МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2017
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2017
Proceedings of the Forty-sixth Spring Conference
of the Union of Bulgarian Mathematicians
Borovets, April 9–13, 2017

ON THE SIZE OF UNIONS OF LINES IN n-DIMENSIONAL VECTOR SPACES OVER FINITE FIELDS SATISFYING THE WOLFF AXIOM*

Hong Wang, Kristian Georgiev

The paper deals with finding unions of lines in $\mathbb{F}^n_{p^{2k}}$ which obey the Wolff axiom and have minimal size. We provide an extension of the constructions for $\mathbb{F}^3_{p^{2k}}$, obtained by Tao in 2002, to a construction for $\mathbb{F}^n_{p^{2k}}$. We determine the size of the union of lines in our construction in $\mathbb{F}^n_{p^{2k}}$ to be $O\left(p^{1.6kn}\right)$. We prove that our construction obeys the Wolff axiom up to a heuristically negligible number of lines.

1. Introduction. Let us consider the *Wolff axiom*, introduced by Thomas Wolff in [1].

Wolff axiom. *Consider the finite field $\mathbb{F}_{p^d}$ of order p^d in the n-dimensional space $\mathbb{F}^n_{p^d}$, where p is a prime and d is a positive integer. A collection $\mathcal{L}$ of lines in $\mathbb{F}^n_{p^d}$ is said to obey the Wolff axiom if for each $2 \le k \le n-1$, every $(k+1)$-dimensional affine subspace $V \subset \mathbb{F}^n_{p^d}$ contains at most $O(|\mathbb{F}_{p^d}|^k)$ lines in $\mathcal{L}$ [2].*

The focus of interest in our research is the following problem.

Conjecture 1.1 (**Main Problem**). *Let $\mathcal{L} = \{l_i\}$ $(i = 1, 2, \dots, p^{dn-d})$ be a set of lines in $\mathbb{F}^n_{p^d}$ obeying the Wolff axiom. Let $\Sigma = \cup l_i$. Then the asymptotic size of Σ is of order p^{dn}. That is, $|\Sigma| = O(p^{dn})$.*

Wolff provided a lower bound for the size of such sets [1] in 1995 using the *hairbrush argument*. The main goal of our project is to determine whether the hairbrush argument bound is sharp in dimensions higher than four. An additional motivation to tackle this problem is a statement of Tao in [2, p. 2]:"It seems of interest to extend this construction to higher dimensions, though perhaps the bound of $|F|^{\frac{n+2}{2}}$ is not necessarily sharp for large n." Here *this construction* refers to the construction showing that the hairbrush argument bound is sharp in four dimensions and "$|F|^{\frac{n+2}{2}}$" is the bound obtained by the hairbrush argument.

We are interested in the asymptotic behavior of $|\Sigma|$ as p tends to infinity, i.e. in finding the exponent ψ for which

$$|\Sigma| = O(p^{\psi}).$$

To this goal we will use genereal definitions and theorems from [3] and [4], as well as the result from Wolff's hairbrush argument [1] and the methods used by Tao [5] and

*2010 Mathematics Subject Classification: 12E20.
Key words: abstract algebra, Wolff axiom, finite fields, Heisenberg group, constructions.

Ellenberg and Hablicsek [6]. We present in Section 2 a construction for vector spaces of dimension four which obeys the Wolff axiom. Then we generalize the construction to dimension n. The proof that this generalized construction satisfies the Wolff axiom, however, is up to a heuristically negligible number of lines.

2. Known lower bound. Sets derived from the Heisenberg group in $\mathbb{F}_{p^d}^n$ for $n > 3$. The hairbrush argument was first introduced by Wolff in [1].

Theorem 2.1. *Let $\mathbb{F}_{p^d}^n$ be a vector space of dimension n over the finite field $\mathbb{F}_{p^d}$. Let $\mathcal{L}$ be a set of $p^{d(n-1)}$ lines l_i obeying the Wolff axiom and the set of points Σ be $\cup l_i$. Then $|\Sigma| \geq O(p^{\frac{d(n+2)}{2}})$.*

It has been shown by Tao and Mockenhaupt that the hairbrush argument (2.1) gives the sharp lower bound for $\mathbb{F}_{p^d}^2$, $\mathbb{F}_{p^{2k}}^3$ [7] and $\mathbb{F}_{p^d}^4$ [2], where p is prime and k and d are positive integers.

