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CAYLEY–BACHARACH PROPERTY A HISTORICAL ACCOUNT: OLD AND BEYOND*

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Basics of configuration of finite set of points in a projective plane is discussed. Using the minimal possible language concerning *super abundance*, a very special case of Cayley–Bacharach Theorem is proved. Based upon these, we realize well-known classical theorems such as Pappus' Theorem and Pascal's Theorem as direct applications of Cayley–Bacharach Theorem. At the end, we exhibit an explicit example related to the Brill-Noether problem in algebraic curves.

1. A foreword and an overview. Around the time of ancient Greeks, some geometric objects such as parabola, ellipse, circle, etc., were quite well understood and realized as conic sections. In modern mathematical point of view which we all share in the 21st century, such objects fall into the class of algebraic varieties, i.e. one of the main objects of study in algebraic geometry which is – in short – a study of loci of polynomial equations in several variables.

Classical theorems in projective geometry including theorems of Pappus, Desargues or other variations of these – which were already known to ancient Greeks – have been studied repeatedly by several mathematicians since 18th century and almost all of them are related to the so-called Cayley–Bacharach property of finite set of points in the projective plane.

In this note – which reflects some of the main ingredients of the talk delivered at the meeting – we will start with introducing the notion of super abundance with minimal prerequisites and machineries and then proceed to go further how one can recover various classical theorems in projective geometry by using these notions.

Finally, we will count precisely how many specific families of rational functions with certain prescribed attributes may exist on curves (either non-singular or singular) in a complex projective plane.

2. What did the ancient Greeks know? There are several geometrical figures in the Euclidean plane $\mathbb{R}^2$ determined by a single equation of degree two in x and y such as parabola, hyperbola, circle, ellipse and if you allow degenerate case, a union of two distinct lines and a line counted twice fall into the same category. The ancient Greeks realized such objects as conic sections (Fig. 1).

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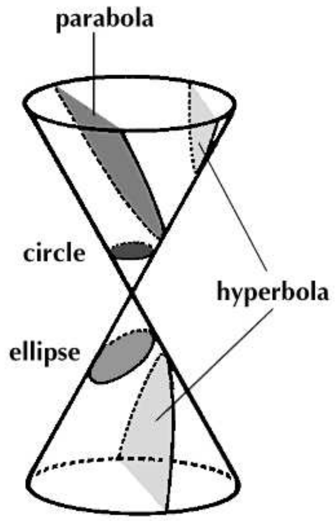

Fig. 1. The kind of conic section produced by the intersection of a plane and conical surface is determined by the angle at which the plane intersects the surface

Explicit Algebraic Example: One can algebraically realize the above conic sections as follows. Let

$$T := \{(x, y, z)|x^2 + y^2 = z^2\} \subset \mathbb{R}^3$$

be a cone in $\mathbb{R}^3$. We consider the following planes defined by several linear equations:

$H_1 = \{(x, y, z)|x = 1\}$, $H_2 = \{(x, y, z)|z = 1\}$,
$H_3 = \{(x, y, z)|z + y = 1\}$, $H_4 = \{(x, y, z)|x + 2y = 1\}$.

Then one sees easily that
$T \cap H_1$ is a **hyperbola** in the plane H_1.
$T \cap H_2$ is a **circle** in H_2.

$$\begin{aligned} T \cap H_3 &= \{(x, y, z)|x^2 + y^2 = z^2, z + y = 1\} \\ &= \{(x, y, z)|x^2 + y^2 = (1 - y)^2, z + y = 1\} \\ &= \{(x, y, z)|x^2 = 1 - 2y, z + y = 1\} \end{aligned}$$

is a **parabola** in H_3 and finally $T \cap H_4$ is an **ellipse** in a similar way.

Degenerate Case: With the same cone $T := \{(x, y, z)|x^2 + y^2 = z^2\} \subset \mathbb{R}^3$, one also gets several degenerate cases when moving around the plane; say

$H_x = \{(x, y, z)|x = 0\} \cap T$ is a **union of two lines**,
$H_z = \{(x, y, z)|z = 0\} \cap T=$ **origin** and
$H_{z-y} = \{(x, y, z)|z - y = 0\} \cap T$ is a line in the yz plane counted twice, which one may call **a double line**.

Another Example: Let for another example

$$S := \{(x, y, z)|xy - z^2 = 0\} \subset \mathbb{R}^3.$$

Consider the planes

$H_1 = \{(x, y, z)|x = 1\}$
$H_2 = \{(x, y, z)|z = 1\}$
$H_3 = \{(x, y, z)|x + y = 4\}$.

