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DISCRETE GENERALIZATION OF BIHARI INEQUALITY WITH "MAXIMA"*

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This paper deals with nonlinear discrete inequalities of Bihari type. The main characteristic of the considered inequalities is the presence of the maximum of the unknown function over a past time interval. These inequalities are base of studying qualitative properties of the solutions of nonlinear difference equations with "maxima".

1. Introduction. In the recent years great attention has been paid to finite difference equations and their applications in modeling of real world problems ([1], [3], [4]). At the same time there are many real world processes in which the present state depends significantly on its maximal value on a past time interval. Adequate mathematical models of these processes are the so called difference equations with "maxima". Meanwhile, this type of equations is not widely studied yet and there are only some isolated results ([2], [6]). The development of the theory of difference equations with "maxima" requires solving of finite difference inequalities that involve the maximum value of the unknown function.

The main purpose of the paper is solving of a new type of nonlinear discrete inequalities with maxima. Some of the solved inequalities are applied to difference equations with "maxima" and bounds of their solutions are obtained.

2. Preliminary Notes and Definitions. Let $h \geq 0$ be a given fixed integer, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Z}$ be the set of all integers, $\mathbb{Z}_{[\alpha,\beta]} = \{\alpha, \alpha+1, \dots, \beta\}$, where $\alpha < \beta$. Denote by $\Delta u(n) = u(n+1) - u(n)$. We will assume that $\sum_{i=n}^{m} = 0$ and $\prod_{i=n}^{m} = 1$ for $m < n$.

We will introduce the following classes of functions.

Definition 1. *The function $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is said to be from the class Ω if:*

(i) ω *is a nondecreasing function and* $\omega(u) > 0$ *for* $u > 0$;

(ii) $t\omega(u) \leq \omega(tu)$ *for* $t \in [0, 1]$;

(iii) $\displaystyle\int^{\infty} \frac{du}{\omega(u)} = \infty$.

Definition 2. *The function $\psi \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ is said to be from the class Λ if*

(i) it is an increasing function, $\psi(0) = 0$ *and* $\lim_{u\to\infty} \psi(u) = \infty$;

(ii) $\psi(tu) \leq t\psi(u)$ *for* $t \in [0, 1]$.

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3. Main Results.

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions* $f_i,\, g_j : \mathbb{Z}_{[0,\,T]} \to \mathbb{R}_+$, $i \in \mathbb{Z}_{[1,\,l]}$, $j \in \mathbb{Z}_{[1,\,m]}$, $l,\, m < +\infty$.
2. *The function* $a : \mathbb{Z}_{[0,\,T]} \to [1,\, +\infty)$ *is nondecreasing.*
3. *The function* $\varphi : \mathbb{Z}_{[-h,\,0]} \to \mathbb{R}_+$ *with* $\max_{s \in \mathbb{Z}_{[-h,\,0]}} \varphi(s) \le a(0)$.
4. *The function* $\psi \in \Lambda$.
5. *The functions* $\omega_i,\, \tilde{\omega}_j \in \Omega$, $i \in \mathbb{Z}_{[1,\,l]}$, $j \in \mathbb{Z}_{[1,\,m]}$, $l,\, m < +\infty$.
6. *The function* $u : \mathbb{Z}_{[-h,\,T]} \to \mathbb{R}_+$ *satisfies the inequalities*

$$\begin{aligned}\psi\big(u(n)\big) \le a(n) + \sum_{s=0}^{n-1} u^p(s) \Big\{ \sum_{i=1}^{l} f_i(s)\omega_i\big(u(s)\big) + \\ + \sum_{j=1}^{m} g_j(s)\tilde{\omega}_j\Big(\max_{\xi \in \mathbb{Z}_{[s-h,\,s]}} u(\xi)\Big)\Big\} \quad \text{for } n \in \mathbb{Z}_{[0,\,T]},\end{aligned} \tag{1}$$

$$u(n) \le \varphi(n) \qquad \text{for } n \in \mathbb{Z}_{[-h,\,0]}, \tag{2}$$

where the constant $p \ge 0$.

