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MENELAUS, EINSTEIN, PEANO AND THE PROOFS IN GEOMETRY*

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On the example of the theorem of Menelaus, we draw attention to some possible drawbacks of synthetic geometric proofs. These are contrasted with the advantages of calculational, vector-based approach in geometry.

Formulational issues. In the following, a number of proofs of the Menelaus' theorem are examined. But before discussing proofs, it is worth noticing that the very formulation of the theorem is, more often than not, problematic. We intend to show eventually that the formulational issues are of the same origin as those we find in the proofs: the relative unsuitability, in the case of this and other theorems, of the synthetic approach in geometry.

A proper statement of the theorem is as follows.

For each three non-collinear points A, B, C and points A_1, B_1, C_1 on the lines BC, CA and AB, respectively, a necessary and sufficient condition for the collinearity of A_1, B_1, C_1 is

$$\lambda\,\mu\,\nu + (1-\lambda)(1-\mu)(1-\nu) = 0, \tag{1}$$

where $\lambda = BA_1{:}BC$, $\mu = CB_1{:}CA$, $\nu = AC_1{:}AB$, and the ratios are positive or negative according to the participating segments having the same or opposite directions.

However, in many textbooks one finds the following formulation instead:

For each three non-collinear points A, B, C, if a line intersects BC, CA and AB at A_1, B_1 and C_1, respectively, then

$$\frac{AC_1}{C_1B}\cdot\frac{BA_1}{A_1C}\cdot\frac{CB_1}{B_1A} = 1. \tag{2}$$

Compared to the former formulation, the latter is limited in three respects:
– none of A_1, B_1 and C_1 can be any of A, B and C;
– the arithmetic sign of the ratios is not taken into account;
– only necessity is considered.

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The vertices A, B and C are excluded from consideration, because otherwise a denominator in (2) would have vanished, rendering (2) meaningless. But this is really a flaw in the particular way the theorem is being stated, as (1) still holds for a line $A_1B_1C_1$ passing through a vertex of $\triangle ABC$.

The arithmetic sign of the ratios, if respected, requires a separate treatment in the synthetic approach.

Sufficiency is stated as a separate theorem because (again, in the synthetic approach) it requires a separate proof anyway.

The reason for using (2) and not (1) in most formulations of the theorem must be that, as it happens and seen from the following, the synthetic proofs themselves do use the ratios in (2) and not those in (1). If that is not the case, the condition (2) can be reformulated, in the spirit of (1), including respecting the sign, as

$$AC_1 . BA_1 . CB_1 + C_1B . A_1C . B_1A = 0. \tag{3}$$

The equation (1) is still preferable, though. In (3) the signs of AC_1 and C_1B must each accord with the direction of AB, and similarly for the other two pairs — a dependence that is not explicitly stated in the equation. The equation (1), on the other hand, promotes the use of signed ratios in place of lengths, thus presenting a simpler relation of only three variables, which it really is.

Besides, (1) easily admits generalization to indefinite magnitudes: e. g. as ν grows indefinitely, (1) transforms into $\lambda+\mu=1$ [2], suggesting that it holds even when $A_1B_1 \| AB$ or $A_1B_1 \equiv AB$.

Einstein's concern. In a 1937 letter to M. Wertheimer, A. Einstein, commenting on two different proofs of the theorem of Menelaus, expressed his belief that *'we are completely satisfied only if we feel of each intermediate concept that it has to do with the proposition to be proved'* [4].

Einstein only considered the necessity part of the theorem, and only segments of positive length. Here is the first proof he quoted.

Proof 1. Let m be the line through A_1, B_1, C_1, and let $n \| m$ be a line through A (Fig. 1). As n intersects BC at, say, X, the ratios $CB_1 : B_1A$ and $AC_1 : C_1B$ satisfy

$$CB_1 : B_1A \ = \ CA_1 : A_1X, \qquad AC_1 : C_1B \ = \ XA_1 : A_1B\,,$$

from which (2) follows.