It is easy to see that

$S \cap H_1$ is a **parabola** with the equation $y - z^2 = 0$.

$S \cap H_2$ is a **hyperbola** with the equation $xy = 1$.

$S \cap H_3$ becomes an **ellipse**: To see this, we rotate $\mathbb{R}^3$ $45°$ around the z-axis by means of the rigid transformation ϕ: $x \mapsto \frac{1}{\sqrt{2}}(x - y)$, $y \mapsto \frac{1}{\sqrt{2}}(x + y)$, $z \mapsto z$. Then one sees that

$$\phi^{-1}(H_3) = \{(x, y, z)|x = 2\sqrt{2}\}$$

$$\phi^{-1}(S) = \{(x, y, z)|\frac{1}{2}(x^2 - y^2) = z^2\}.$$

Finally we get an ellipse

$$\phi^{-1}(H_3) \cap \phi^{-1}(S) = \{(x, y, z)|z^2 + \frac{1}{2}y^2 = 4, x = 2\sqrt{2}\}.$$

3. Projective Space, a Unifier. The conic sections can also be regarded as shadows of the surfaces S or T considered above on various moving planes in $\mathbb{R}^3$ where the light emanates from the origin. The mathematical object which formalizes this simple idea is the notion of **projective space** $\mathbb{P}^r$ defined as follows:

$$\mathbb{P}^r := \mathbb{R}^{r+1} - \mathbb{O}/\sim = \{\text{dimension one subspaces of } \mathbb{R}^{r+1}\}$$

In other words, $\mathbb{P}^r$ is a collection of $(r + 1)$-tuples $(x_0, \ldots, x_r) \neq (0, \ldots, 0)$ modulo the equivalence relation;

$$(x_0, \ldots, x_r) \sim (y_0, \cdots, y_r) \text{ if and only if } \exists\, \lambda \in \mathbb{R}^* \text{such that } x_i = \lambda y_i \text{ for all } i.$$

We note that $\mathbb{P}^r = V_r \cup V_\infty$ where

$$V_r = \{(x_0, \ldots, x_r) \in \mathbb{P}^r|x_0 \neq 0\}, V_\infty = \{(x_0, \ldots, x_r) \in \mathbb{P}^r|x_0 = 0\}.$$

We also note that

$$V_r = \{(x_0, \ldots, x_r) \in \mathbb{P}^r|x_0 = 1\}$$

is a copy of $\mathbb{R}^r$. Hence $\mathbb{P}^r$ is a space containing the usual **Euclidean space** $\mathbb{R}^r$ together with the extra piece V_∞ which is called a **hyperplane at infinity**.

In our first example, all the conic sections can be unified as the locus

$$U := \{x, y, z) \in \mathbb{P}^2|x^2 + y^2 = z^2\} \subset \mathbb{P}^2,$$

where

$$U \cap \{(x, y, z)|x = 1\} = U \cap V_2$$

is the hyperbola in the Euclidean space $V_2 = \mathbb{R}^2$ and

$$U \cap \{(x, y, z)|z = 1\}$$

is a circle in another Euclidean piece of the projective plane $\mathbb{P}^2$, etc.

4. Super abundance with some elementary linear algebra. Let

$$\mathbb{R}[x, y, z] = \{\text{polynomials in } x, y, z \text{ with real coefficients}\}$$

be the polynomial ring with three indeterminates over $\mathbb{R}$ and let

$$P(d) := \{\text{homogeneous polynomials in } x, y, z \text{ of degree } d\} \cup 0.$$

be the space of homogeneous polynomials of degree $d \in \mathbb{N}$, which is a vector space over the field $\mathbb{R}$ of dimension $\binom{d+2}{2}$.

In particular, $P(2)$ is the vector space with basis $\{x^2, y^2, z^2, xy, yz, xz\}$ and hence $\dim P(2) = 6$. We now choose a point $q = (q_0, q_1, q_2) \in \mathbb{P}^2$, and consider a non-zero homogeneous polynomial of degree two

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx \in P(2) \subset \mathbb{R}[x, y, z]$$

as well as its **zero locus**

$$C := \{(x, y, z) \in \mathbb{P}^2 | Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz = 0\} \subset \mathbb{P}^2.$$

Note that

$$q \in C$$

$$\Updownarrow$$

$$Aq_0^2 + Bq_1^2 + Cq_2^2 + Dq_0q_1 + Eq_1q_2 + Fq_0q_2 = 0.$$

and hence we see that the requirement for the point q to be included in the locus C induces (or imposes) one non-trivial linear relation among the coefficients $A, \dots, F$ of homogeneous polynomials in $P(2)$. From this observation, one can measure the dimension of the subspace $V(q)$ of $P(2)$ consisting of polynomials such that its zero locus C contains the given point q:

$$\dim V(q) = \dim P(2) - 1.$$

Now we choose another point $s \in \mathbb{P}^2$ and ask:

What is the value of $\dim V(q, s)$?

