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NEW BOUNDS ON THE VERTEX FOLKMAN NUMBER $F_v(2,2,2,3;4)$*

Aleksandar Bikov

For a graph G the expression $G \overset{v}{\rightarrow} (a_1, \dots, a_s)$ means that for every coloring of the vertices of G in s colors there exists $i \in \{1, \dots, s\}$ such that there is a monochromatic a_i-clique of color i. The vertex Folkman number $F_v(a_1, \dots, a_s; q)$ is defined as

$$F_v(a_1, \dots, a_s; q) = \min\{|\operatorname{V}(G)| : G \overset{v}{\rightarrow} (a_1, \dots, a_s) \text{ and } K_q \not\subseteq G\}.$$

In this paper we improve the known bounds on the number $F_v(2,2,2,3;4)$ by proving with the help of a computer that $20 \le F_v(2,2,2,3;4) \le 22$.

1. Introduction. Only finite, non-oriented graphs without loops and multiple edges are considered in this paper. The vertex set and the edge set of a graph G are denoted by $\operatorname{V}(G)$ and $\operatorname{E}(G)$ respectively. K_n stands for a complete graph or a clique on n vertices, $\omega(G)$ – for the clique number of G, and $\alpha(G)$ – for the independence number of G. For two given graphs G_1 and G_2 we denote by $G_1 + G_2$ the graph G obtained by making every vertex of G_1 adjacent to every vertex of G_2. All undefined terms can be found in [19].

The Ramsey number $R(p,q)$ is defined as the least positive integer n such that every graph on n vertices contains either a p-clique or an independent set of size q. More information on the Ramsey numbers used in this paper can be found in [16].

Let $a_1, \dots, a_s$ be positive integers. The expression $G \overset{v}{\rightarrow} (a_1, \dots, a_s)$ means that for every coloring of $\operatorname{V}(G)$ in s colors (s-coloring) there exists $i \in \{1, \dots, s\}$ such that there is a monochromatic a_i-clique of color i.

Define:

$\mathcal{H}_v(a_1, \dots, a_s; q) = \left\{G : G \overset{v}{\rightarrow} (a_1, \dots, a_s) \text{ and } \omega(G) < q\right\}.$

$\mathcal{H}_v(a_1, \dots, a_s; q; n) = \{G : G \in \mathcal{H}_v(a_1, \dots, a_s; q) \text{ and } |\operatorname{V}(G)| = n\}.$

The vertex Folkman number $F_v(a_1, \dots, a_s; q)$ is defined as

$F_v(a_1, \dots, a_s; q) = \min\{|\operatorname{V}(G)| : G \in \mathcal{H}_v(a_1, \dots, a_s; q)\}.$

Folkman proves in [5] that

$$F_v(a_1, \dots, a_s; q) \text{ exists } \Leftrightarrow q > \max\{a_1, \dots, a_s\}. \tag{1.1}$$

Obviously $F_v(a_1, \dots, a_s; q)$ is a symmetric function of $a_1, \dots, a_s$, and if $a_i = 1$, then

$$F_v(a_1, \dots, a_s; q) = F_v(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_s; q).$$

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Therefore, it is enough to consider only such Folkman numbers $F_v(a_1, \dots, a_s; q)$ for which $2 \le a_1 \le \dots \le a_s$. In the case $a_r = 2$, for convenience, instead of $G \overset{v}{\to} (\underbrace{2, \dots, 2}_{r}, a_{r+1}, \dots, a_s)$ we can write $G \overset{v}{\to} (2_r, a_{r+1}, \dots, a_s)$.

In [9] Luczak and Urbański defined for arbitrary positive integers $a_1, \dots, a_s$:

$$m(a_1, \dots, a_s) = m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad p = \max\{a_1, \dots, a_s\}.$$

If $q \ge m + 1$, then $F_v(a_1, \dots, a_s; q) = m$, since $K_m \overset{v}{\to} (a_1, \dots, a_s)$ and $K_{m-1} \overset{v}{\nrightarrow} (a_1, \dots, a_s)$.

If $q = m$, by (1.1), $F_v(a_1, \dots, a_s; m)$ exists if and only if $m \ge p + 1$. If $m \ge p + 1$, then $F_v(a_1, \dots, a_s; m) = m + p$ and $K_{m-p-1} + \overline{C}_{2p+1}$ is the only $(m + p)$-vertex graph in $\mathcal{H}_v(a_1, \dots, a_s; m)$, [9], [8].