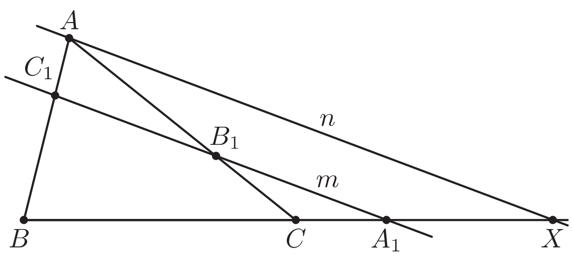

Fig. 1

This proof Einstein called *'ugly'* and *'not satisfying'*, for *'it uses an auxiliary line which has nothing to do with the content of the proposition to be proved, and the proof* *favors, for no reason, the vertex A, although the proposition is symmetrical in relation to A, B and C'.*

In relation to Einstein's first objection we may add the following two observations: we do not know *in advance* (a) what auxiliary object(s) to construct, and (b) how to use those objects in order to obtain a proof.

Indeed, such a proof inevitably leaves the feeling of pulling a rabbit out of a hat.

The second proof, which Einstein called 'elegant', and is apparently his own invention, is as follows.

Proof 2. Each two of $\triangle AB_1C_1$, $\triangle BC_1A_1$ and $\triangle CA_1B_1$ have two angles that are either equal or supplementary (Fig. 2). Then the ratio of the areas of each pair of triangles equals the ratio of the products of the adjacent sides:

$$\begin{aligned} S_{AB_1C_1} : S_{BC_1A_1} &= AC_1.B_1C_1 : C_1B.C_1A_1, \\ S_{BC_1A_1} : S_{CA_1B_1} &= BA_1.C_1A_1 : A_1C.A_1B_1, \\ S_{CA_1B_1} : S_{AB_1C_1} &= CB_1.A_1B_1 : B_1A.B_1C_1. \end{aligned}$$

Multiplying the three equalities establishes (2).

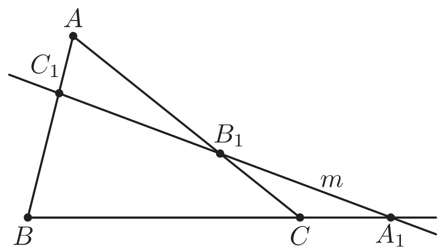

Fig. 2

The proof, although free of auxiliary objects, does not avoid introducing concepts without apparent relation to the problem being solved. Einstein admits this in a second letter to his friend, saying that *'A certain uneasiness remains, however, even in the better proof, since we don't see a priori what the segments on the transversal are supposed to have to do with the matter. We use them in the proof, and then they are cancelled out.'* He then concludes: *'I believe every proof leaves a certain quantum of uneasiness behind it'.*

Not mentioned by Einstein, but no less artificial is the consideration of triangle areas, which, too, enter the scene as if without a relation to the goal being pursued — and only in order to disappear a moment later. Besides, how do we know in advance that multiplying the ratios of the products of the adjacent sides has anything to do with obtaining (2)?

We now consider several other synthetic proofs of the Menelaus' theorem. Similar to the above two, each is characterized by one or another degree of artificiality.

Other proofs. The following proof, known to the author from [8], is somewhat similar to Einstein's 'elegant' one by using ratios of triangle areas and not requiring construction of auxiliary objects.

Proof 3. If the segments from the ratios in (2) are the bases of triangles with the same altitudes, then each such ratio equals the ratio of the areas of the respective triangles:

$$\begin{aligned} AC_1 : C_1B &= S_{AC_1A_1} : S_{BC_1A_1}, \\ BA_1 : A_1C &= S_{BA_1B_1} : S_{CA_1B_1}, \\ CB_1 : B_1A &= S_{CB_1A_1} : S_{AB_1A_1}. \end{aligned}$$

Multiplying the right-hand sides and substituting now ratios of triangle bases for the ratios of triangle areas yields

$$(S_{AC_1A_1}:S_{AB_1A_1}).(S_{BA_1B_1}:S_{BC_1A_1}) = (C_1A_1:A_1B_1).(A_1B_1:C_1A_1) = 1\,.$$

Therefore (2) holds.

The proof is 'more natural' than Einstein's in that it works by transforming the equality being aimed at, rather than arriving at it 'miraculously'. The (smaller) element of surprise that it does contain is related to bringing triangle areas into consideration.

One cannot fail to observe the vexing asymmetry in the right-hand side of the third equality above: C_1 should be present in place of A_1 for all the three equalities to remain cyclic. In fact, doing so brings us to the following symmetric and more direct proof.