Then for $n \in \mathbb{Z}_{[0,\,\beta_1]}$ *the inequality*

$$u(n) \le a(n) \left\{ \psi^{-1}\left\{ \Psi^{-1}\left[W^{-1}\Big(W\big(\Psi(1)\big) + A(n)\Big)\right]\right\}\right\} \tag{3}$$

holds, where Ψ^{-1} *and* W^{-1} *are the inverse functions of*

$$\Psi(r) = \int_{r_0}^{r} \frac{ds}{\big[\psi^{-1}(s)\big]^p}\,, \quad 0 < r_0 < 1, \quad r_0 \le r, \tag{4}$$

$$W(r) = \int_{r_1}^{r} \frac{ds}{\bar{w}\big[\psi^{-1}(\Psi^{-1}(s))\big]}\,, \quad 0 < r_1 < \Psi(1), \quad r_1 \le r, \tag{5}$$

$$\bar{w}(n) = \max\Big(\max_{1 \le i \le l} \omega_i(n),\, \max_{1 \le j \le m} \tilde{\omega}_j(n)\Big), \tag{6}$$

$$A(n) = \sum_{s=0}^{n-1} a^p(s) \Big\{ \sum_{i=1}^{l} f_i(s) + \sum_{j=1}^{m} g_j(s)\Big\}, \tag{7}$$

$$\begin{aligned}\beta_1 = \sup\Big\{ & n_1 \in \mathbb{Z}_{[0,\,T]} : \big(W\big(\Psi(1)\big) + A(n)\big) \in \mathrm{Dom}\,(W^{-1}), \\ & W^{-1}\big(W\big(\Psi(1)\big) + A(n)\big) \in \mathrm{Dom}\,(\Psi^{-1}), \\ & \Psi^{-1}\Big[W^{-1}\big(W\big(\Psi(1)\big) + A(n)\big)\Big] \in \mathrm{Dom}\,(\psi^{-1}) \text{ for all } n \in \mathbb{Z}_{[0,\,n_1]}\Big\}.\end{aligned}$$

Proof. From condition 5 of the theorem and $0 < \dfrac{1}{a(n)} \le 1$ it follows that inequalities

(1), (2) could be written in the following form

$$\tag{8} \begin{aligned} \psi\left(\frac{u(n)}{a(n)}\right) \le 1 + \sum_{s=0}^{n-1} u^p(s) \Bigg\{ \sum_{i=1}^{l} f_i(s)\omega_i\left(\frac{u(s)}{a(s)}\right) + \\ + \sum_{j=1}^{m} g_j(s)\tilde{\omega}_j\left(\frac{\max_{\xi\in\mathbb{Z}_{[s-h,\,s]}} u(\xi)}{a(s)}\right)\Bigg\} \quad \text{for } n \in \mathbb{Z}_{[0,\,T]}, \end{aligned}$$

$$\tag{9} u(n) \le \frac{\varphi(n)}{a(0)} \le 1 \qquad \text{for } n \in \mathbb{Z}_{[-h,\,0]}.$$

From the monotonicity of the function $a(n)$ it follows

$$\tag{10} \frac{\max_{\xi\in\mathbb{Z}_{[n-h,\,n]}} u(\xi)}{a(n)} = \frac{u(\eta)}{a(n)} \le \frac{u(\eta)}{a(\eta)} \le \max_{\xi\in\mathbb{Z}_{[n-h,\,n]}} \frac{u(\xi)}{a(\xi)} \quad \text{for } \eta \in \mathbb{Z}_{[n-h,\,n]},\ n \in \mathbb{Z}_{[0,\,T]}.$$

Define a function $v : \mathbb{Z}_{[-h,\,T]} \to \mathbb{R}_+$ by the equalities

$$\tag{11} v(n) = \begin{cases} \dfrac{u(n)}{a(n)} & \text{for } n \in \mathbb{Z}_{[0,\,T]}, \\ \dfrac{u(n)}{a(0)} & \text{for } n \in \mathbb{Z}_{[-h,\,0]}. \end{cases}$$

