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MEASURABILITY OF GENERALISED SEMICONTINUOUS SINGLE- AND SET-VALUED MAPPINGS*

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We present some interrelations between measurability and generalised semicontinuity of single- and set-valued mappings. They are in the form of Lusin type theorems and their corollaries such as Scorza Dragoni type theorems.

1. Introduction. One of the most important results clarifying the relation between continuity and measurability of a real-valued function is the classical Lusin's theorem (named after the prominent Russian mathematician Nikolai Lusin, cf. [10]):

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be measurable. Then, for every $\eta > 0$ there exists a closed $I_\eta \subset I$ with $\mu(I \setminus I_\eta) < \eta$ such that $f|_{I_\eta}$ is continuous.*

In the above written theorem, as well as throughout the paper, it is assumed that *measurable* means Lebesgue measurable, μ is the Lebesgue measure and I is a compact interval.

Lusin's theorem has been developed and generalised in many directions over the years. One of these directions concerns functions of two variables satisfying the Carathéodory property. The following theorem was stated and proved by Scorza Dragoni [12] and nowadays bears his name:

Theorem 1.2. *Let $D \subset \mathbb{R}$ be a compact interval and $f : D \times I \to \mathbb{R}$ be continuous in the first variable and measurable in the second variable. Then, for every $\eta > 0$ there exists a closed $I_\eta \subset I$ with $\mu(I \setminus I_\eta) < \eta$ such that $f|_{D\times I_\eta}$ is continuous.*

There are many generalisations of this theorem, most of which are based on Lusin type theorems for functions or set-valued mappings (cf. [8]) and the *projectivity* of the Lebesgue measure:

Theorem 1.3 (Theorem III.23 in [4])**.** *Let $D \subset \mathbb{R}^n$ be Borel measurable and $A \subset D\times I$ be $\mathcal{B} \otimes \mathcal{L}$-measurable. Then, the natural projection from $D \times I$ to I of A*

$$Pr_t A := \{t \in I : (x, t) \in A \text{ for some } x \in D\}$$

is Lebesgue measurable.

The product σ-algebra $\mathcal{B} \otimes \mathcal{L}$ is the smallest σ-algebra, generated by $\{B \times J : B \in \mathcal{B},\ J \in \mathcal{L}\}$ ($\mathcal{B}$ is the Borel σ-algebra and $\mathcal{L}$ is the Lebesgue σ-algebra).

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Scorza Dragoni type theorems have numerous applications not only in measure theory, but also in control theory (e.g. "the famous Filippov Lemma [6]) and differential inclusions (cf. [11], [7], [9] and [5]).

Some relations between measurability and semicontinuity of single-valued and set-valued mappings have been established in [4], [9], [1] and the results obtained therein proved to be useful in the theory of differential inclusions on many occasions. In this paper we study the interrelations between measurability and a generalisation of semicontinuity – ε-semicontinuity, where ε is a fixed positive real. This study is a vital part of a bigger project concerning a nonautonomous Olech type theorem (cf. [2] and [3]). We hope that our results could be used in problems in control theory and differential inclusions theory when some „tolerance“ in the data or some small perturbations are allowed.

The paper is organised as follows: In Section 2 the definitions for ε-semicontinuity of single- and set-valued mappings are stated and some basic facts about them are proven. Section 3 contains the main theorems in the paper and their proofs.

2. Generalised semicontinuity of single- and set-valued mappings. From now on, $D \subset \mathbb{R}^n$ is assumed to be Borel measurable.

Definition 2.1. *A mapping $f : D \to \mathbb{R}$ is said to be ε-lower semicontinuous (ε-lsc for short) [ε-upper semicontinuous (ε-usc)] at $x_0 \in D$, if for every $\eta > 0$ there exists $\delta > 0$ such that*

$$f(x) > f(x_0) - \varepsilon - \eta \quad [f(x) < f(x_0) + \varepsilon + \eta]$$

for all $x \in D \cap B_\delta(x_0)$. This is equivalent to the statement that for every sequence $\{x_n\}$ with $x_n \to x_0$ it is true that $\liminf_{n\to\infty} f(x_n) \geq f(x_0) - \varepsilon$ [$\limsup_{n\to\infty} f(x_n) \leq f(x_0) + \varepsilon$].

