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SOME WAYS TO CREATE PROBLEMS FOR STUDENT OLYMPIADS

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We give some ways to construct problems for Olympiads for university students. The material is organized into 3 sections, tracing different aspects of composing such problems: 1. Using mistakes, 2. Using known problems and 3. Passing from "elementary mathematics" to "higher" one and asking "What happens if?".

We first mention that unfortunately Olympiads for secondary schools are much more popular than student ones. Probably this is a result of a political decision and it is not so bad. What is not so good is that increasing the number and different forms of competitions leads to some kind of populism. Anyway, we share here some ways to create Olympiad problems for University Students.

1. Using mistakes. Most teachers have some experience with exotic student mistakes. Below we describe some of them.

Mistake 1 (*incorrect division*, problem offered at IMC'2012 [3])**.** Sometimes one can find the following "equality" in exam tests

$$\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}. \tag{1}$$

Obviously this does not hold in general. But relation (1) may be used to generate real mathematical problems.

Problem 1.1 a) (*P. Stoev*, *Vl. Todorov*)**.** Given a number $a\neq 0$ find all numbers b for which (1) holds.

For this purpose solve the equation $\dfrac{1}{a+x}=\dfrac{1}{a}+\dfrac{1}{x}$, or $x^2+ax+a^2=0$, which gives $x_{1,2}=\dfrac{-1\pm i\sqrt{3}}{2}a$. Note that a may not be real.

b) (*Vl. Todorov*). Suppose that (1) holds. Prove that for every integer n the identity $\dfrac{1}{a^{2^n}+b^{2^n}}=\dfrac{1}{a^{2^n}}+\dfrac{1}{b^{2^n}}$ is valid.

These problems are not difficult. For example, the answer of b) can be obtained from the identity $\dfrac{1}{z}+z=-1$ whenever $z^2+z+1=0$. Then $\dfrac{1}{z^2}+z^2=-1$ and so on. This should be true for every field of characteristic different from 2^n for any $n\in\mathbb{N}$. This condition holds for arbitrary nonzero $a,b\in\mathbb{C}$.

Problem 1.2 (*P. Stoev, Vl. Todorov*)**.** Is it possible to find a field $\mathbb{F}$ for which (1) is an identity?

If we replace a and b by 1 we get $1 = 4$; $3 = 0$ in $\mathbb{F}$. So, $\mathbb{F}$ must be of characteristic ≤ 3. It is easy to see that if $\mathbb{F} = \mathbb{Z}_3$, then (1) is an identity.

The relation (1) holds also for $\mathbb{F} = \mathbb{Z}_2$ which is surprising to some extent. It is true because in $\mathbb{Z}_2$ the identity (1) does not make sense. Hence it remains true in this case since the empty set admits arbitrary properties. This is a reason to consider the factor space $\mathbb{F} = \mathbb{Z}_2[x]/(x^2 + x + 1)$. It is easy to verify that $\mathbb{F}$ is a field having four elements. Moreover, (1) holds for every two admissible elements of $\mathbb{F}$. This problem was prepositional for IMC'2012 but it was rejected as "too hard", and we could not discuss it in more details.

Relation (1) can generate even more nontrivial problems. Consider two of them.

Problem 1.3 (*P. Stoev, M. Konstantinov, Vl. Todorov*)**.** For which $n \times n$ real or complex matrices A and B does the equality

$$(A + B)^{-1} = A^{-1} + B^{-1}$$

hold?

Setting $X = A^{-1}B$ we obtain the matrix equation

$$X^2 + X + I = 0, \tag{2}$$

where I is the identity matrix. The general solution Σ_n of (2) may be very involved. The set Σ_1 has two elements $x_{1,2} = (-1 \pm i\sqrt{3})/2$. The set Σ_2 contains four isolated diagonal solutions, two 1-parametric families of solutions and one 2-parametric family of solutions.

Problem 1.4 (*P. Stoev, M. Konstantinov, Vl. Todorov*)**.** The matrices A and B are given such that $AB = BA$ and $(A + B)^{-1} = A^{-1} + B^{-1}$. Prove that $(A^{2^n} + B^{2^n})^{-1} = A^{-2^n} + B^{-2^n}$.

Mistake 2 (*incorrect differentiation*)**.** One can see the fake "equality" $(f(x)g(x))' = f'(x)g'(x)$ in some student papers. However, it may help to generate infinitely many real mathematical problems.