The hairbrush argument provides us with a non-trivial lower bound which is valid for $\mathbb{F}_{p^d}^n$ for all $n \in \mathbb{N}$. However, it has not been proven that this bound is sharp in the general case. Constructions have been provided only for vector spaces of dimension $n \leq 4$. The construction presented by Tao in [7] relies on the Heisenberg group and on the fact that in fields of the type $\mathbb{F}_{p^{2k}}$, the map $\Phi : \alpha \mapsto \alpha^{p^k}$ is an involution. To discuss our construction, we introduce the Heisenberg group.

Definition 2.1. *The* Heisenberg group $\mathbb{H} \in \mathbb{F}_{p^{2k}}^3$ *is defined by*

$$\mathbb{H} = \left\{ (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{F}_{p^{2k}} \quad : \quad \zeta_1 - \zeta_1^{p^k} - \zeta_2 \zeta_3^{p^k} + \zeta_2^{p^k} \zeta_3 = 0 \right\}.$$

In this section we examine the sharpness of the bound obtained by the hairbrush argument in vector spaces of dimension $n > 3$. The main goal is to find a series of constructions for smaller n ($n = 4, 5, 6, \ldots$) that follow particular patterns that will be then generalized. Thus, we will be able to expand the constructions for vector spaces of arbitrarily large dimension.

We use the following definitions.

Definition 2.2. *The prime dimension* $\dim_p(\mathbb{F}_{p^d}^n)$ *of a space* $\mathbb{F}_{p^d}^n$ *over a finite field* $\mathbb{F}_{p^d}$ *is defined by* $\dim_p(\mathbb{F}_{p^d}^n) = d \cdot (\dim(\mathbb{F}_{p^d}^n))$*, provided that a field extension* $\mathbb{F}_{p^d}/\mathbb{F}_p$ *can be represented as a d-dimensional vector space over* $\mathbb{F}_p$.

In analogy with the Galois conjugates of an element α of $\mathbb{F}_{p^d}$, we define

Definition 2.3. *The polynomials* $(f)^{p^i}$, $i = 1, 2, \ldots, d-1$ *are the* conjugate polynomials *of the polynomial f with coefficients in* $\mathbb{F}_{p^d}$.

Definition 2.4. *Let $\mathbb{F}_{p^d}$ be a finite field with $d = a \cdot k$. We define the k-trace of an element $\alpha \in \mathbb{F}_{p^d}$ as*

$$Tr_k(\alpha) = \textstyle\sum_{i=1}^{a} \alpha^{p^{i \cdot k}}.$$

Remark. We denote α^{p^k} by $\bar{\alpha}$ because the map $\Phi : \alpha \mapsto \alpha^{p^k}$ is an involution in $\mathbb{F}_{p^{2k}}$ We use this notation to describe our constructions more concisely.

Corollary 2.1. *For all $\alpha \in \mathbb{F}_{p^d}$, we have* $\mathrm{Tr_k}(\alpha) \in \mathbb{F}_{p^k}$.

2.1. Construction in four dimensions. We start with extending the sets derived from the Heisenberg group to vector spaces of dimension four.

Our construction proves that the hairbrush bound is sharp only for fields of the type $\mathbb{F}_{p^d}$ where d is even. For odd d the size of our construction is no longer equal to the hairbrush bound. For that reason, we build the construction only for fields of that type.

The generalization to $\mathbb{F}_{p^d}$, where $d \in \mathbb{N}$ is straightforward and the result is analogous to the one obtained in [6].