A mapping $f : D \to \mathbb{R}$ is said to be ε-lsc [ε-usc] on $D' \subset D$, if it is ε-lsc [ε-usc] for every $x_0 \in D'$.

It is obvious that f is ε-upper semicontinuous if and only if $-f$ is ε-lower semicontinuous.

For $f : D \to \mathbb{R}$ let us denote its epigraph

$$Epif := \{(x, r) \in D \times \mathbb{R} \ : \ r \geq f(x)\} .$$

Proposition 2.2. *A mapping $f : D \to \mathbb{R}$ is ε-lower semicontinuous [ε-upper semicontinuous] at $x_0 \in D$ if and only if*

$$\overline{Epif} \cap \{(x_0, r_0) \ : \ r_0 < f(x_0) - \varepsilon\} = \emptyset$$

$$[\overline{\{(x, r) \in D \times \mathbb{R} \ : \ r \leq f(x)\}} \cap \{(x_0, r_0) \ : \ r_0 > f(x_0) + \varepsilon\} = \emptyset] .$$

Proof. Let f be ε-lower semicontinuous at $x_0 \in D$. Let us assume the contrary - there exists $(x_0, y_0) \in \overline{Epif} \cap \{(x_0, r) \ : \ r < f(x_0) - \varepsilon\}$. Therefore, there exists a sequence $(x_n, y_n) \to (x_0, y_0)$, $(x_n, y_n) \in Epif$. From $y_n \geq f(x_n)$ and ε-lower semicontinuity of f it follows that $y_0 \geq \liminf_{n\to\infty} f(x_n) \geq f(x_0) - \varepsilon$. This is a contradiction with $(x_0, y_0) \in \{(x_0, r) \ : \ r < f(x_0)) - \varepsilon\}$.

For the converse, let $x_n \to x_0$, $y_n = f(x_n)$ and $y_0 = \liminf_{n\to\infty} f(x_n)$. Then, $(x_0, y_0) \in \overline{Epif}$. Since $\overline{Epif} \cap \{(x_0, r) \ : \ r < f(x_0) - \varepsilon\} = \emptyset$, we have that $y_0 \geq f(x_0) - \varepsilon$ and therefore $\liminf_{n\to\infty} f(x_n) \geq f(x_0) - \varepsilon$. $\square$

Lemma 2.3. *Let $f : D \times I \to \mathbb{R}$ be $\mathcal{B} \otimes \mathcal{L}$-measurable. Then, the set*

$$E := \{t \in I \ : \ f(\cdot, t) \text{ is } \varepsilon\text{-uppper semicontinuous}\}$$

is measurable.

Proof. Let $G : I \rightrightarrows D \times \mathbb{R}$ be given by

$$G(t) = \overline{\{(x,r) \in D \times \mathbb{R} \; : \; r \le f(x,t)\}} \cap \{(x,r) \; : \; r > f(x,t) + \varepsilon\}$$

and $H : D \times I \rightrightarrows \mathbb{R}$ be given by

$$H(x,t) := (Pr_r \overline{\{(x,r) \in D \times \mathbb{R} \; : \; r \le f(x,t)\}}) \cap \{r \; : \; r > f(x,t)+\varepsilon\} = \{r \; : \; (x,r) \in G(t)\},$$

where Pr_r is the natural projection from $D \times \mathbb{R}$ to $\mathbb{R}$.

If a mapping $K : I \rightrightarrows \mathbb{R}^m$ is measurable, then the mapping $\overline{K}$, which assigns to t the set $\overline{K(t)}$, is measurable as well. Indeed, let K_n be given by $t \mapsto K(t) + \frac{1}{n}B$. Since K is measurable, K_n is measurable and then, using Proposition III.4 from [4], we obtain that the mapping $t \mapsto \cap_{n=1}^{\infty} K_n(t) = \overline{K(t)}$ is measurable.

