МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2013
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2013
Proceedings of the Forty Second Spring Conference
of the Union of Bulgarian Mathematicians
Borovetz, April 2–6, 2013

ABOUT METHODS OF CONSTRUCTING PLANE SECTIONS OF SPACIAL FIGURES*

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A universal method of constructing plane sections of prisms and pyramids is presented. This method is compared to the method of constructing plane sections of prisms and pyramids by using the traces of the plane in the projection plane. Two examples are given using axonometric projection.

1. Introduction. Different problems of constructing plane sections of pyramids and prisms date back to ancient times. The general formulation of these problems is the following:

Problem 1.1. *Let F be a spacial figure (solid) and α be a plane given by three non-collinear points L, M and N. Find the contour of the intersection of F and α.*

In the problem above it is assumed that the figure F and the plane α are depicted on a drawing plane π which can be regarded as a leaf of a book, a handbook, a desktop or a drawing board. This problem is a particular case of the positional problem of finding the mutual points of two or more different figures. For successfully solving it one needs to depict correctly the solids on a plane. Drawing spacial figures and solving positional problems is the subject of Descriptive Geometry. As a rule the figure F is drawn by its projections onto one, two or more orthogonal planes. For the sake of clarity only one projecting plane will be used in this paper. This case is sufficient for finding plane sections of solids. Only the parallel projection will be considered, as the central projection is analogous. Some problems are solved using axonometric projection.

2. Depicting spacial figures. Let π be the projection (drawing) plane. A line l non-parallel to π defines the projection direction onto π. Every line parallel to l is called a *projection line* (ray). Through an arbitrary point M there passes a projection ray l_M intersecting the plane π in the point M'. The point M' is called the projection of the point M onto π along the projection ray l_M (see Fig. 1).

Every plane α parallel to l is called projecting plane. The projection of α in π is a line α' which coincides with the intersection line of α and π. This line is also called the trace of α in π (see Fig. 1).

The projection F' of a figure F consists of the projections of all points of F (see Fig. 2). That way the depiction of a figure F is the pair $F(F')$, where F is the image of the given figure and F' is the image of its projection onto π.

The following notion is important for solving positional problems:

*Key words: solid, plane section, pyramid, axonometric projection.

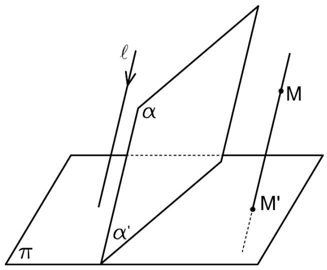

Fig. 1

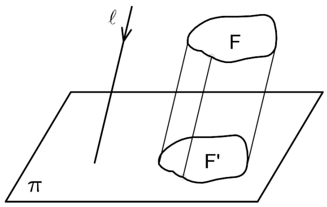

Fig. 2

Definition 2.1. *The depiction of a figure $F(F')$ is complete if for every point M from F its projection M' onto the plane π is either given, or can be determined uniquely.*

Proposition 2.2. *A point A is depicted completely if and only if its projection A' is depicted.*

Proposition 2.3. *A line d is depicted completely if and only if the projections A' and B' of two points A and B lying on d are depicted. That way the line d' passing through A' and B' is the projection of d onto π.*

Proof. The case when d is a projecting ray is obvious. Let now d and l be non-parallel, where l is the projection direction. Then d lies on a plane α containing the points A, B and parallel to l. The line d' passing through A' and B' is the intersecting line of α and π and the projection of d onto π. Let M be an arbitrary point from d. A line l_M parallel to l is passing through M and its intersection with d' is the uniquely determined point M' (see Fig. 3). The point M' is the projection of the point M. $\square$

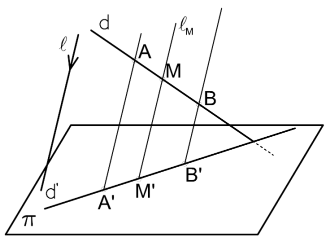

Fig. 3

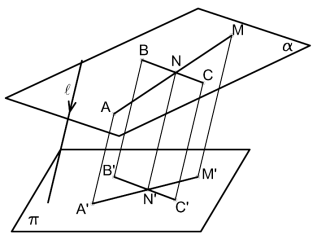

Fig. 4

Proposition 2.4. *A plane α is depicted completely if and only if the projections A', B' and C' of three non-collinear points A, B and C lying on α are depicted.*

Proof. By virtue of Proposition 2.3 the lines $m = (AB),\ n = (BC)$ and $p = (AC)$ passing through the points $A, B;\ B, C$, and A, C respectively are depicted completely. The case when α is parallel to l is obvious.