Problem 1.5 (*P. Stoev, Vl. Todorov*)**.** Let $f(x) = x^n$. Find all functions g for which $(f(x)g(x))' = f'(x)g'(x)$ holds (this problem may be appropriate for student Olympiads).

The solution is not too hard, just replace $f(x)$ by x^n to obtain

$$nx^{n-1}g(x) + x^n g'(x) = nx^{n-1}g'(x);\ ng(x) + xg'(x) = ng'(x);\ \frac{g'(x)}{g(x)} = \frac{n}{n-x}.$$

Thus $(\ln g(x))' = \dfrac{n}{n-x}$ and we have

$$g(x) = C \exp\left(\frac{n}{n-x}\right),$$

where C is a constant. We may also choose $f(x)$ to be different from x^n. We leave to the reader to find g if $f(x) = \sin x$.

Similar problems may be inspired by other widespread mistakes like

$$\int f(x)g(x)dx = \int f(x)dx \int g(x)dx \tag{3}$$

or

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)}{g'(x)}.$$

For example, if $f(x) = \sin x$ then (3) holds when

$$g(x) = C\exp\left(\frac{x}{2} + \frac{1}{4}\sin 2x\right).$$

Thus it is possible to construct a lot of problems using such mistakes.

Mistake 3 (*improper use of the* (ε,δ)*–language*)**.** Many students have problems with the (ε,δ)–definitions of continuity for a function $f : T \to \mathbb{R}$, where $T \subset \mathbb{R}$ contains at least two points (for students we assume that T is an interval). When using this language they often make mistakes and, instead of the desired function property, define something else. A typical error is to forget one or both of the inequalities $\delta > 0$, $\varepsilon > 0$.

Problem 1.6 (*M. Konstantinov, At. Hamamdjiev*)**.** Let S_1 be the set of functions $f : T \to \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \le \delta$ implies $|f(x) - f(x_0)| \le \varepsilon$ (the inequality $\delta > 0$ is missing). Prove that S_1 is the set of all functions.

Hint. Taking $\delta = -1$ we see that the inequality $|x - x_0| \le -1$ is never fulfilled and implies anything.

Problem 1.7 (*M. Konstantinov, At. Hamamdjiev*)**.** Consider the set S_2 of functions $f : T \to \mathbb{R}$ such that for any ε there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \le \delta$ implies $|f(x) - f(x_0)| \le \varepsilon$ (the inequality $\varepsilon > 0$ is missing). Prove that S_2 is the empty set.

Hint. Take $\varepsilon = -1$ and choose the corresponding $\delta = \delta(-1)$. Then for $x = x_0$ we have $0 = |x - x| \le \delta$ which implies $0 = |f(x) - f(x)| \le -1$ which is a contradiction.

Consider now the following definition. The function $f : T \to \mathbb{R}$ is uniformly continuous if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|x - x_0| \le \delta$ implies $f(x) - f(x_0) \le \varepsilon$ (the modulus signs are missing as it happens in student papers). This modification seems fake but it is correct (interchange the arguments x and x_0). This definition is more "economical" and should be recognized as preferable. A student attempt to define uniform continuity leads to a nontrivial result as follows.

Problem 1.8 (*M. Konstantinov*)**.** Let S_3 be the set of functions $f : T \to \mathbb{R}$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, x_0 \in T$ the inequality $|f(x) - f(x_0)| \le \delta$ implies $|x - x_0| \le \varepsilon$ (x and x_0 are replaced by $f(x)$ and $f(x_0)$, respectively). Prove that S_3 is the set of functions having uniformly continuous inverse.

Problems of type 1.6–1.8 as well as other similar problems had been used by the Russian Mathematical School of *A. Myshkis*.

2. Using known problems. Ancient Egyptians used to write rational numbers as sums of aliquot fractions of type $\frac{1}{k}$ with different divisors (probably for philosophical or religion reasons). For example, $\frac{2}{7} = \frac{1}{7} + \frac{1}{8} + \frac{1}{56}$ follows from the equality $\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}$. Replacing here numbers by matrices, we obtain interesting problems.

Problem 2.1 (*P. Stoev and Vl. Todorov*, SEEMOUS'2010 [2])**.** Let A and B be $n \times n$ matrices with integer entries (integer n-matrices) such that that B^{-1} exists and

$A \neq 0$. Prove that AB^{-1} can be represented as

$$AB^{-1} = \sum_{k=1}^{m} N_k^{-1},$$

where N_k is an integer matrix for every k and $N_i \neq N_j$ for $i \neq j$.