Proof 4. Multiplying

$$\begin{aligned} AC_1 : C_1B &= S_{AC_1A_1} : S_{BC_1A_1}, \\ BA_1 : A_1C &= S_{BA_1B_1} : S_{CA_1B_1}, \\ CB_1 : B_1A &= S_{CB_1C_1} : S_{AB_1C_1} \end{aligned}$$

and rearranging the r.h.s. by the first vertex of each triangle gives

$$\begin{aligned} & (S_{AC_1A_1} : S_{AB_1C_1}).(S_{BA_1B_1} : S_{BC_1A_1}).(S_{CB_1C_1} : S_{CA_1B_1}) \\ = \;& (C_1A_1 : C_1B_1).(A_1B_1 : C_1A_1).(B_1C_1 : A_1B_1) = 1\,, \end{aligned}$$

thus establishing (2).

Each of the following three proofs is well known from the college-level geometry textbooks. Beside being concerned only with the necessity part of the Menelaus' theorem and disregarding the arithmetic sign of the ratios, all three resort to auxiliary lines bearing no relation to the theorem's content. Therefore they all qualify as 'ugly', in the same sense that Einstein assessed Proof 1.

This one can be found e. g. in [1] and appears to be one of the most widely referred to.

Proof 5. Let $n \| BC$ be a line passing through A and $X = m \cap n$ (Fig. 3). From the similarlities $\triangle CA_1B_1 \sim \triangle AXB_1$ and $\triangle BA_1C_1 \sim \triangle AXC_1$ it follows

$$CB_1 : B_1A = CA_1 : AX, \quad AC_1 : C_1B = AX : BA_1$$

and substituting these in (2) shows that the latter holds.

Like Proof 1, this proof is asymmetric with respect to A, B, C. An additional disadvantage is the need to consider separately the case of m intersecting all three sides of $\triangle ABC$ externally: the triangle similarities hold in both cases, but for different reasons.

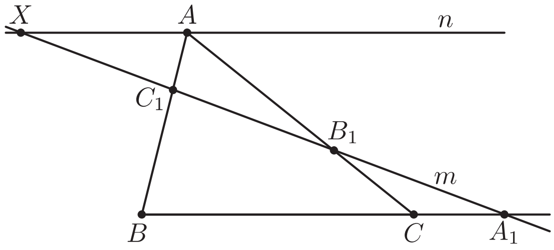

Fig. 3

Proof 6. Let n be a line such that $n \nparallel m$, and A_2, B_2, C_2 — the points of intersection with n of the lines through A, B, C parallel to m (Fig. 4). Then

$$\begin{aligned} AC_1 : C_1B &= A_2X : XB_2, \\ BA_1 : A_1C &= B_2X : XC_2, \\ CB_1 : B_1A &= C_2X : XA_2, \end{aligned}$$

and by multiplying the three equalities, (2) is obtained.

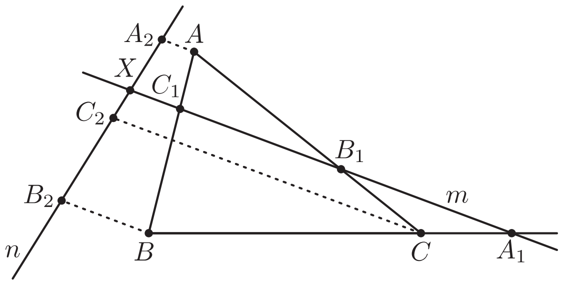

Fig. 4

Along with resorting to an auxiliary line, the arbitrariness of the latter is confusing. Being irrelevant to the result, it leaves a sense of superfluity and inadequacy. (If one is using a wrench and can choose its size arbitrarily, perhaps a wrench is not the proper tool for doing the job.)

Proof 7. Let A_2, B_2, C_2 be the feet of the perpendiculars from A, B, C to m (Fig. 5). Then, from the similarlities $\triangle AC_1A_2 \sim \triangle BC_1B_2$, $\triangle BA_1B_2 \sim \triangle CA_1C_2$, $\triangle CB_1C_2 \sim \triangle AB_1A_2$ there follows

$$\begin{aligned} AC_1 : C_1B &= AA_2 : BB_2, \\ BA_1 : A_1C &= BB_2 : CC_2, \\ CB_1 : B_1A &= CC_2 : AA_2, \end{aligned}$$

which, after multiplying and cancelling out on the r.h.s., produces (2).