Therefore, the $\mathcal{B} \otimes \mathcal{L}$-measurability of f and the projectivity of the Lebesgue measure imply $\mathcal{B} \otimes \mathcal{L}$-measurability of H and measurability of G, since $G(t) = graph\, H(\cdot, t)$.

From Proposition 2.2 we have that $G(t) = \emptyset$ iff $t \in E$. Hence,

$$I \setminus E = \{t \in I \; : \; G(t) \ne \emptyset\} = Pr \; graph\, G$$

is measurable due to the projectivity of the Lebesgue measure (here Pr is the natural projection from $I \times D \times \mathbb{R}$ to I). $\square$

Definition 2.4. *A mapping $F : D \rightrightarrows \mathbb{R}^n$ is said to be ε-lower semicontinuous (ε-lsc) [ε-upper semicontinuous (ε-usc)] at $x_0 \in D$, if for every $\eta > 0$ there exists $\delta > 0$ such that $F(x_0) \subset F(x) + (\varepsilon + \eta)B$ [$F(x) \subset F(x_0) + (\varepsilon + \eta)B$] for all $x \in D \cap B_\delta(x_0)$.*

A mapping $F : D \rightrightarrows \mathbb{R}^n$ is said to be ε-lower semicontinuous (ε-lsc) [ε-upper semicontinuous (ε-usc)] on $D' \subset D$, if it is ε-lsc [ε-usc] for every $x_0 \in D'$.

It is straight-forward to check that if a single- or set-valued mapping is ε-lower semicontinuous [ε-upper semicontinuous] for each $\varepsilon > 0$, then it is lower semicontinuous [upper semicontinuous].

The following proposition establishes the connection between ε-semicontinuity of single- and set-valued mappings:

Proposition 2.5. *Let $F : D \rightrightarrows \mathbb{R}^n$ be compact-valued. Then F is ε-lower semicontinuous at $x_0 \in D$ if and only if $\varphi(\cdot) = dist(y, F(\cdot))$ is ε-upper semicontinuous at x_0 for every $y \in \mathbb{R}^n$ (or only for all y_i, $i \in \mathbb{N}$ such that $\{y_i\}_{i \in \mathbb{N}}$ is dense in $\mathbb{R}^n$).*

Proof. Let F be ε-lower semicontinuous at x_0 and $y \in \mathbb{R}^n$. From the closed-valuedness of F it follows that there exists $y_0 \in F(x_0)$ such that $\varphi(x_0) = \|y - y_0\|$. From the ε-lower semicontinuity of F at x_0 for $\eta > 0$ we get $\delta > 0$ such that for all $x' \in D \cap B_\delta(x_0)$ there exists $y' \in F(x')$ with $\|y' - y_0\| < \varepsilon + \eta$. Therefore,

$$\varphi(x') = \operatorname{dist}(y, F(x')) \le \|y - y'\| \le \|y - y_0\| + \|y_0 - y'\| < \varphi(x_0) + \varepsilon + \eta.$$

For the converse, let us assume the contrary – F is not ε-lsc at x_0. This means that there exist $\eta > 0$ and sequences $\{x_k\} \subset D$, $x_k \to x_0$ and $\{z_k\} \subset F(x_0)$ with $z_k \notin F(x_k) + (\varepsilon + \eta)B$. Without loss of generality, we may assume that $z_k \to z_0 \in F(x_0)$ ($F(x_0)$ is compact) and therefore $\|z_k - z_0\| < \frac{\eta}{3}$ for $k \ge k_0$.

Due to the triangle inequality, we have:

$$\operatorname{dist}(z_0, F(x_k)) \ge -\|z_k - z_0\| + \operatorname{dist}(z_k, F(x_k)) > -\frac{\eta}{3} + (\varepsilon + \eta) = \varepsilon + \frac{2}{3}\eta.$$

Since $\{y_i\}_{i \in \mathbb{N}}$ is dense in $\mathbb{R}^n$, we can find $i_0 \in \mathbb{N}$ such that $\|y_{i_0} - z_0\| < \frac{\eta}{3}$. Using the