Let us provide a solution of this problem (as usual it may have many solutions). We note that if A and $A+I$ are invertible then $A^{-1} = (A+I)^{-1} + (A(A+I))^{-1}$, where I is the identity matrix. Note that this observation does not lead immediately to similar considerations as in the Egypt case.

Problem 2.2 (*Vl. Todorov*)**.** Every power series $\sum\limits_{n=0}^{\infty} a_n(x-a)^n$ has a radius of convergence. What shall happen if we replace $(x-a)^n$ with something else, e.g. with $(x-b_n)^n$, where $\{b_n\}$ is a given sequence?

It is not difficult to prove that if $\{b_n\}$ is an arithmetic progression, then the series $\displaystyle\sum_{n=0}^{\infty} a_n(x-b_n)^n$ does not converge for any x provided $a_n \geq \dfrac{1}{n!}$. What if a_n's are arbitrary?

On the other hand, it is not very hard to realize that the set $\mathbb{Q}$ of rational numbers can be rearranged as $\mathbb{Q} = \{q_0, q_1, \dots\}$ so that the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x-q_n)^n}{n!}$ is convergent for any x. As above, we may ask (for $\{a_n\}$ being a given sequence) whether it is possible to rearrange the set $\mathbb{Q}$ so that the series $\displaystyle\sum_{n=0}^{\infty} a_n(x-q_n)^n$ is convergent *for all* x? The answer seems to be "yes".

Problem 2.3 (*Vl. Todorov*)**.** Now we are going to replace the summands in given power series by something else. For example, it seems reasonable to replace the numbers with matrices. But we are going to replace the powers with the cross product of vectors in $\mathbb{R}^3$. The obstacle here is that the cross product of the vector with itself gives the zero vector. That is why we define the power of vector variable $\vec{r}$ under some basic fixed non-zero vector $\vec{a} \in \mathbb{R}^3$. For an arbitrary vector $\vec{r} \in \mathbb{R}^3$ we set $\vec{a} \times \vec{r}^{[0]} = \vec{a}$. Next, for every integer $n \in \mathbb{N}$ let us define

$$\vec{r}_{\vec{a}}^{\,n} = \left(\vec{a} \times \vec{r}^{(n-1)}\right) \times \vec{r},$$

where $\times$ denotes the cross product.

Note that in this way we may obtain a lot of problems. For example, one may consider the analogues of an arbitrary analytic function by replacing x^n by "the powers of vectors". Thus, one can consider functions of vector arguments like

$$\vec{f}_{\vec{a}}(\vec{r}) = \exp_{\vec{a}}(\vec{r}) = \sum_{n=0}^{\infty} \frac{\vec{r}_{\vec{a}}^{\,n}}{n!} = \cos|\vec{r}|\,\vec{a} + \left(\frac{\vec{a}.\vec{r}}{\vec{r}^2} - \frac{\vec{a}.\vec{r}}{\vec{r}^2}\cos|\vec{r}|\right)\vec{r} + \frac{\sin|\vec{r}|}{|\vec{r}|}\,\vec{a} \times \vec{r}$$

or

$$\vec{g}_{\vec{a}}(\vec{r}) = \sin_{\vec{a}}(\vec{r}) = \sum_{n=0}^{\infty} (-1)^n \frac{\vec{r}_{\vec{a}}^{\,(2n+1)}}{(2n+1)!} = \operatorname{sh}|\vec{r}|(\vec{a} \times \vec{r})$$

and so on. We can also ask what the sets $\vec{f}_{\vec{a}}(\mathbb{R}^3)$ and $\vec{g}_{\vec{a}}(\mathbb{R}^3)$ are (as well as other images of subsets of $\mathbb{R}^3$).

3. Passing from "Elementary" to "Higher Mathematics" and asking "What happens if?" In this section we use school problems to obtain Olympiad problems for students (as a rule, the problems obtained in this way are not very hard).