In the case $q = m - 1$, thanks to the contributions of different authors, the exact values of all the numbers $F_v(a_1, \dots, a_s; m - 1)$ for which $p \le 6$ are known. For more information about these numbers see [1] and [2].

Not much is known about the numbers $F_v(a_1, \dots, a_s; q)$ when $q \le m - 2$. In the case $q = m - 2$ we know all the numbers where $p = 2$, i.e. all the numbers of the form $F_v(2_r; r-1)$. Nenov proved in [12] that $F_v(2_r; r-1) = r + 7$ if $r \ge 8$, and later in [13] he proved that the same equality holds when $r \ge 6$. Another proof of the same equality was given in [14]. By a computer aided research, Jensen and Royle [6] obtained the result $F_v(2_4; 3) = 22$. The last remaining number of this form is $F_v(2_5; 4) = 16$. The upper bound was proved by Nenov in [13] and the lower bound was proved by Lathrop and Radziszowski in [7] with the help of a computer.

The exact values of the numbers $F_v(a_1, \dots, a_s; m - 2)$ for which $p \ge 3$ are unknown and no good general bounds have been obtained. According to (1.1), $F_v(a_1, \dots, a_s; m-2)$ exists if and only if $m \ge p + 3$. If $p = 3$, it is easy to see that in the border case $m = 6$ there are only two numbers of the form $F_v(a_1, \dots, a_s; m - 2)$, namely $F_v(2, 2, 2, 3; 4)$ and $F_v(2, 3, 3; 4)$. Since $G \overset{v}{\to} (2, 3, 3) \Rightarrow G \overset{v}{\to} (2, 2, 2, 3)$, it follows that

$$F_v(2, 2, 2, 3; 4) \le F_v(2, 3, 3; 4).$$

Computing and obtaining bounds on the numbers $F_v(2, 2, 2, 3; 4)$ and $F_v(2, 3, 3; 4)$ has an important role in relation to computing and obtaining bounds on the other numbers of the form $F_v(a_1, \dots, a_s; m - 2)$ for which $p = 3$. For example, it is easy to prove that if $G \overset{v}{\to} (a_1, .., a_s)$, then $K_t + G \overset{v}{\to} (2_t, a_1, \dots, a_s)$, and therefore

$$F_v(2_r, 3; r + 1) \le F_v(2, 2, 2, 3; 4) + r - 3, r \ge 3. \tag{1.2}$$

Also, according to Theorem 4.5 and Theorem 7.2 ($m_0 = 6, p = 3, q = 4$) from [1]

$$F_v(a_1, \dots, a_s; m - 2) \le F_v(2, 3, 3; 4) + m - 6, \text{ if } m \ge 6 \text{ and } \max\{a_1, \dots, a_s\} = 3. \tag{1.3}$$

Shao, Xu and Luo [18] proved in 2009 that

$$18 \le F_v(2, 2, 2, 3; 4) \le F_v(2, 3, 3; 4) \le 30.$$

In 2011 Shao, Liang, He and Xu [17] raised the lower bound on $F_v(2, 3, 3; 4)$ to 19. In 2016 Nenov and the current author [3] obtained the bounds

$$20 \le F_v(2, 3, 3; 4) \le 24. \tag{1.4}$$

In this paper, we improve the bounds on the number $F_v(2,2,2,3;4)$ by proving the following

Main Theorem. $20 \le F_v(2,2,2,3;4) \le 22$.

We prove the lower bound from the Main Theorem in Section 2 and the upper bound in Section 3. Substituting $F_v(2,2,2,3;4)$ with its new upper bound in (1.2) gives the following

Corollary 1.1. $F_v(2_r,3;r+1) \le r+19, r \ge 3$.

Let us also note the following corollary from (1.4) and (1.3), which gives a general upper bound on the numbers $F_v(a_1,\dots,a_s;m-2)$ for which $p=3$

Corollary 1.2. $F_v(a_1,\dots,a_s;m-2) \le m+18$, *if* $m \ge 6$ *and* $\max\{a_1,\dots,a_s\}=3$.