Construction for $\mathbb{F}^4_{p^{2k}}$. Let A be the algebraic variety in $\mathbb{F}^4_{p^{2k}}$ cut out by the polynomial

$$f_A = \mathrm{Tr}_k(\alpha_1) + \mathrm{Tr}_k(\alpha_2) + \mathrm{Tr}_k(\alpha_3) + \mathrm{Tr}_k(\alpha_4) \tag{1}$$

with coefficients in $\mathbb{F}_{p^{2k}}$. If we consider f_A as a map $\mathbb{F}_{p^{2k}} \to \mathbb{F}_{p^k}$, f_A takes every value of $\mathbb{F}_{p^k}$ exactly p^k times. Therefore, A has prime dimension $\dim_p(A) = 2nk - k$ and as a result it contains p^{2nk-k} points. We now find the number of lines that lie in A. However, we consider only the lines that intersect the $\alpha_1\alpha_2\alpha_3$ plane transversely. This allows us to parameterize the lines and thus verify the Wolff axiom with less effort. The lines we examine are of the form

$$\mathcal{L}_{(a,b,c,u,v,w)} = \left\{(a,b,c,0) + t(u,v,w,1) \middle|\ t \in \mathbb{F}_{p^d}\right\}. \tag{2}$$

We now parameterize the lines in the surface. We get that $\alpha_1 = a + ut$, $\alpha_2 = b + vt$, $\alpha_3 = c + wt$ and $\alpha_4 = t$. If A contains the lines of the form $\mathcal{L}$, then the quadruples $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ are solutions of equation (1) for any value of t. We get the following constraints from equation (1) for the coefficients c_j of the jth power of t.

$$\begin{cases} c_0 = \mathrm{Tr}_k(a) + \mathrm{Tr}_k(b) + \mathrm{Tr}_k(c) \\ c_{t^p} = u^p + v^p + w^p \\ c_{\bar{t}} = \bar{u} + \bar{v} + \bar{w}. \end{cases} \tag{3}$$

Because the coefficients vanish, we get that $c_0 = c_t = c_{t^p} = c_{\bar{t}} = 0$. We note that $\mathrm{Tr}_k(a) \in \mathbb{F}_{p^k}$ and $\mathrm{Tr}_k(a)$ attains each value of $\mathbb{F}_{p^k}$ exactly p^k times. The polynomials determining $c_{\bar{t}^i}$ are conjugate to each other. Hence, without loss of generality, we examine only c_t. In total, for each of a, b, u and v we have p^{2k} choices, for c we have p^k choices and w is fixed by u and v. Altogether, there are $p^{2k} \cdot p^{2k} \cdot p^{2k} \cdot p^{2k} \cdot p^k = p^{9k}$ lines lying in A.

Now, let us consider the the hypersurface B in $\mathbb{F}^4_{p^{2k}}$ cut out by the polynomial

$$f_B = \mathrm{Tr}_k(\alpha_1) + \mathrm{Tr}_k(\alpha_2) + \mathrm{Tr}_k(\alpha_3\bar{\alpha_4}) \tag{4}$$

with coefficients in $\mathbb{F}_{p^{2k}}$. Note that B also contains p^{2nk-k} points. Analogously, we examine only the lines that intersect the $\alpha_1\alpha_2\alpha_3$ plane transversely. They are of the form (2). Using analogous techniques, we get the following system

$$\begin{cases} c_0 = \mathrm{Tr}_k(a) + \mathrm{Tr}_k(b) \\ c_{t^p} = u^p + v^p + \bar{c} \\ c_{\bar{t}} = \bar{u} + \bar{v} + c \\ c_{\bar{t}+1} = \mathrm{Tr}_k(w). \end{cases} \tag{5}$$

Here c_{t^i} is the coefficient of t^i, $\alpha_1 = a + ut$, $\alpha_2 = b + vt$, $\alpha_3 = c + wt$, and $\alpha_4 = t$. We note that the polynomials determining c_{t^p} and $c_{\bar{t}}$ are conjugate to each other and again we can examine only c_{t^p} without loss of generality. In total, for a, u and v we have p^{2k} choices, for b and w we have p^k choices, and c is fixed by u and v. Altogether, the hypersurface B contains p^{8k} lines that intersect the $\alpha_1\alpha_2\alpha_3$ plane transversely.