ε-upper semicontinuity of $\operatorname{dist}(y_{i_0}, F(\cdot))$ at x_0 and the previous inequality, we obtain:

$$\begin{aligned}\operatorname{dist}(y_{i_0}, F(x_0)) &\geq \limsup_{k\to\infty} \operatorname{dist}(y_{i_0}, F(x_k)) - \varepsilon \\ &\geq -\|y_{i_0} - z_0\| + \limsup_{k\to\infty} \operatorname{dist}(z_0, F(x_k)) - \varepsilon \\ &> -\frac{\eta}{3} + (\varepsilon + \frac{2}{3}\eta) - \varepsilon = \frac{\eta}{3}.\end{aligned}$$

But on the other hand, we have $\operatorname{dist}(y_{i_0}, F(x_0)) \leq \|y_{i_0} - z_0\| < \dfrac{\eta}{3}$, which is a contradiction. $\square$

3. Scorza Dragoni type theorems. Using the Lusin's theorem corollaries from [9], we prove Scorza Dragoni type theorems for ε-semicontinuous single- and set-valued mappings:

Theorem 3.1. *Let $f : D \times I \to \mathbb{R}$ be $\mathcal{B} \otimes \mathcal{L}$-measurable and $f(\cdot, t)$ be ε-lower semicontinuous [ε-upper semicontinuous]. Then, for every $\eta > 0$ there exists a closed $I_\eta \subset I$ with $\mu(I \setminus I_\eta) < \eta$ such that $f|_{D\times I_\eta}$ is ε-lower semicontinuous [ε-upper semicontinuous].*

Proof. Let $G_1(t) = \overline{Epi f(\cdot, t)}$, $G_2(t) = \{(x, r) \in D \times \mathbb{R} \ : \ r < f(x, t) - \varepsilon\}$ and $G(t) = G_1(t) \cap G_2(t)$. According to Proposition 2.2, $G(t) = \emptyset$. The measurability of G_1 and G_2 is verified in the same way as in the proof of Lemma 2.3.

Applying Corollary 2 from [9] to G_1, for any $\eta > 0$ we find a closed measurable $I_\eta \subset I$ with $\mu(I \setminus I_\eta) < \eta$ such that $G_1|_{I_\eta}$ is upper semicontinuous. Therefore its graph

$$A := graph\, G_1|_{I_\eta} = \{(t, x, r) \in I_\eta \times D \times \mathbb{R} \ : \ \exists (x_n, r_n) \to (x, r),\ r_n \geq f(x_n, t)\}$$

is closed. We claim that A coincides with

$$B := \{(t, x, r) \in I_\eta \times D \times \mathbb{R} \ : \ \exists (t_n, x_n, r_n) \to (t, x, r),\ r_n \geq f(x_n, t_n)\}.$$

The inclusion $A \subset B$ is obvious. For the converse, take $(t, x, r) \in B$. Then, there exists a sequence $(t_n, x_n, r_n) \to (t, x, r)$ for which $r_n \geq f(x_n, t_n)$. Using that A is closed and $(t_n, x_n, r_n) \in A$, we obtain $(t, x, r) \in A$.

In fact, we have established that

$$\begin{aligned}&\overline{\{(t, x, r) \in I_\eta \times D \times \mathbb{R} : r \geq f(x, t)\}} \cap \{(t, x, r) \in I_\eta \times D \times \mathbb{R} : r < f(x, t) - \varepsilon\} = \\ &= graph\, G_1|_{I_\eta} \cap graph\, G_2|_{I_\eta} = graph\, G|_{I_\eta} = \emptyset\end{aligned}$$

which implies that $f|_{D\times I_\eta}$ is ε-lower semicontinuous (due to Proposition 2.2). $\square$

Theorem 3.2. *Let $F : D \times I \rightrightarrows \mathbb{R}^n$ be compact-valued and $\mathcal{B} \otimes \mathcal{L}$-measurable and let $F(\cdot, t)$ be ε-lower semicontinuous for every $t \in I$. Then for every $\eta > 0$ there exists a closed $I_\eta \subset I$ with $\mu(I \setminus I_\eta) < \eta$ such that $F|_{D\times I_\eta}$ is ε-lower semicontinuous.*