2. Proof of the lower bound $F_v(2,2,2,3;4) \ge 20$. We will prove with the help of a computer that $\mathcal{H}_v(2,2,2,3;4;19)=\emptyset$. Even for a very powerful machine it is practically impossible to accomplish this task by an exhaustive search through the set of all 19-vertex graphs. Instead, the computational approach is based on the following useful fact, which is easy to prove (see Proposition 2.2 from [2]):

Proposition 2.1. *Let $G \in \mathcal{H}_v(a_1,\dots,a_s;q;n)$, $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G and $H=G-A$. Then, $H \in \mathcal{H}_v(a_1-1,a_2,\dots,a_s;q;n-|A|)$.*

According to Proposition 2.1, if $G \overset{v}{\rightarrow} (2,3,3)$ and the graph H is obtained by removing an independent set of vertices from G, then $H \overset{v}{\rightarrow} (3,3)$. Shao, Liang, He and Xu showed in [17] that if $G \in \mathcal{H}_v(2,3,3;4;18)$, then G can be obtained by adding 4 independent vertices to 111 of the 153 graphs in $\mathcal{H}_v(3,3;4;14)$, which were obtained in [15]. With the help of a computer, the authors showed that such extension does not produce any graphs in $\mathcal{H}_v(2,3,3;4;18)$ and proved the lower bound $F_v(2,3,3;4) \ge 19$. Similarly, they showed that if $G \in \mathcal{H}_v(2,2,2,3;4;18)$, then G can be obtained by adding 4 independent vertices to 12064 of the 12227 graphs in $\mathcal{H}_v(2,2,3;4;14)$, which were obtained in [4]. Computing the bound $F_v(2,2,2,3;4) \ge 19$ is harder because of the larger number of graphs that have to be extended.

In [3] Nenov and the current author obtained with the help of a computer all 2081214 graphs in $\mathcal{H}_v(3,3;4;15)$ and proved the lower bound $F_v(2,3,3;4) \ge 20$. In this section we prove the stronger inequality $F_v(2,2,2,3;4) \ge 20$, which requires much more computational effort. To obtain this result we modify Algorithm 2.4 from [3].

We will need the following auxiliary definitions and propositions. The graph G is called a maximal graph in $\mathcal{H}_v(2_r,3;4;n)$ if $\omega(G+e)=4$ for every $e \in \mathrm{E}(\overline{G})$. We say that G is a $(+K_3)$-graph if $G+e$ contains a new 3-clique for every $e \in \mathrm{E}(\overline{G})$. Suppose that G is a maximal graph in $\mathcal{H}_v(2_r,3;4;n)$ and $\alpha(G)=s$. Let $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G, $|A|=s$ and $H=G-A$. By Proposition 2.1, we have $H \in \mathcal{H}_v(2_{r-1},3;4;n-s)$. From $\omega(G+e)=4, \forall e \in \mathrm{E}(\overline{G})$ it follows that H is a $(+K_3)$-graph, and obviously, from $\alpha(G)=s$ it follows that $\alpha(H) \le s$. Thus, we proved the following

Proposition 2.2. *Let G be a maximal graph in $\mathcal{H}_v(2_r,3;4;n)$ and $\alpha(G)=s$. Let $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G, $|A|=s$ and $H=G-A$. Then H is a $(+K_3)$ graph in $\mathcal{H}_v(2_{r-1},3;4;n-s)$ with independence number not greater than s.*

The graph G is called a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some pair of vertices $u,v \in \mathrm{V}(G)$. If $G \in \mathcal{H}_v(2_r,3;4;n)$ is a Sperner graph and $N_G(u) \subseteq N_G(v)$, then $G-u \in \mathcal{H}_v(2_r,3;4;n-1)$. In the special case when G is a maximal Sperner graph in

$\mathcal{H}_v(2_r, 3; 4; n)$, from $N_G(u) \subseteq N_G(v)$ we derive $N_G(u) = N_G(v)$, and it follows that $G - u$ is a maximal graph in $\mathcal{H}_v(2_r, 3; 4; n-1)$. Thus, we proved the following

Proposition 2.3. *Every maximal Sperner graph in $\mathcal{H}_v(2_r, 3; 4; n)$ is obtained by duplicating a vertex in some of the maximal graphs in $\mathcal{H}_v(2_r, 3; 4; n-1)$.*

All maximal non-Sperner graphs with independence number s in $\mathcal{H}_v(2_r, 3; 4; n)$ are obtained very efficiently with the following algorithm which is a slightly modified variant of Algorithm 2.4 from [3] (the set $\mathcal{L}(n; p; s)$ is replaced by the set $\mathcal{H}_v(2_r, 3; 4; n)$ and the last check $K_p + G \overset{e}{\rightarrow} (3, 3)$ is replaced by $G \overset{v}{\rightarrow} (2_r, 3)$). As noted in [3], this algorithm can be used not only to prove $F_e(3, 3; 4) \geq 20$, but also to obtain other results. At the end of [3], one such example is given by proving $F_v(2, 3, 3; 4) \geq 20$. In this paper we use the following modification of this algorithm to obtain another new result ($F_v(2, 2, 2, 3; 4) \geq 20$).