Problem 3.1 (*Vl. Todorov*)**.** Let $a = BC$, $b = CA$ and $c = AB$ be the sides of the triangle ABC with area S and angles α, β, γ at the vertices A, B, C.

a) Prove that if $\alpha < \pi/2$ then

$$\frac{4S}{b^2+c^2-a^2}\left(1-\frac{4S}{b^2+c^2-a^2}\right) < \alpha < \frac{4S}{b^2+c^2-a^2}.$$

This follows easily from the inequalities $x - x^2/4 \le \arctan x \le x$.

b) Prove that for $\gamma \le \pi/2$ we have the inequalities

$$\sqrt{2}\sin\frac{\gamma}{2} \leqslant \frac{c}{\sqrt{a^2+b^2}} \le \exp\left(-\frac{ab\cos\gamma}{a^2+b^2}\right).$$

Note that this is obtained as a corollary of $\ln(1+x) \le x$.

c) (NSOM'09) Suppose that $\gamma = \pi/2$ and $a < b$. Set

$$\alpha_n = \frac{a}{b} - \frac{a^3}{3b^3} \pm \cdots + (-1)^{n+1}\frac{a^{2n-1}}{(2n-1)\,b^{2n-1}}.$$

Prove that $\lim\limits_{n\to\infty} \alpha_n = \alpha$.

Problem 3.2 (*Vl. Todorov*, NSOM'06 [1])**.** Let $\mathbb{K}_i$, $i \ge 1$, be squares with vertices A_i, B_i, C_i, D_i, where each $\mathbb{K}_{i+1}$ is inscribed in K_i so that $A_{i+1} \in A_iB_i$, $B_{i+1} \in B_iC_i$, $C_{i+1} \in C_iD_i$, $D_{i+1} \in D_iA_i$, $A_iA_{i+1} = \lambda A_{i+1}B_i$, where $\lambda \ge 0$.

Prove that if for some $i \ne j$ the sides of $\mathbb{K}_i$ and $\mathbb{K}_j$ are parallel, then the number $\dfrac{1}{\pi}\arctan\lambda$ is rational.

This should be an easy problem. Clearly, this problem may generate more complicate ones (for example, in the case of higher dimensional cubes).

Problem 3.3 (*Vl. Todorov*, SEEMOUS'07 [2])**.** It is well known that every rational number $a \in [0,1]$ can be represented as a periodical decimal fraction

$$a = 0.a_0a_1\cdots a_k\,(a_{k+1}a_{k+2}\cdots a_{k+p})\,.$$

If $a \in [0,1]$ and its decimal representation are fixed, we define $f_n(x)$, $x \in (-1,1)$, by

$$f_n(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

a) Prove that the limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists.

b) Prove that $f(x)$ is a rational function: $f(x) = P(x)/Q(x)$, where P and Q are polynomials.

c) Prove that these polynomials may be chosen with integer coefficients.

Hint. To solve c) it is sufficient to see that $f(0.1) = a$.

Problem 3.4 (*Vl. Todorov*, NSOM'06 [1])**.** Suppose that $\alpha = \{a_1, a_2, \ldots\}$ is a permutation of the integers a_k and consider the expression $g_n(\alpha) = \dfrac{1}{a_1} + \dfrac{1}{2a_2} + \cdots + \dfrac{1}{na_n}$.

a) Prove that $g(\alpha) = \lim\limits_{n\to\infty} g_n(\alpha)$ exists.

b) Evaluate $\inf\limits_{\alpha} g(\alpha)$ and $\sup\limits_{\alpha} g(\alpha)$ (answer: $\inf\limits_{\alpha} g(\alpha) = 0$ and $\sup\limits_{\alpha} g(\alpha) = \pi^2/6$).

Problem 3.5 (*S. Stefanov*, NSOM'07 [1])**.** a) Prove that there exists a non-constant (continuous) periodic function $f : \mathbb{R} \to \mathbb{R}$ for which

(4) $$f(x+1) + f(x-1) = pf(x)$$

if and only if $|p| \leq \sqrt{2}$.

b) If $p = \sqrt{2}$ find a solution with an integer period.

Note. In the original problem there was a mistake – the assumption "continuous" was missing. How does this change the problem?

As usual, we do not provide the precise solutions and limit ourselves with hints only. To solve the "continuous" part consider equation (4) over the integers $\mathbb{Z}$ and put $f(n) = a_n$. Then (4) may be written as a recurrent equation

(5) $$a_{n+1} - pa_n + a_{n-1} = 0.$$

Next we note that each continuous periodic function is bounded which means that the characteristic equation $\lambda^2 - p\lambda + 1 = 0$ has discriminant $D = p^2 - 4 \leq 0$ – recall that every solution of (5) is of the type $a_n = C_1\lambda_1^n + C_2\lambda_2^n$ where $C_{1,2}$ are constants and $\lambda_{1,2}$ are the roots of the characteristic equation. Furthermore, it is easy to see that the solution of (4) is $C_1 \cos \alpha x + C_2 \sin \alpha x$ where $\alpha = \arccos \dfrac{p}{2}$ for $|p| < 2$. We leave to the reader of this paper to find a solution for $|p| = 2$ and turn to the case $|p| > 2$.