Consider the surface C obtained by intersecting A and B. We are interested in the number of points in C and the number of lines that intersect the $\alpha_1\alpha_2\alpha_3$ plane transversely in C. The lines, again, are of the form (2). From (3) and (5) we get the following system of equations.

$$\begin{cases} \mathrm{Tr}_k(a) + \mathrm{Tr}_k(b) = 0 \\ \mathrm{Tr}_k(c) = 0 \\ u + v + w = 0 \\ w = c^p. \end{cases} \tag{6}$$

Remark. Only non-conjugate polynomials are included in this system of equations.

Here, for a and u we have p^{2k} choices, for c and v we have p^k choices, and w is fixed by c. Altogether, we have p^{6k} lines lying in C that intersect the $\alpha_1\alpha_2\alpha_3$ plane transversely.

The surface C contains fewer lines than either A or B and therefore $C \not\equiv A$ and $C \not\equiv B$. Therefore, from Theorem 6.29 in [8, p. 82] we get that $\dim_p(C) = \dim_p(A) - k = 2nk - 2k$. Hence, C contains p^{2nk-2k} points.

Let us summarize what we have so far. The surface C contains p^{2nk-2k} points and p^{6k} lines. The hairbrush bound yields that the minimal size of a set obeying the Wolff axiom is $|\Sigma| = O(p^{3d})$. Thus, we have a sufficient number of lines in C and when $d = 2k$ the bound is attained. What is left is only to verify that the lines in C satisfy the Wolff axiom. We do that thoroughly in the full version of the paper. The result is that the Wolff axiom is indeed satisfied.

In conclusion, our union of p^{6k} lines which obey the Wolff axiom is contained in a set with p^{2nk-2k} points.

We have shown that sets derived from the Heisenberg group can be extended to four dimensions. An important note, however, is that the construction presented here is valid only for fields of the type $\mathbb{F}_{p^{2k}}$. In the next subsection, we consider the n-dimensional case.

2.2. Construction in n dimensions. Before we examine the general case, we investigate some properties of the surfaces we examined that make them suitable for our construction. Choosing to work only with k-traces, we ensure that the number of equations in the system that determines the number of lines depends only on the smallest non-trivial divisor s of d for fields of the type $\mathbb{F}_{p^d}$. Because we are examining only fields of the type $\mathbb{F}_{p^{2k}}$, in our construction we take $s = 2$. Consequently, we have fewer restrictions for the lines. This results in more lines on the surface. This makes such surfaces useful for our purposes because we want to intersect as many surfaces as possible to obtain a size as small as possible and each intersection reduces the number of points and consequently the number of lines.

Now we generalize our construction to vector spaces of dimension n over $\mathbb{F}_{p^{2k}}$. The construction procedure is as follows.

(i) Find a general form of the hypersurfaces.

(ii) Determine the number of lines contained in each hypersurface.

Remark. In $\mathbb{F}^n_{p^{2k}}$ we only examine lines that intersect the $\alpha_1\alpha_2\dots\alpha_{n-1}$ plane transversely. The reasoning is analogous to the one applied in the construction for $\mathbb{F}^4_{p^{2k}}$.

(iii) Determine the number of hypersurfaces that we have to intersect.

(iv) Verify that the selected lines satisfy the Wolff axiom.

Construction for $\mathbb{F}^n_{p^{2k}}$. Let us examine a set of algebraic varieties $\{X_i\}$ in $\mathbb{F}^n_{p^{2k}}$ such that X_i, $i = 0, 1, \ldots, \lfloor n/2 \rfloor$, are cut out by the polynomials

$$Y_i = A(n-2i) + M(2i),$$

where $A(n-2i)$ is the *addition* part, defined as $A(n-2i) = \sum_{l=1}^{n-2i} \mathrm{Tr}_k(\alpha_l)$, and $M(2i)$ is the *multiplication* part, defined as $M(2i) = \sum_{j=0}^{i-1} \mathrm{Tr}_k(\alpha_{n+2(j-i)+1} \cdot \bar{\alpha}_{n+2(j-i)+2})$. Each of the hypersurfaces X_i has p^{2nk-k} points. In the full version of the paper we explicitly determine the number of lines that each X_i contains. The result is that there are a total of $p^{k(4n-7)}$ lines through X_0 and $p^{4k(n-2)}$ in X_i when $i > 0$.