Algorithm 2.4. *Finding all maximal non-Sperner graphs with independence number s in $\mathcal{H}_v(2_r, 3; 4; n)$, for fixed n, r and s.*

1. *The input of the algorithm is the set $\mathcal{A}$ of all $(+K_3)$-graphs in $\mathcal{H}_v(2_{r-1}, 3; 4; n-s)$ with independence number not greater than s. The obtained graphs will be output in the set $\mathcal{B}$, let $\mathcal{B} = \emptyset$.*

2. *For each graph $H \in \mathcal{A}$:*

2.1. *Find the family $\mathcal{M}(H) = \{M_1, \dots, M_t\}$ of all maximal K_3-free subsets of $\mathrm{V}(H)$.*

2.2. *Find all s-element subsets $N = \{M_{i_1}, M_{i_2}, \dots, M_{i_s}\}$ of $\mathcal{M}(H)$, which fulfill the conditions:*

(*a*) $M_{i_j} \neq N_H(v)$ *for every* $v \in \mathrm{V}(H)$ *and for every* $M_{i_j} \in N$.

(*b*) $K_2 \subseteq H[M_{i_j} \cap M_{i_k}]$ *for every* $M_{i_j}, M_{i_k} \in N$.

(*c*) $\alpha(H - \bigcup\limits_{M_{i_j} \in N'} M_{i_j}) \leq s - |N'|$ *for every* $N' \subseteq N$.

2.3. *For each s-element subset $N = \{M_{i_1}, M_{i_2}, \dots, M_{i_s}\}$ of $\mathcal{M}(H)$ found in step 2.2 construct the graph $G = G(N)$ by adding new independent vertices $v_1, v_2, \dots, v_s$ to $\mathrm{V}(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \dots, s$. If G is not a Sperner graph and $\omega(G+e) = 4, \forall e \in \mathrm{E}(\overline{G})$, then add G to $\mathcal{B}$.*

3. *Remove the isomorph copies of graphs from $\mathcal{B}$.*

4. *Remove from $\mathcal{B}$ all graphs G for which $G \overset{v}{\nrightarrow} (2_r, 3)$.*

We will prove the correctness of Algorithm 2.4 with the help of the following

Lemma 2.5 ([3]). *After the execution of step 2.3 of Algorithm 2.4, the obtained set $\mathcal{B}$ coincides with the set of all maximal K_4-free non-Sperner graphs G with $\alpha(G) = s$ which have an independent set of vertices $A \subseteq \mathrm{V}(G), |A| = s$ such that $G - A \in \mathcal{A}$.*

Theorem 2.6. *After the execution of Algorithm 2.4, the obtained set $\mathcal{B}$ coincides with the set of all maximal non-Sperner graphs with independence number s in $\mathcal{H}_v(2_r, 3; 4; n)$.*

Proof. Suppose that, after the execution of Algorithm 2.4, $G \in \mathcal{B}$. According to Lemma 2.5, G is a maximal K_4-free non-Sperner graph with independence number s. From step 4 it follows that $G \in \mathcal{H}_v(2_r, 3; 4; n)$.

Conversely, let G be an arbitrary maximal non-Sperner graph with independence number s in $\mathcal{H}_v(2_r, 3; 4; n)$. Let $A \subseteq \mathrm{V}(G)$ be an independent set of vertices of G, $|A| = s$ and $H = G - A$. According to Proposition 2.2, $H \in \mathcal{A}$, and from Lemma 2.5 it follows that, after the execution of step 2.3, $G \in \mathcal{B}$. Clearly, after step 4, G remains in $\mathcal{B}$. $\square$

We shall note the important role of the *nauty* programs [11] in this work. We use them for fast generation of non-isomorphic graphs and for graph isomorph rejection.