Clearly, in this case (4) does not have a continuous solution. But it is possible to obtain some interesting solutions. To do this consider the set

(6) $$\mathbb{M} = \{x |\, x = m + n\sqrt{2};\, m, n \in \mathbb{Z}\}.$$

Note that $\mathbb{M}$ is an additive subgroup of $\mathbb{R}$ and $x \in \mathbb{M}$ implies $x \pm 1 \in \mathbb{M}$. Moreover, if $x \notin \mathbb{M}$ then $x \pm 1 \notin \mathbb{M}$. Next, take a number λ with $\lambda + \dfrac{1}{\lambda} = p$. It is obvious that one of solutions is the function $f(x) = 0$ if $x \notin \mathbb{M}$ and $f(x) = \lambda^m$ if $x = m + n\sqrt{2} \in \mathbb{M}$. Check that f is a solution of (4) with period $\sqrt{2}$. Generally, $\mathbb{M}$ and f admit *exotic* properties. For example, f is unbounded at every point because $\mathbb{M}$ is a dense subset of $\mathbb{R}$. Changing p with a given concrete number one can obtain many interesting problems.

Problem 3.6 (*P. Stoev, Vl. Todorov*)**.** Suppose that every number $x \in [0, 1]$ is written as a decimal fraction $x = 0.x_1x_2\cdots$. Some x may be written in two ways, e.g. $0.2300\cdots = 0.2299\cdots$. Here we shall consider only fractions with infinite "tails" of 9, so we write $0.9999\cdots$ instead of 1 (except for $x = 0$).

Define the function $f : [0, 1] \to [0, 1]$ by $f(0) = 0$ and $f(x) = 0.(x_1 + x_2)_{10}(x_3 + x_4)_{10}\cdots$ if $0 < x = 0.x_1x_2x_3\cdots$. Here $(a + b)_{10} = a + b$ if $a + b < 10$ and $(a + b)_{10} = a + b - 10$ otherwise. The function f has really some *exotic* properties.

a) (NSOM'04) Prove that f is continuous at every point different from $k\,10^{-n}$.

b) Prove that f is not monotone on any non-degenerate subinterval of $[0, 1]$.

c) Let $a \in [0, 1]$ and $\mathbb{A} = f^{-1}(a)$. Prove that there is a map $g : \mathbb{A} \to \mathbb{R}$, which is "onto".

Hint. Consider the set $\mathbb{A}_1 = f^{-1}(1)$. Part of $\mathbb{A}_1$ consists only of fractions, formed consecutively by the groups of pairs of digits (81) and (72). Next, assign 0 to the group (81) and 1 to (72) and consider the binary fraction formed by the above correspondence. It follows now that some subset of $\mathbb{A}_1$ has the same power as the interval $[0, 1]$.

d) (NSOM'04) It follows from a) that f is integrable. Prove that $\displaystyle\int_0^1 f(x)\,dx = 0.5$.

Problem 3.7 (*Vl. Babev*, NSOM'07 [1]). Let $f : [1, \infty) \to (0, \infty)$ be a continuous function such that for every $a > 0$ the equation $f(x) = ax$ has a solution in $[1, \infty)$.

a) Prove that for each $a > 0$ the equation $f(x) = ax$ has infinitely many solutions.

b) Does assertion a) remain true if f is strictly increasing? (the answer is "yes")

Remark. The authors apologize to the colleagues who have created other interesting problems which are not included in this exposition.

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ВЪРХУ НЯКОИ НАЧИНИ ЗА СЪСТАВЯНЕ НА ЗАДАЧИ ЗА СТУДЕНТСКИ ОЛИМПИАДИ

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Представени са някои начини за съставяне на задачи за студентски олимпиади. Материалът е представен в 3 секции, посветени на различни аспекти от конструирането на такива задачи: 1. Използване на грешки; 2. Използване на известни задачи и 3. Преход от „елементарна“ към „висша“ математика и питане „Какво ще стане, ако?“.