So far, we have completed the first two steps (i) and (ii) of the construction plan. Now we start intersecting the surfaces we described. We do this step by step to see the number of lines at each step and determine whether there are sufficiently many.

The system of equations determining the lines in the intersection $X_0 \cap X_1$ is as follows.

$$\begin{cases} \sum_{j=1}^{n-2} \mathrm{Tr}_k(r_j) = 0 \\ \sum_{j=1}^{n-1} s_j = 0 \\ s_{n-1} = \bar{r}_{n-1} \\ \mathrm{Tr}_k(s_{n-1}) = 0. \end{cases} \tag{7}$$

Here, we have p^k choices for r_{n-2} and r_{n-1}. Additionally, s_{n-1} is fixed by r_{n-1}, and we have one choice for s_{n-2}. Altogether, we have $p^{2k(2n-5)}$ lines in $X_0 \cap X_1$. Furthermore, $\dim_p(X_0 \cap X_1) = 2nk - 2k$.

Remark. The construction in $\mathbb{F}^4_{p^{2k}}$ is exactly the intersection of $X_0 \cap X_1$ and indeed when $n = 4$, $2k(2n-5) = 6k$.

The system of equations determining the lines in the intersection $X_0 \cap X_1 \cdots \cap X_l$ after a series of routine operations is as follows.

$$\begin{cases} \sum_{j=1}^{n-2} \mathrm{Tr}_k(r_j) = 0 \\ \mathrm{Tr}_k(s_{n-1}) = 0 \\ \mathrm{Tr}_k(r_{n-3} \cdot \bar{r}_{n-2}) = \mathrm{Tr}_k(r_{n-3}) + \mathrm{Tr}_k(r_{n-2}) \\ \ldots \\ \mathrm{Tr}_k(r_{n-2l+1} \cdot \bar{r}_{n-2l+2}) = \mathrm{Tr}_k(r_{n-2l+1}) + \mathrm{Tr}_k(r_{n-2l+2}) \\ s_{n-1} = \bar{r}_{n-1} \\ \sum_{j=1}^{n-1} s_j = 0 \\ \mathrm{Tr}_k(s_{n-3} \cdot \bar{s}_{n-2}) = 0 \\ \ldots \\ \mathrm{Tr}_k(s_{n-2l+1} \cdot \bar{s}_{n-2l+1}) = 0 \\ s_{n-3} + s_{n-2} = \bar{r}_{n-2} \cdot s_{n-3} + \bar{r}_{n-3} \cdot s_{n-2} \\ \ldots \\ s_{n-2l+1} + s_{n-2l+2} = \bar{r}_{n-2l+2} \cdot s_{n-2l+1} + \bar{r}_{n-2l+1} \cdot s_{n-2l+2}. \end{cases} \tag{8}$$

We explicitly analyze the number of lines in the resulting surface in the full version of the paper. We conclude that in an n-dimensional vector space the resulting surface $X_0 \cap X_1 \cap \cdots \cap X_l$ of the intersection of $l+1$ surfaces contains $p^{k(4n-5l-5)}$ lines. From this we see that l should be at most $\dfrac{2n-3}{5}$. Otherwise, the resulting surface would have an insufficient number of lines. Because we want the set of points to be of minimal size, we want the resulting surface to be an intersection of as many X_i as possible. Therefore, for l we obtain

$$l = \left\lfloor \frac{2n-3}{5} \right\rfloor .$$

With that, we have completed the third step (2.2) of the construction plan. It remains to be shown that the union of lines we selected satisfies the Wolff axiom. We do this in detail in the full version of the project. However, our proof is not valid for a particular number of m-lines. We conclude that this number is negligible heuristically, provided with the result for the construction in dimension four. Additionally, we have checked explicitly the number of lines that ought to be removed in the cases $n = 4, 5$. For these cases the heuristic argument holds.