Let M be a point from α and the line (AM) intersects the line n in the point N. Then the projection N' of N is uniquely determined and is on the line $B'C'$ (see Fig. 4). Then the line $(A'N')$ intersects the projection ray l_M in the point M'. Thus the point M' is the projection of M onto π and is uniquely determined. The cases when the lines (BM) and p or (CM) and m are not parallel are analogous. At least one of these three cases holds. □

3. Elementary positional problems. The following three problems are the main instruments for constructing plane sections.

Problem 3.1. *Assume that the line d passes through the points A and B and their projections onto π are A' and B' respectively. Depict the trace of d on π.*

Solution. The trace D of the line d on π is the intersection point of the lines $d = (AB)$ and $d' = (A'B')$. If the lines $d = (AB)$ and $d' = (A'B')$ are parallel then the trace D is the infinite point. □

Problem 3.2. *Assume that the plane α passes through the non-collinear points A, B and C and their projections onto π are A', B' and C' respectively. Depict the trace ρ of α on π.*

Solution. Let L be the trace of the line (AB) on π and M be the trace of the line (AC) on π (By virtue of the solution of Problem 3.1 L and M can be depicted). Then $\rho = (LM)$ is the trace of α on π (see Fig. 5). If the points L and M are out of our drawing board then auxiliary points $D, E, \ldots$ from α and their projections $D', E', \ldots$ are needed. Then the traces of the lines $(AD), (AE), (BD), (BE), \ldots$ should be found. □

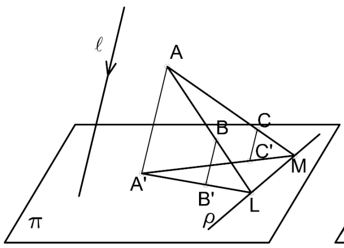

Fig. 5

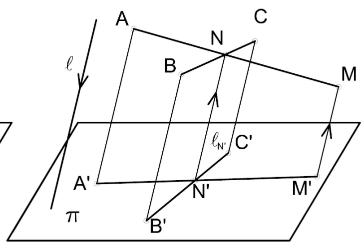

Fig. 6

Problem 3.3. *Assume that the plane α passes through the non-collinear points A, B and C and their projections onto π are A', B' and C' respectively and the point M' is from the projection plane π. Depict the point M on α and on the projecting ray $l_{M'}$.*

Solution. One of the pair of lines $((A'M')$ and $(B'C')), ((B'M')$ and $(A'C'))$ and $((C'M')$ and $(A'B'))$ intersect on a point in the boarders of our drawing board. Assume the lines $(A'M')$ and $(B'C')$ meet on the point N' (see Fig. 6). Now the projecting ray $l_{N'}$ passing through N' can be drawn. The line $l_{N'}$ intersects the line (BC) in the point N. The lines (AN) and $l_{M'}$ lay in one plane and meet in the point M. □

Problem 3.4. *Assume the plane β passes through the non-collinear points A, B and C with projections onto π A', B' and C' respectively and the line d passes through the points D and E with projections D' and E' in π. Depict the point M which is the intersection of β and d. If $\beta \nparallel l \nparallel d$ draw the projection d'' of d in β along the direction l.*

Solution. The following three cases will be considered:

Case 1. $\beta \parallel l$ and $d \parallel l$. Then the points A', B' and C' are on the intersection line b of β and π and $D' = E'$. If D' lies on b then the line d lies on β. If D' does not lie on b then the line d is parallel to β and $d \cap \beta = \emptyset$.

Case 2. $\beta \parallel l$ and $d \nparallel l$. Then the $d' = (D'E')$ is the projection of d. Now if $d' \parallel b$ then $d \parallel \beta$. If $d' \nparallel b$ denote by M' the intersection point of d' and b. Draw the line l_M

parallel to l and passing through M'. Now the point M is the intersection of l_M and d (see Fig. 7).

Case 3. $\beta \nparallel l$ and $d \nparallel l$. Then the method in the solution of Problem 3.3 can be applied. Find the point D'' which is the intersection of the ray $l_D = (D'D)$ and β and the point E'' which is the intersection of the ray $l_E = (E'E)$ and β. The line $d'' = (D''E'')$ is the projection of d on β. Thus the intersection point M of d and d'' is the intersection of β and d (see Fig. 8). $\square$

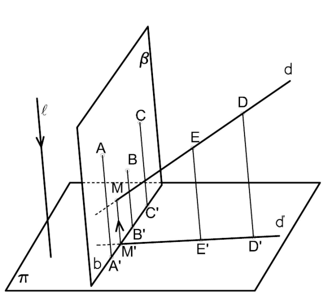

Fig. 7

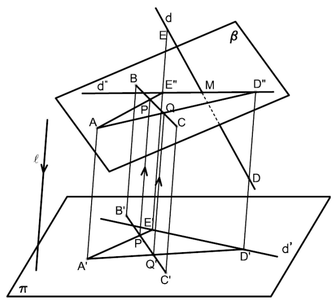

Fig. 8

4. Methods of constructing plane sections of prisms. The methods proposed in Problems 3.2 and 3.3 are important for finding plane sections of a prism. The first one enables us to construct the trace of the section plane and then find the contour of the plane section (see [2]). This method is well known and widely used by students and teachers. Unfortunately it is not possible to use it in every case as the traces cannot be always constructed. The method developed in Problem 3.3 is the bases of the method of "the projecting rays". The latter method is always applicable. The disadvantage of it is the accumulation of great number of projecting rays on the drawing board.