This proof is considered by some the 'standard' one (see e.g. [7], [3] and the English Wikipedia). Like most others, it does not avoid using auxiliary objects and introducing ephemeral values whose only role is to be cancelled out.

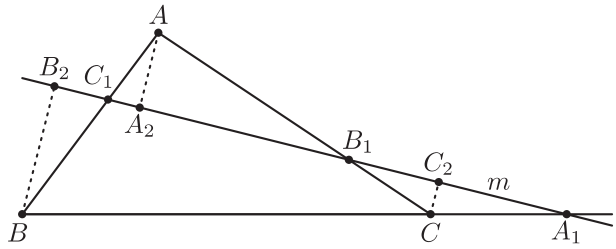

Fig. 5

Proving by calculating. A fundamentally different approach to proving theorems and solving problems in general is calculation. In the case of the Menelaus' theorem, as in others, this approach is superior to the synthetic one.

Indeed, for the problem at hand, what is asked is proving the equivalence of two relations. One is an equality and thus algebraic. The other is a collinearity, which, using a proper language, also readily admits algebraic expression. What then is more natural than calculation in order to show the two relations equivalent?

G. Peano's 1888 book [5] is a masterful display of the application of H. Grassmann's so called *extension theory* to geometry — something in which Grassmann himself was notoriously unsuccessful. Among the many examples, there is the following proof of the theorem of Menelaus.

With λ, μ and ν as defined in the beginning, in Peano's (and Grassmann's) algebra of points it holds that

$$\begin{aligned} A_1 &= (1-\lambda)B + \lambda\, C, \\ B_1 &= (1-\mu)C + \mu\, A, \\ C_1 &= (1-\nu)A + \nu\, B. \end{aligned}$$

Points can be multiplied, the product of two points being a line (i. e., a line segment), and that of three points — a (triangular planar) surface. The latter product vanishes whenever the three points are collinear. Taking these, as well as $BCA = CAB = ABC$ in mind, by multiplying the above equalities we obtain

$$A_1B_1C_1 = (\lambda\,\mu\,\nu + (1-\lambda)(1-\mu)(1-\nu))ABC.$$

As A, B, C are not collinear, the necessary and sufficient condition for the product $A_1B_1C_1$ to vanish, hence A_1, B_1, C_1 to be collinear, is that the coefficient of ABC be zero, i. e. (1).

In fact, the above is a proof of a more general theorem, one of Routh, stating that the oriented area of $\triangle A_1B_1C_1$ is related to that of $\triangle ABC$ by a coefficient of $\lambda\,\mu\,\nu + (1-\lambda)(1-\mu)(1-\nu)$.

Proofs of Routh's and Menelaus' theorems using vectors but otherwise similar to Peano's proof of the Menelaus' theorem were given in [2]. The oriented area of $\triangle A_1B_1C_1$ is expressed as an area product: $2\,S_{A_1B_1C_1} = \mathbf{A_1B_1} \times \mathbf{B_1C_1}$. As the vectors $\mathbf{A_1B_1}$ and $\mathbf{B_1C_1}$ are easily seen to be

$$\begin{aligned} \mathbf{A_1B_1} &= (1-\lambda)\,\mathbf{BC} + \mu\,\mathbf{CA}, \\ \mathbf{B_1C_1} &= (1-\mu)\,\mathbf{CA} + \nu\,\mathbf{AB}, \end{aligned}$$

multiplying them while keeping in mind $\mathbf{BC}\times\mathbf{CA}=\mathbf{CA}\times\mathbf{AB}=\mathbf{AB}\times\mathbf{BC}=2\,S_{ABC}$ and $\mathbf{BC}\times\mathbf{BC}=0$ yields

$$S_{A_1B_1C_1} = (\lambda\,\mu\,\nu + (1-\lambda)(1-\mu)(1-\nu))\,S_{ABC},$$

which is what Routh's theorem says, and of which the Menelaus's theorem is the obvious direct corollary.