Proof. Let $\{y_i\}_{i\in\mathbb{N}}$ be dense in $\mathbb{R}^n$. Define $f_i : D \times I \rightrightarrows \mathbb{R}$ by $f_i(x, t) = \operatorname{dist}(y_i, F(x, t))$. Then f_i is $\mathcal{B} \otimes \mathcal{L}$-measurable (Remark 5 in [9], Theorem III.9 in [4]) and $f_i(\cdot, t)$ is ε-upper semicontinuous on I (Proposition 2.5). Using Theorem 3.1 for f_i, we find a closed $I_i \subset I$ with $\mu(I \setminus I_i) < \dfrac{\eta}{2^{i+1}}$ such that $f_i|_{D\times I_i}$ is ε-upper semicontinuous.

Let $I_\eta := \cap_{i\in\mathbb{N}} I_i$. The set I_η is closed with $\mu(I \setminus I_\eta) < \eta$ such that $f_i|_{D\times I_\eta}$ is ε-upper semicontinuous for every $i \in \mathbb{N}$. Hence, due to Proposition 2.5, $F|_{D\times I_\eta}$ is ε-lower semicontinuous. $\square$

Proposition 3.3. *Let $F : D \times I \rightrightarrows \mathbb{R}^n$ be compact-valued and $\mathcal{B} \otimes \mathcal{L}$-measurable. Let us define $\widetilde{D} := \{(x, t) \in D \times I :$ there exists a relatively open in D neighborhood U of x such that $F(\cdot, t)|_U$ is ε-lower semicontinuous$\}$. Then, $\widetilde{D}$ and $F|_{\widetilde{D}}$ are $\mathcal{B} \otimes \mathcal{L}$-measurable.*

Proof. Let $\{B_i\}_{i\in\mathbb{N}}$ be a countable base of the topology on $\mathbb{R}^n$ (e.g. open balls with rational centre and rational radius) and define $E_i := \{t \in I \ : \ F(\cdot,t)|_{B_i\cap D}$ is ε-lsc$\}$. Then $F_i := F|_{(B_i\cap D)\times I}$ is $\mathcal{B}\otimes\mathcal{L}$-measurable and $\widetilde{D} = \cup_{i\in\mathbb{N}}(B_i\cap D)\times E_i$.

Let $\{y_j\}_{j\in\mathbb{N}}$ be a dense sequence in $\mathbb{R}^n$, $f_{i,j} := \operatorname{dist}(y_j, F_i(x,t))$ and $E_{i,j} := \{t \in I \ : \ f_{i,j}(\cdot,t)$ is ε-usc$\}$. Proposition 3.1 from [5] implies that $f_{i,j}$ is $\mathcal{B}\otimes\mathcal{L}$-measurable and therefore due to Lemma 2.3 the sets $E_{i,j}$ are measurable. Then, $E_i = \cap_{j\in\mathbb{N}}E_{i,j}$ is measurable and hence $\widetilde{D}$ is $\mathcal{B}\otimes\mathcal{L}$-measurable.

Let W be an open set in $\mathbb{R}^n$. Then $F|_{\widetilde{D}}^{-1}(W) = \widetilde{D}\cap F^{-1}(W)$ is $\mathcal{B}\otimes\mathcal{L}$-measurable. Therefore, $F|_{\widetilde{D}}$ is $\mathcal{B}\otimes\mathcal{L}$-measurable. $\square$

Proposition 3.4. *Let $F : D\times I \rightrightarrows \mathbb{R}^n$ be compact-valued, $\widetilde{D} := \{(x,t)\in D\times I :$ there exists a relatively open in D neighborhood U of x such that $F(\cdot,t)|_U$ is ε-lsc$\}$ and $F|_{\widetilde{D}}$ be $\mathcal{B}\otimes\mathcal{L}$-measurable. Then, for every $\eta>0$ there exists a closed $I_\eta\subset I$ with $\mu(I\setminus I_\eta)<\eta$ and a relatively open $\Omega\subset D\times I$ such that $\widetilde{D}\cap(D\times I_\eta) = \Omega\cap(D\times I_\eta)$ and $F|_{\Omega\cap(D\times I_\eta)}$ is ε-lower semicontinuous.*