Use inequalities (8), (9), (10), the definition of $v(n)$ and get

$$\tag{12} \begin{aligned} \psi\big(v(n)\big) \le 1 + \sum_{s=0}^{n-1} a^p(s)v^p(s) \Bigg\{ \sum_{i=1}^{l} f_i(s)\omega_i\big(v(s)\big) + \\ + \sum_{j=1}^{m} g_j(s)\tilde{\omega}_j\left(\max_{\xi\in\mathbb{Z}_{[s-h\,s]}} v(\xi)\right)\Bigg\} \quad \text{for } n \in \mathbb{Z}_{[0,\,T]}, \end{aligned}$$

$$\tag{13} v(n) \le 1 \qquad \text{for } n \in \mathbb{Z}_{[-h,\,0]}.$$

According to Theorem 1 given in [5] from inequalities (12), (13) it follows

$$\tag{14} v(n) \le \psi^{-1}\left\{\Psi^{-1}\left[W^{-1}\Big(W\big(\Psi(1)\big) + A(n)\Big)\right]\right\} \quad \text{for } n \in \mathbb{Z}_{[0,\,\beta_1]},$$

where the function $A(n)$ is defined by equality (7).

From inequality (14) and the definition of $v(n)$ we get the required inequality (3).

□

Note that the inequalities (1), (2) could have another type of solution as follows, which is simpler than (3) but the used integral function is more complicated.

Theorem 2. *Let the conditions of Theorem* 1 *be satisfied. Then for* $n \in \mathbb{Z}_{[0,\,\beta_2]}$ *the following inequality is valid*

$$\tag{15} u(n) \le a(n)\left\{\psi^{-1}\left\{\Psi_1^{-1}\Big(\Psi_1(1) + A(n)\Big)\right\}\right\},$$

where the functions $\bar{w}(n)$, $A(n)$ *are defined by (6), (7), respectively, and* Ψ_1^{-1} *is the inverse function of*

$$\tag{16} \Psi_1(r) = \int_{r_2}^{r} \frac{ds}{\big[\psi^{-1}(s)\big]^p \bar{w}\big[\psi^{-1}(s)\big]}, \quad 0 < r_2 < 1, \quad r_2 \le r,$$

$$\beta_2 = \sup\Big\{n_2 \in \mathbb{Z}_{[0,\,T]} : \Big(\Psi_1(1) + A(n)\Big) \in \operatorname{Dom}\big(\Psi_1^{-1}\big),$$
$$\Psi_1^{-1}\Big(\Psi_1(1) + A(n)\Big) \in \operatorname{Dom}\big(\psi^{-1}\big) \ \ \textit{for all} \ \ n \in \mathbb{Z}_{[0,\,n_2]}\Big\}.$$

Proof. Following the proof of Theorem 1, we obtain inequalities (12), (13). According to Theorem 2 given in [5] for the function $v(n)$ we get

$$v(n) \le \psi^{-1}\Big\{\Psi_1^{-1}\Big(\Psi_1(1) + A(n)\Big)\Big\}, \tag{17}$$

where the function $A(n)$ is defined by equality (7).

From inequality (17) and the definition of $v(n)$ we get the required inequality (15).