Worth noting is that the area product of vectors is a simpler construct than Peano's (Grassmann's) product of points: there are always only two objects to multiply, and the product itself is a number rather than a geometric figure.

The above examination of synthetic and calculational proofs leads to the following conclusions in favour of the latter kind.

A calculational proof does not, in principle, involve introduction of auxiliary constructs or concepts, or other such inventions. It works straightforwardly by formulating the goal in an algebraic language and doing the necessary calculations. No part of it leaves the feeling of arbitrariness, artificiality, or unrelatedness to the goal.

A calculational proof reliably and tacitly handles signed values and oriented geometric objects. There is no need to relegate orientational issues to accompanying proofs (and such proofs are often difficult, subtle and error-prone in the synthetic approach).

As long as a calculational proof is essentially an equivalence transform of an arithmetic expression, there is no need (and no sense) to split the respective theorem into 'necessity' and 'sufficiency' parts.

A calculational proof is reliable as long as the calculations are carried out correctly. But the fundamental operations that constitute those calculations are only a few, simple, and the same for all the problems to solve. A synthetic proof, on the contrary, requires paying attention to specific details, seeing that no assumption is left unproven, is easy to get wrong, and, eventually, is only rarely complete. For example, in Proof 2 above, we used without proving that each triangle pair has equal or supplementary angles.

A calculational proof is general — for such is the underlying algebraic language — saving the need to consider different variants of mutual location of the objects under consideration.

It is also general in that it does not need to avoid 'special cases', such as the line passing through a vertex of the triangle, which is by no means special in the essence.

A calculational proof has the potential to reveal intrinsic generality. In the case of the Menelaus' theorem, we are interested in collinearity, but, as the same algebraic expression yields both collinearity (when 0) and signed area, it makes sense to consider the general meaning of the expression from the outset — thus we discover the more general theorem of Routh, even though we might not have intended to.

A huge advantage of the calculational approach over the synthetic one is that in the former the goal need not be formulated explicitly. With all the synthetic proofs of the Menelaus' theorem, we need to know in advance the formula, such as (1) or (2), the validity of which we want to establish. With a calculational proof, we only write down what it means for A_1, B_1, C_1 to be collinear, then simplify. The result is whatever equation for λ, μ, ν we arrive at: rather than proving a given formula, we are *discovering* one. Consequently, any student can discover the theorem on one's own.

Final remarks. A geometric proof may use an algebraic method without being a calculational proof. Correspondingly, it does not necessarily bring the benefits of a calculational proof. An example of a proof of the Menelaus' theorem that uses vectors but is not calculational is one of D. Pedoe [6]. In that proof, a necessary and sufficient condition for collinearity is used that contains unknowns: the points X, Y and Z are collinear if and only if there exist numbers p, q and r, not all 0, such that $p+q+r=0$ and $p\mathbf{X}+q\mathbf{Y}+r\mathbf{Z}=\mathbf{0}$ ($\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ being the position vectors of X, Y, Z). The presence of unknowns prevents the possibility of directly transforming a condition relating A_1, B_1, C_1 into one relating λ, μ, ν. Some roundabout ways must be followed rather than simply doing calculations. In effect, some of the disadvantages of the synthetic proofs reappear: having to prove the reverse theorem (sufficiency) separately, to consider particular cases, etc. One can conclude that an algebraic approach works best only when in a calculational form.

We have chosen to use the Menelaus' theorem as an example, but there are great many other problems in geometry that might be dissected in a similar manner, showing how calculation outperforms the more conventional synthetic method. Ceva's theorem must be mentioned in this respect, especially because it abounds with incorrect published 'proofs' [2].

Certainly, by far not all geometry problems are best served by the calculational solving method. But when they are, this must be recognized and put to use. School and college teaching of geometry, among others, can greatly benefit from this.

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МЕНЕЛАЙ, АЙНЩАЙН, ПЕАНО И ГЕОМЕТРИЧНИТЕ ДОКАЗАТЕЛСТВА

Бойко Бл. Банчев

Чрез примера на теоремата на Менелай се обръща внимание на някои възможни недостатъци на синтетичните доказателства в геометрията. За сравнение се изтъкват предимствата на аналитичния, основан на смятане с вектори подход.