Problem 4.1. *Let $ABCDEA'B'C'D'E'$ be a given prism and M, N, L be given points on the lateral faces of the prism with projections M', N', L' on the plane of the bases*

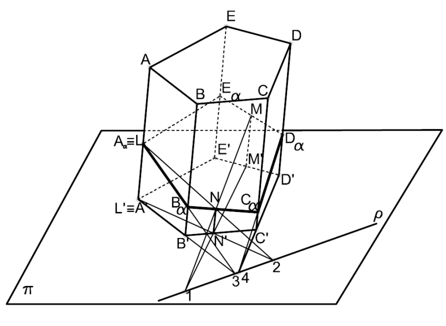

Fig. 9

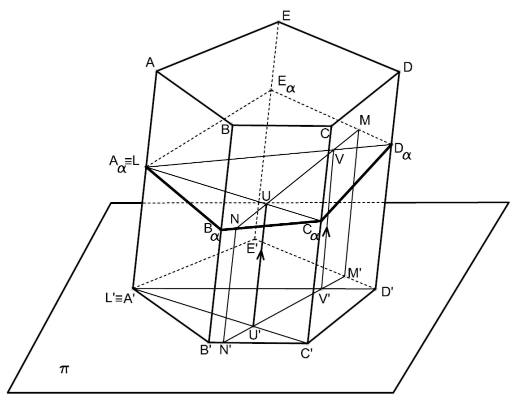

Fig. 10

$A'B'C'D'E'$. *Draw the section of the prism with the plane* β *passing through the points* M, N, L.

Solution. The plane of the bases $A'B'C'D'E'$ will be considered as the projection plane π and the projection direction will be the lateral edge (AA').

Case 1. The trace of β is in the boarders of our board (see Fig. 9).

Case 2. The trace of β is out of the boarders of our drawing board (see Fig. 10). It is sufficient to find the intersection points C_α and D_α of β and the lines $(C'C)$ and $(D'D)$ respectively.

The following construction can be applied:

1. Find $U' = (L'C') \cap (N'M')$ and $V' = (L'D') \cap (N'M')$;

2. Draw the projecting rays l_U and l_V passing through U' and V' respectively and parallel to the direction $(A'A)$ and find their intersections with the line (MN) – the points U and V respectively;

3. Find $C_\alpha = (LU) \cap (C'C)$ and $D_\alpha = (LV) \cap (D'D)$;

4. Find $E_\alpha = (D_\alpha M) \cap (E'E)$ and $B_\alpha = (C_\alpha N) \cap (B'B)$;

5. The wanted plane section is $A_\alpha B_\alpha C_\alpha D_\alpha E_\alpha$.

Example 4.2. In cavalier projection (see [1, 3]) draw a prism with the first base the triangle $A(7, 1.5, 0)$, $B(0, 2, 0)$ and $C(4, 5, 0)$ if the point $A_0(5, 0, 8)$ is a vertex from the other base. Draw the plane section of the prism with α passing through $M(9, 4, 7)$, $N(2, 7, 5)$, $P(0, 0, 5)$ (see Fig. 11).

For solving the problem above, first the projections M'_l, N'_l and P'_l of M, N and P respectively, onto μ along the projection direction $l = (AA_0)$, are found. Then the method described in Problem 4.1 is applied.

The previous example and the example from the next section can be used in classes of Descriptive Geometry for students of Architecture or Civil Engineering specialities.

5. Constructing plane sections of pyramids. For finding plane sections of pyramids a central projection is used. In that case the projecting rays and planes are passing through the apex S of the pyramid. Analogous methods and propositions of complete depiction are valid. Let $SA_1A_2A_3\ldots A_n$ be a given pyramid then the apex S is the center of the projection and the projection plane is the plane of the bases. As in the case of prism, problems of finding plane sections can be solved in an analogous way (see Fig. 12).

