \le \dfrac{2\beta}{3(\beta+1)}$ *for example, this is the function*

$$f_0(z) = \frac{1}{2(\beta+1)}\left[\left(\frac{1+z}{1-z}\right)^{\beta} - 1\right].$$

Proof. Since $f(z) \in K(\beta)$, it follows that we can write

$$zf'(z) = g(z)h(z)^{\beta} \tag{3.2}$$

for $g(z) \in S^*$ and $h(z) \in P$. Equating coefficients in (3.2) we obtain $2a_2 = \beta c_1 + b_2$ and $3a_3 = \frac{\beta(\beta-1)}{2}c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3$ so that

(3.3)

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3}\left[\frac{\beta(\beta-1)}{2}c_1^2 + \beta c_2 + \beta c_1 b_2 + b_3\right] - \frac{\mu}{4}\left(\beta c_1 + b_2\right)^2 = \\ &= \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{\beta}{3}\left[c_2 + \left(\frac{\beta(2-3\mu)}{4} - \frac{1}{2}\right)c_1^2\right] + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)c_1 b_2. \end{aligned}$$

We consider at first the case $\dfrac{2\beta}{3(\beta+1)} \le \mu \le \dfrac{2}{3}$. Equation (3.3) gives

$$\left|a_3 - \mu a_2^2\right| \le \left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{\beta}{3}\left|c_2 - \frac{1}{2}c_1^2\right| + \beta^2\frac{(2-3\mu)}{12}|c_1|^2 + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)|c_1||b_2|.$$

Now from Theorems 2.1 and 2.4 and the fact that $|b_2| \le 2$ for $g(z) \in S^*$ we have

$$\left|a_3 - \mu a_2^2\right| \le 1 - \mu + \frac{\beta}{3}\left(2 - \frac{|c_1|^2}{2}\right) + \frac{\beta^2(2-3\mu)}{12}|c_1|^2 + \frac{\beta(2-3\mu)}{3}|c_1| = \Phi(x)$$

say with $x = |c_1|$. $\Phi(x)$ attains its maximum value at the point $x_0 = \dfrac{2(2-3\mu)}{2-\beta(2-3\mu)}$. Thus

$$\Phi(x_0) = 1-\mu+\frac{2\beta}{3}-\frac{2\beta(2-3\mu)^2}{3\left[2-\beta(2-3\mu)\right]^2}+\frac{\beta^2(2-3m)^2}{3\left[2-\beta(2-3\mu)\right]^2} = 1-\mu+\frac{2\beta}{3}+\frac{\beta(2-3\mu)^2}{3\left[2-\beta(2-3\mu)\right]}$$

and so

$$\left|a_3 - \mu a_2^2\right| \le \Phi(x_0)$$

which proves the Theorem if $\mu \le \dfrac{2}{3}$ and $\beta \ge 0$. Choosing $c_1 = \dfrac{2(2-3\mu)}{2-\beta(2-3\mu)}$, $c_2 = 2$, $b_2 = 2$ and $b_3 = 2$ in (3.3) we see that the result is sharp. Since $|c_1| \le 2$ (Theorem 2.4), it follows that $\mu \ge \dfrac{2\beta}{3(\beta+1)}$.

Next, consider the case $\mu \le \dfrac{2\beta}{3(\beta+1)}$. Since $K(0) = C$, we may assume that $\beta > 0$. Again (3.3) gives

$$\begin{aligned} \left|a_3 - \mu a_2^2\right| &\le \frac{3\mu(\beta+1)}{2\beta}\left|a_3 - \frac{2\beta}{3(\beta+1)}a_2^2\right| + \left[1 - \frac{3\mu(\beta+1)}{2\beta}\right]|a_3| \le \\ &\le \frac{3\mu(\beta+1)}{2\beta}\left(1 + \frac{2\beta}{3}\right) + \left[1 - \frac{3\mu(\beta+1)}{2\beta}\right]\left[\frac{2\beta(\beta+2)}{3} + 1\right] \\ &= 1 - \mu + \frac{\beta(2-3\mu)(\beta+2)}{3} \end{aligned}$$

for $\beta > 0$, where we have used the result already proved in the case $\mu = \dfrac{2\beta}{3(\beta+1)}$ and the fact in Theorem 2.3 that for $f(z) \in K(\beta)$, the inequality $|a_3| \le 1 + \dfrac{2\beta(\beta+2)}{3}$ holds. Equality is attained on choosing $c_1 = c_2 = b_2 = 2$ and $b_3 = 3$. The cases $\dfrac{2}{3} \le \mu \le \dfrac{2(\beta+2)}{3(\beta+1)}$ and $\mu \ge \dfrac{2(\beta+2)}{3(\beta+1)}$ are proven in a similar manner. As the calculations are rather long, they are not given here.

REFERENCES

[1] W. Kaplan. Close-to-convex schlicht functions. *Mich. Math. J.*, **1** (1952), 169–185.

[2] M. Fekete, G. Szegö. Eine Bemerkung über ungerade schlichte Funktionen. *J. London Math. Soc.*, **8** (1933), 85–89.

[3] A. Pfluger. The Fekete-Szegö inequality for complex parameters. *Complex Variables*, **7** (1986), 149–160.

[4] F. Keogh, E. Merkes. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.*, **20** (1969), 8–12.

[5] W. Haynman. *Multivalent functions.* Cambridge, 1958.

[6] W. Koepf. On the Fekete-Szegö problem for close-to-convex functions. *Proc. Amer. Math. Soc.*, **101** (1987), 89–95.

[7] D. Brannan, W. Kirwan. On some classes of bounded univalent functions. *J. London Math. Soc.*, **1** (1969), 431–443.

[8] D. Brannan, J. Clunie, W. Kirwan. Coefficient estimates for a class of starlike functions. *Canad. J. Math.*, **22** (1970), 476–485.

[9] J. Stankiewicz. Quelques problems extremaux dans les classes des fonctions α-angulairement étoilees. *Ann. Univ. Marie Curie-Sklodowska,* Sect. A, **20** (1966), 39–75.

[10] Ch. Pommerenke. *Univalent functions.* Vandenhoeck and Ruprecht Göttingen, 1977.

[11] W. Keopf. On the Fekete-Szegö problem for close-to-convex functions, *Arch. Math.*, **49** (1987), 420–433.

[12] W. Koepf. Close-to-convex functions, univalence criteria and quasi-conformal extension. *Ann. Univ. Mariae-Curie-Sklodowska*, Sect. A, **9** (1986), 97–102.

[13] J. Noonan. On close-to-convex functions of order β. *Pacific J. Math.*, **44** (1973), 263–280.

Donka Zheleva Pashkouleva
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Block 8
1113 Sofia, Bulgaria
e-mail: donka_zh_vasileva@abv.bg

БЕЛЕЖКА ВЪРХУ КЛАСОВЕТЕ ОТ СИЛНО ЗВЕЗДНИ И СИЛНО ПОЧТИ ИЗПЪКНАЛИ ФУНКЦИИ

Донка Пашкулева

Целта на тази статия е да получи резултати за проблема на Фекете-Сегьо за класовете, състоящи се от силно звездни и силно почти изпъкнали функции.