We performed various tests to check the correctness of the implementation of Algorithm 2.4. For example, we used the algorithm to reproduce all 12227 graphs in $\mathcal{H}_v(2,2,3;4;14)$, which were obtained in [4]. A lot of different tests of step 2 of the algorithm were performed in [3]. All computations were completed in about a week on a personal computer.

Proof of the bound $F_v(2,2,2,3;4) \geq 19$. It is enough to prove that $\mathcal{H}_v(2,2,2,3;4;18) = \emptyset$. From $R(4,4) = 18$ it follows that there are no graphs with independence number less than 4 in $\mathcal{H}_v(2,2,2,3;4;18)$, and from Proposition 2.1 and $F_v(2,2,3;4) = 14$ [4] we derive that there are no graphs with independence number more than 4 in this set. Since $F_v(2,2,2,3;4) \geq 18$, there are no Sperner graphs in $\mathcal{H}_v(2,2,2,3;4;18)$. It remains to be proved that there are no non-Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,2,3;4;18)$. We apply Algorithm 2.4 ($n = 18, r = 3, s = 4$). In step 1, $\mathcal{A}$ is the set of all 11844 ($+K_3$)-graphs with independence number less than 5 in $\mathcal{H}_v(2,2,3;4;14)$. Let us remind, that all 12227 graphs in $\mathcal{H}_v(2,2,3;4;14)$ were obtained in [4]. After the execution of step 3, 130923 graphs remain in the set $\mathcal{B}$. None of these graphs belong to $\mathcal{H}_v(2,2,2,3;4)$, and therefore after step 4 we have $\mathcal{B} = \emptyset$. Now, from Theorem 2.6 we conclude that there are no non-Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,2,3;4;18)$, which completes the proof.

Proof of the bound $F_v(2,2,2,3;4) \geq 20$. It is enough to prove that $\mathcal{H}_v(2,2,2,3;4;19) = \emptyset$. Again, from $R(4,4) = 18$ it follows that there are no graphs with independence number less than 4 in $\mathcal{H}_v(2,2,2,3;4;18)$, and from Proposition 2.1 and $F_v(2,2,3;4) = 14$ [4] we derive that there are no graphs with independence number more than 5 in this set. Since $F_v(2,2,2,3;4) \geq 19$, there are no Sperner graphs in $\mathcal{H}_v(2,2,2,3;4;19)$.

We use Algorithm 2.4 ($n = 19, r = 3, s = 5$) to prove that there are no non-Sperner graphs with independence number 5 in $\mathcal{H}_v(2,2,2,3;4;19)$. In step 1, out of the 12227 graphs in $\mathcal{H}_v(2,2,3;4;14)$ from [4], there are 11989 ($+K_3$)-graphs with independence number less than 6. After the execution of step 3, 27433657 graphs remain in the set $\mathcal{B}$. Since none of these graphs are in $\mathcal{H}_v(2,2,2,3;4)$, after step 4 we have $\mathcal{B} = \emptyset$. Now, from Theorem 2.6 we conclude that there are no non-Sperner graphs with independence number 5 in $\mathcal{H}_v(2,2,2,3;4;19)$.

It remains to be proved that there are no non-Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,2,3;4;19)$. By an exhaustive computer search we find all 3423631 ($+K_3$)-graphs in $\mathcal{H}_v(2,3;4;11)$ with independence number not greater than 4. We apply Algorithm 2.4 ($n = 15, r = 2, s = 4$) to obtain all 165614 maximal non-Sperner graphs in $\mathcal{H}_v(2,2,3;4;15)$ with independence number 4. According to Proposition 2.3, all maximal Sperner graphs in $\mathcal{H}_v(2,2,3;4;15)$ are obtained by duplicating a vertex in the maximal graphs in $\mathcal{H}_v(2,2,3;4;14)$. In this way, we find all 4603 maximal Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,3;4;15)$. There are exactly 640 15-vertex K_4-free graphs with independence number less than 4, which are available on [10]. Among them, 35 are maximal graphs in $\mathcal{H}_v(2,2,3;4;15)$. Thus, we found all 170252 maximal graphs in $\mathcal{H}_v(2,2,3;4;15)$ with independence number less than 5. By removing edges from these graphs, we find all 68783156 ($+K_3$)-graphs in $\mathcal{H}_v(2,2,3;4;15)$ with independence number less than 5. Now, we apply Algorithm 2.4 ($n = 19, r = 3, s = 4$) to obtain all maximal non-Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,2,3;4;19)$. After the execution of step 3, 347307340 graphs remain in the set $\mathcal{B}$. Similarly to the previous case, none of these graphs are in $\mathcal{H}_v(2,2,2,3;4)$, and after step 4 we have $\mathcal{B} = \emptyset$. By Theorem 2.6, it follows that there are no non-Sperner graphs with independence number 4 in $\mathcal{H}_v(2,2,2,3;4;19)$, which completes the proof.