□

In the special case $p = 0$ the following result is valid:

Corollary 1. *Let the conditions* 1–5 *of Theorem* 1 *be satisfied and the function* $u : \mathbb{Z}_{[-h,\,T]} \to \mathbb{R}_+$ *satisfy the inequalities*

$$\psi\big(u(n)\big) \le a(n) + \sum_{s=0}^{n-1}\Big\{\sum_{i=1}^{l} f_i(s)\omega_i\big(u(s)\big) + \sum_{j=1}^{m} g_j(s)\tilde{\omega}_j\Big(\max_{\xi\in\mathbb{Z}_{[s-h,\,s]}} u(\xi)\Big)\Big\} \ \textit{for}\ n \in \mathbb{Z}_{[0,\,T]}, \tag{18}$$

$$u(n) \le \varphi(n) \qquad \textit{for}\ n \in \mathbb{Z}_{[-h,\,0]}. \tag{19}$$

Then for $n \in \mathbb{Z}_{[0,\,\beta_3]}$ *the inequality*

$$u(n) \le a(n)\Big\{\psi^{-1}\Big\{W_1^{-1}\Big(W_1(1) + A_1(n)\Big)\Big\}\Big\} \tag{20}$$

holds, where $\bar{w}(n)$ *is defined by* (6) *and* W_1^{-1} *is the inverse function of*

$$W_1(r) = \int_{r_3}^{r} \frac{ds}{\bar{w}\big[\psi^{-1}(s)\big]}, \quad 0 < r_3 < 1, \quad r_3 \le r, \tag{21}$$

$$A_1(n) = \sum_{s=0}^{n-1}\Big\{\sum_{i=1}^{l} f_i(s) + \sum_{j=1}^{m} g_j(s)\Big\}, \tag{22}$$

$$\beta_3 = \sup\Big\{n_3 \in \mathbb{Z}_{[0,\,T]} : W_1(1) + A_1(n) \in \operatorname{Dom}\big(W_1^{-1}\big),$$
$$W_1^{-1}\Big(W_1(1) + A_1(n)\Big) \in \operatorname{Dom}\big(\psi^{-1}\big) \ \ \textit{for all} \ \ n \in \mathbb{Z}_{[0,\,n_3]}\Big\}.$$

4. Applications.

Example. Consider the following difference equation with "maxima"

$$\Delta u^b(n) = F\Big(n,\, u(n),\, \max_{\xi\in\mathbb{Z}_{[n-h,\,n]}} u(\xi)\Big) + v(n) \quad \text{for } n \in \mathbb{Z}_{[0,\,T]}, \tag{23}$$

with the initial condition

$$u(n) = \varphi(n) \qquad \text{for } n \in \mathbb{Z}_{[-h,\,0]}, \tag{24}$$

where the functions $u : \mathbb{Z}_{[-h,\,T]} \to \mathbb{R},\ \varphi : \mathbb{Z}_{[-h,\,0]} \to \mathbb{R},\ F : \mathbb{Z}_{[0,\,T]} \times \mathbb{R}^2 \to \mathbb{R},\ v : \mathbb{Z}_{[0,\,T]} \to \mathbb{R}$ is nondecreasing, $\Delta u^b(n) = u^b(n+1) - u^b(n)$, the constant $b \in \mathbb{Z}_1$ and $h \in \mathbb{Z}_0$ is a given fixed integer.

Theorem 3. *Let the following conditions be fulfilled:*

1. *The functions* $F : \mathbb{Z}_{[0,\,T]} \times \mathbb{R}^2 \to \mathbb{R},\ R,\, Q : \mathbb{Z}_{[0,\,T]} \to \mathbb{R}_+$ *satisfy*

$$\big|F(n,\, x,\, y)\big| \le |x|^q \Big[R(n)|x|^t + Q(n)|y|^t\Big] \quad \textit{for } n \in \mathbb{Z}_{[0,\,T]},\ x,\, y \in \mathbb{R},$$

where the constants q, t *are such that* $0 \le q < b,\ 0 \le t < b$ *and* $b - q - t > 0$.

2. *The function* $\varphi : \mathbb{Z}_{[-h,\,0]} \to \mathbb{R}$ *and*

$$M = \begin{cases} 1 & \textit{for} \quad \max\limits_{n \in \mathbb{Z}_{[-h,\,0]}} \big|\varphi(n)\big| \le 1, \\ \max\limits_{n \in \mathbb{Z}_{[-h,\,0]}} \big|\varphi(n)\big| & \textit{for} \quad \max\limits_{n \in \mathbb{Z}_{[-h,\,0]}} \big|\varphi(n)\big| > 1. \end{cases}$$

3. *The function* $v : \mathbb{Z}_{[0,\,T]} \to \mathbb{R}$ *is nondecreasing.*
4. *The IVP* (23), (24) *has at least one solution, defined for* $n \in \mathbb{Z}_{[-h,\,T]}$.