Proof. Let us fix $\eta>0$. Let the closed-valued $G : I \rightrightarrows D$ be given by $G(t) = \{x \in D \ : \ (x,t)\in(D\times I)\setminus\widetilde{D}\}$. Since G is measurable (its graph is $\mathcal{L}\otimes\mathcal{B}$-measurable), we can apply Corollary 2 from [9] and find a closed $I_0\subset I$ with $\mu(I\setminus I_0) < \dfrac{\eta}{2}$ such that $G|_{I_0}$ is upper semicontinuous.

Let us define a closed-valued $F_0 : D\times I \rightrightarrows \mathbb{R}^n$ by $F_0(x,t) = F(x,t)$ if $(x,t)\in\widetilde{D}\cap(D\times I_0)$ and $F_0(x,t)=\emptyset$ otherwise. Then F_0 is $\mathcal{B}\otimes\mathcal{L}$-measurable and $F_0(\cdot,t)$ is ε-lower semicontinuous on I. Using Theorem 3.2 for F_0, we find a closed $I_1\subset I$ with $\mu(I\setminus I_1) < \dfrac{\eta}{2}$ such that $F_0|_{D\times I_1}$ is ε-lower semicontinuous.

Let $A := \{(x,t)\in D\times I_\eta \ : \ x\in G(t)\}$. Let $I_\eta := I_0\cap I_1$ and $\Omega := (D\times I)\setminus\{(x,t)\in D\times I_\eta \ : \ x\in G(t)\}$. Then, I_η is closed with $\mu(I\setminus I_\eta)<\eta$ and Ω is open in $D\times I$ since *graph* $G|_{I_\eta}$ is closed. In fact, $\Omega = \widetilde{D}\cup(D\times(I\setminus I_\eta))$ and therefore $\Omega\cap(D\times I_\eta) = \widetilde{D}\cap(D\times I_\eta)$ and $F|_{\Omega\cap(D\times I_\eta)} = F_0|_{\widetilde{D}\cap(D\times I_\eta)}$ is ε-lower semicontinuous. $\square$

Definition 3.5. *Let $D\subset\mathbb{R}^n$ and $I\subset\mathbb{R}$ be a compact interval. A mapping $F : D\times I \rightrightarrows \mathbb{R}^n$ is said to be almost ε-lower semicontinuous on $D'\subset D\times I$, if there exists a closed J_ε such that $\mu(I\setminus J_\varepsilon)<\varepsilon$ and $F|_{D\times J_\varepsilon}$ is ε-lower semicontinuous on D'.*

If a mapping is almost ε-lower semicontinuous for each $\varepsilon>0$, then it is almost lower semicontinuous in the sense of the corresponding definition in [5].

Lemma 3.6. *Let $F : D\times I \rightrightarrows \mathbb{R}^n$ be compact-valued, $\mathcal{B}\otimes\mathcal{L}$-measurable and $F(\cdot,t)$ be ε-lower semicontinuous for almost every $t\in I$. Then F is almost ε-lower semicontinuous.*

If F is almost ε-lower semicontinuous for every $\varepsilon>0$, the converse is also true.

Proof. We find a closed I_0 with $\mu(I\setminus I_0) < \dfrac{\varepsilon}{2}$ such that $F(\cdot,t)$ is ε-lower semicontinuous for every $t\in I_0$ and apply Proposition 3.4 to $F|_{D\times I_0}$ with $\eta = \dfrac{\varepsilon}{2}$. $\square$

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ИЗМЕРИМОСТ НА ОБОБЩЕНИ ПОЛУНЕПРЕКЪСНАТИ ЕДНО- И МНОГОЗНАЧНИ ИЗОБРАЖЕНИЯ

Мира Изак Бивас

Представяме някои връзки между измеримостта и обобщената полунепрекъснатост на едно- и многозначни изображения под формата на теореми от тип на Лузин и следствията им – теореми от тип на Скорца Драгони.