3. Proof of the upper bound $F_v(2,2,2,3;4) \le 22$. We need to construct a 22-vertex graph in $\mathcal{H}_v(2,2,2,3;4)$. First, we find a large number of 24-vertex and 23-vertex maximal graphs in $\mathcal{H}_v(2,2,2,3;4)$.

All 352366 24-vertex graphs G with $\omega(G) \le 3$ and $\alpha(G) \le 4$ were found by McKay, Radziszowski and Angeltveit (see [10]). Among these graphs there are 3903 maximal graphs in $\mathcal{H}_v(2,2,2,3;4;24)$.

By removing one vertex from the obtained 24-vertex maximal graphs we find 6 graphs in $\mathcal{H}_v(2,2,2,3;4;23)$. Let us note that the removal of two vertices from the 24-vertex maximal graphs does not produce any graphs in $\mathcal{H}_v(2,2,2,3;4;22)$. Out of the 6 obtained graphs in $\mathcal{H}_v(2,2,2,3;4;23)$, 5 are maximal. One more maximal graph is obtained by adding one edge to the 6th graph, thus we have 6 maximal graphs in $\mathcal{H}_v(2,2,2,3;4;23)$. We find 192 more maximal graphs in $\mathcal{H}_v(2,2,2,3;4;23)$ by applying Procedure 3.1, which is defined below. The same procedure was used in [1] to find the graph $\Gamma_2 \in \mathcal{H}_v(4,5;7;17)$.

By removing one vertex from the obtained 23-vertex maximal graphs we find the graph $\Gamma \in \mathcal{H}_v(2,2,2,3;4;22)$, which is displayed on Fig. 1. By removing one edge from the graph Γ, which is a maximal graph in $\mathcal{H}_v(2,2,2,3;4;22)$, we obtain two more graphs in $\mathcal{H}_v(2,2,2,3;4;22)$.

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Fig. 1. 22-vertex graph $\Gamma \in \mathcal{H}_v(2,2,2,3;4)$

Procedure 3.1 ([1])**.** *Extending a set of maximal graphs in $\mathcal{H}_v(a_1,\dots,a_s;q;n)$.*

1. *Let $\mathcal{A}$ be a set of maximal graphs in $\mathcal{H}_v(a_1,\dots,a_s;q;n)$.*

2. *By removing edges from the graphs in $\mathcal{A}$, find all their subgraphs which are in $\mathcal{H}_v(a_1,\dots,a_s;q;n)$. This way a set of non-maximal graphs in $\mathcal{H}_v(a_1,\dots,a_s;q;n)$ is obtained.*

3. *Add edges to the non-maximal graphs to find all their supergraphs which are maximal in $\mathcal{H}_v(a_1,\dots,a_s;q;n)$. Extend the set $\mathcal{A}$ by adding the new maximal graphs.*

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Aleksandar Bikov
Faculty of Mathematics and Informatics
Sofia University "St. Kliment Ohridski"
5, James Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: asbikov@fmi.uni-sofia.bg

НОВИ ГРАНИЦИ ЗА ВЪРХОВОТО ЧИСЛО НА ФОЛКМАН $F_v(2, 2, 2, 3; 4)$

Александър Биков

За граф G символът $G \overset{v}{\rightarrow} (a_1, \ldots, a_s)$ означава, че при всяко оцветяване на върховете на G в s цвята съществува $i \in \{1, \ldots, s\}$, такова че съществува едноцветна a_i-клика от i-я цвят. Върховото число на Фолкман $F_v(a_1, \ldots, a_s; q)$ се дефинира с равенството

$$F_v(a_1, \ldots, a_s; q) = \min\{|\operatorname{V}(G)| : G \overset{v}{\rightarrow} (a_1, \ldots, a_s) \text{ и } K_q \not\subseteq G\}.$$

В тази работа ние подобряваме известните граници за числото $F_v(2, 2, 2, 3; 4)$, като доказваме с помощта на компютър, че $20 \leq F_v(2, 2, 2, 3; 4) \leq 22$.