Then for $n \in \mathbb{Z}_{[0,\,T]}$ *the solution of IVP* (23), (24) *satisfies the inequality*

$$|u(n)| \le V(n) \sqrt[b-q-t]{1 + \frac{b-q-t}{b} \sum_{s=0}^{n-1} V^q(s)\Big[R(s) + Q(s)\Big]}, \tag{25}$$

where

$$V(n) = M^b + \sum_{s=0}^{n-1} |v(s)| \qquad \textit{for } n \in \mathbb{Z}_{[0,\,T]}. \tag{26}$$

Proof. From condition 1 of the theorem for the norm of the solution $u(n)$ of IVP (23), (24) it follows

$$\begin{aligned} |u(n)|^b &\le M^b + \sum_{s=0}^{n-1} |v(s)| + \sum_{s=0}^{n-1} |u(s)|^q \Big[R(s)|u(s)|^t + Q(s)\big|\max_{\xi \in \mathbb{Z}_{[s-h,\,s]}} u(\xi)\big|^t\Big] \\ &= V(n) + \sum_{s=0}^{n-1} |u(s)|^q \Big[R(s)|u(s)|^t + Q(s)\big|\max_{\xi \in \mathbb{Z}_{[s-h,\,s]}} u(\xi)\big|^t\Big] \quad \text{for } n \in \mathbb{Z}_{[0,\,T]}, \end{aligned} \tag{27}$$

$$|u(n)| \le M \qquad \text{for } n \in \mathbb{Z}_{[-h,\,0]}, \tag{28}$$

where the nondecreasing function $V(n)$ is defined by equality (26).

Set $|u(n)| = U(n)$ for $n \in \mathbb{Z}_{[-h,\,T]}$. According to Theorem 2 for $u(n) \equiv U(n)$, $a(n) \equiv V(n)$, $\psi(U) \equiv U^b$, $p \equiv q$, $l \equiv m = 1$, $f(n) \equiv R(n)$, $g(n) \equiv Q(n)$ for $n \in \mathbb{Z}_{[0,\,T]}$ and $\omega(U) \equiv \tilde{\omega}(U) \equiv U^t$, $\Psi_1(r) = \displaystyle\int_0^r \frac{ds}{s^{\frac{q+t}{b}}} = \frac{b\, r^{\frac{b-q-t}{b}}}{b-q-t}$, $\Psi_1^{-1} = \Big[\Big(\dfrac{b-q-t}{b}\Big) r\Big]^{\frac{b}{b-q-t}}$, $\mathrm{Dom}\Psi_1^{-1} = \mathbb{R}_+$ we obtain for $n \in \mathbb{Z}_{[0,\,T]}$

$$U(n) \le V(n) \sqrt[b-q-t]{1 + \frac{b-q-t}{b} \sum_{s=0}^{n-1} V^q(s)\Big[R(s) + Q(s)\Big]}. \tag{29}$$

From inequality (29) and the definition of the function $U(n)$ we obtain the required inequality (25).

$\square$

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ДИСКРЕТНО ОБОБЩЕНИЕ С МАКСИМУМИ НА НЕРАВЕНСТВОТО НА БИХАРИ

Кремена Василева Стефанова, Лиляна Васкова Ванкова

В тази статия са решени някои нелинейни дискретни неравенства от типа на Бихари. Основната характеристика на тези неравенства е присъствието на максимума на неизвестната функция върху отминал интервал от време. Някои от получените резултати са приложени върху конкретни примери за изследване на качествените свойства на решенията на диференчните уравнения с максимуми.