In conclusion, the size of our construction is

$$O\left(p^{2k\left(n-\left(\frac{\left\lfloor\frac{2n-3}{5}\right\rfloor+1}{2}\right)\right)}\right) = O\left(p^{2k\left(\frac{4}{5}n\right)}\right).$$

Thus, we provide a counterexample to Conjecture 1.1. However, $O\left(p^{2k\left(\frac{4}{5}n\right)}\right) > O\left(p^{2k\left(\frac{n+2}{2}\right)}\right)$ and therefore we do not prove that the hairbrush bound is sharp for all n.

3. Conclusion. In this project we have constructed a general form of unions of lines in $\mathbb{F}_{p^{2k}}^n$ based on the Heisenberg group. They satisfy the Wolff axiom up to a heuristically negligible number of lines. Additionally, the four dimensional case is examined and the Wolff axiom is verified for it.

In particular, the algebraic structure of the unions of lines satisfying the Wolff axiom may be advantageous for understanding the structure of sets satisfying the Kakeya conjecture.

Acknowledgments. We would like to thank Dr. Tanya Khovanova, Prof. Larry Guth, Prof. David Jerison, Dr. John Rickert, Dr. Evgenia Sendova, Shashwat Kishore, Daniel Vitek, Bertrand Stone and Katerina Velcheva for their invaluable help.

We would also like to express our gratitude towards the CEE, RSI, MIT and HSSIMI as well as Dr. Petar Kenderov, America for Bulgaria Foundation and St. Cyril and St. Methodius International Foundation.

REFERENCES

[1] T. Wolff. An improved bound for Kakeya type maximal functions. *Revista Matematica Iberoamericana*, **11** (1995), 651–674.

[2] T. Tao. A new bound for finite field Besicovitch sets in four dimensions. arXiv:math/0204251, 2002.

[3] R. Lidl, H. Niederreiter. Introduction to finite fields and their applications. Cambridge, Cambridge Univewrsity Press, 1986.

[4] K. Conrad. Finite fields, 2013, http://www.math.uconn.edu/~kconrad/blurbs/galoistheory/finitefields.pdf

[5] T. Tao. Stickiness, graininess, planiness, and a sum-product approach to the Kakeya problem. IPAM workshop on "The Kakeya Problem, Restriction Problem, and Sum-product Theory", 2014.

[6] J. S. Ellenberg, M. Hablicsek. An incidence conjecture of Bourgain over finite fields of positive characteristic. arXiv:1311.1479, 2013.

[7] T. Tao, G. Mockenhaupt. Restriction and Kakeya phenomena for finite fields. arXiv:math/0204234, 2010.

[8] D. Harari. Géométrie algébrique. Course fondamentale de M2, Orsay.

Hong Wang
Massachusetts Institute of Technology
Boston, USA
e-mail: hongwang@mit.edu

Kristian Georgiev
61, Dunav Blvd
Plovdiv, Bulgaria
e-mail: georgieff.ch@gmail.com

ВЪРХУ ГОЛЕМИНАТА НА ОБЕДИНЕНИЯ ОТ ПРАВИ В n-МЕРНИ ВЕКТОРНИ ПРОСТРАНСТВА НАД КРАЙНИ ПОЛЕТА, ИЗПЪЛНЯВАЩИ АКСИОМАТА НА ВОЛФ

Хонг Уанг, Кристиан Георгиев Георгиев

В разработка се разглежда намирането на обединения от прави в $\mathbb{F}_{p^{2k}}^n$, които изпълняват аксиомата на Волф и имат минимална големина. Ние представяме разширение на конструкциите за $\mathbb{F}_{p^d}^3$, получени от Тао през 2002 и от Еленберг и Хабличек през 2013, до конструкция за $\mathbb{F}_{p^{2k}}^n$. Определяме големината на обединенията в конструкцията ни в $\mathbb{F}_{p^{2k}}^n$ да бъде $O\left(p^{1.6kn}\right)$. Доказваме, че нашата конструкция изпълнява аксиомата на Волф за $n = 4$ и доказваме, че конструкцията ни изпълнява аксиомата на Волф до евристично пренебрежим брой прави, в случаите когато $n > 4$.

Ключови думи: абстрактна алгебра, аксиома на Волф, крайни полета, група на Хайзенберг, конструкции.