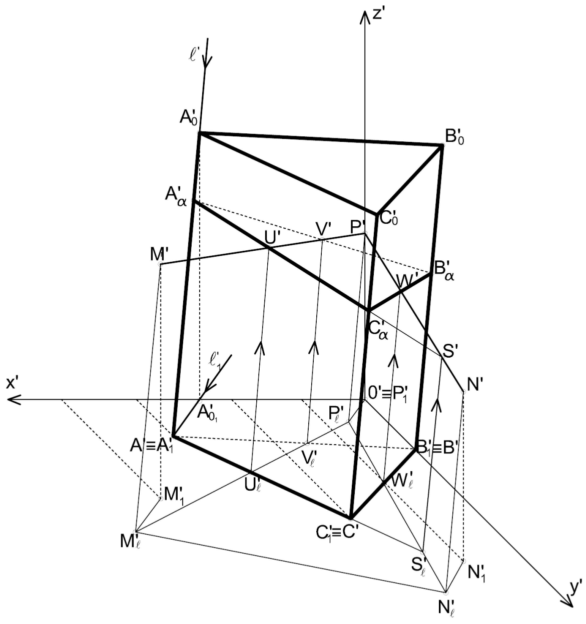

Fig. 11

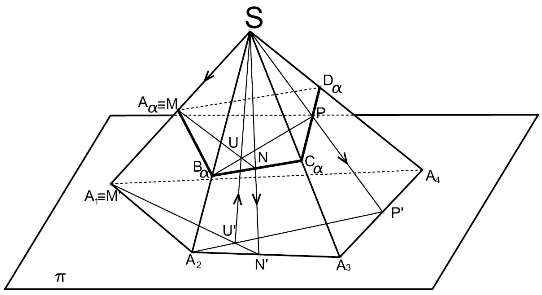

Fig. 12

Example 5.1. In augmented orthogonal isometric projection (see [1]) draw pyramid with apex $V(4, 5, 10)$ and bases $A(7, 2, 0)$, $B(7, 7, 0)$, $C(2.5, 8, 0)$ and $D(0.5, 3, 0)$, and its plane section with α passing through the midpoint P of VD, the centroid G of the triangle ABV and the midpoint Q of the median to the side BC of the triangle BCV (see Fig. 13).

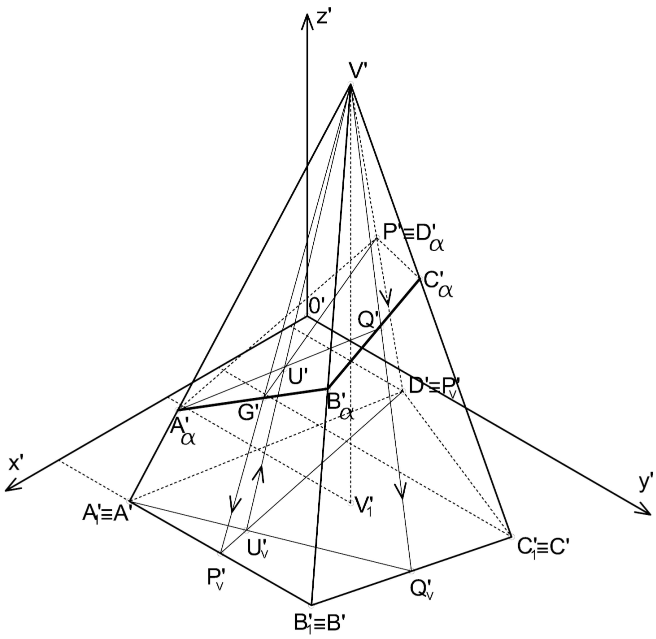

Fig. 13

In the solution above, the developed method and a central projection with center V are used.

6. Conclusions. The present paper is appropriate for teachers as well as for students. The problems of finding plane sections are usually regarded as difficult ones and the role of elementary positional problems is not considered. The ability of drawing plane sections makes stereometric problems easier and helps understanding the logic of geometrical facts. For this reason the methods of finding plane sections of solids and their understanding is interesting, motivating and helps in acquiring mathematical notions.

Acknowledgements. We would like to express our gratitude to Zlatina Tsolova who made the Figures on AUTOCAD!

REFERENCES

[1] N. Danailova, L. Velinova, M. Kolev. Handbook on Axonometry. UACG, ISBN -978-954-724-035-4 (in Bulgarian).

[2] V. Radulov, I. Parvanova. Handbook on descriptive geometry – Plane sections. UACG, ISBN -978-954-745-220-6 (in Bulgarian).

[3] V. Milousheva, J. Davidov. Descriptive Geometry (Lecture Notes). Higher School of Civil Engineering "Lyuben Karavelov", Sofia, 2011.

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ОТНОСНО МЕТОД ЗА ИЗОБРАЗЯВАНЕ НА РАВНИННИ СЕЧЕНИЯ НА ПРИЗМИ И ПИРАМИДИ

Митрофан Чобан, Екатерина Михайлова, Ирена Първанова

Представен е универсален метод за намиране на равнинни сечения на призми и пирамиди. Методът е приложен за решаването на две задачи, при които е използвана аксонометрия. Този метод е сравнен с широко използвания метод на следите, който не винаги е приложим заради ограничеността на „чертожната дъска“.