Since there is a degree two homogeneous polynomial which is a product of two linear forms vanishing at q but not at s, we see that $V(q) \supsetneq V(q, s)$ and hence $\dim V(q, s) = 6 - 2 = 4$.

We choose a third point $t \in \mathbb{P}^2$ and keep asking what the value of $\dim V(q, s, t)$ is. In the same vein we see that

$$\dim V(q, s, t) = \dim P(2) - 3,$$

just because $V(q) \supsetneq V(q, s) \supsetneq V(q, s, t)$ and at each stage the dimension drops by one. So far the dimension of the subspace passing through any chosen points drops by the number of points.

If we choose a fourth point $r \in \mathbb{P}^2$, the situation gets a bit complicated. We first note that the condition $V(q, s, t) = V(q, s, t, r)$ is equivalent to the condition

"a conic passing through q, s, t also passes through r".

Since there is always a conic passing through q, s, t and since r could be arbitrary in case the three points q, s, t are not collinear, in which case $V(q, s, t) \supsetneq V(q, s, t, r)$, we see that q, s, t should be collinear under the assumption $V(q, s, t) = V(q, s, t, r)$ and furthermore the fourth point r should be on the line on which q, s, t lies. Therefore we see that

$$\dim V(q, s, t, r) = \dim P(2) - 4 \;\Leftrightarrow q,\; s,\; t,\; r \;\text{ are not collinear},$$

or equivalently

$$\dim V(q, s, t, r) = \dim P(2) - 3 \;\Leftrightarrow q,\; s,\; t,\; r \;\text{ are collinear}.$$

By using classical language in algebraic geometry, we say that **four points in a projective plane fail to impose independent conditions on conics** if and only if they are collinear.

Now choose five points $\{q(1), \ldots, q(5)\} \subset \mathbb{P}^2$ such that no three are collinear. We claim that

$$\dim V(q(1), \cdots, q(5)) = 6 - \text{number of points} = 1.$$

Indeed, since there is a chain of subspaces of the vector space $P(2)$ of dimension 6, at each step the dimension decreases at most one,

$$P(2) \supseteqq V(q(1)) \supseteqq V(q(1), q(2))) \supseteqq V(q(1), q(2), q(3))$$
$$\supseteqq V(q(1), q(2), q(3), q(4)) \supseteqq V(q(1), \ldots, q(5)).$$

It suffices to check that each inclusion is a strict inclusion, which is very easy by using the fact that no three among five are collinear, e.g. in the last step one may take two lines each passing through $q(1)$, $q(2)$ and $q(3), q(4)$, respectively but not through $q(5)$.

As we have seen,

$$\dim V(q(1), \cdots q(5)) = 6 - \text{number of points} = 1,$$

under the assumption that no three among $\{q(1), \cdots, q(5)\}$ are collinear. Thus here exists a nontrivial homogeneous polynomial in this set and any such polynomial is a scalar multiple of the other. Let $Q(x, y, z)$ be such a **degree two** polynomial and

$$C := \{(x, y, z) \in \mathbb{P}^2 | Q(x, y, z) = 0\}$$

Hence C is the unique **conic** (a zero locus of degree two homogeneous polynomial) containing all the five given points in **(linear) general position**, i.e. points such that no three are collinear.

We pose the next question which may seem to be rather silly:

Can one draw this conic in a projective plane in a synthetic way

using only rulers and compass (possibly infinitely many times)? The answer to this question will be provided in the next section after we go further and prepare adequate machineries regarding super abundance on the the system of cubics.

We now consider $P(3)$ – the vector space of homogeneous polynomials of degree 3, with basis

$$\{x^3, y^3, z^3, x^2y, x^2z, y^2z, xy^2, xz^2, yz^2, xyz\}$$

and

$$\dim P(3) = 10.$$

We choose eight points $\{q(1), \ldots, q(8)\}$ in a projective plane and would like to measure the dimension of the subspaces

$$\dim V(q(1), \ldots, q(i)) \text{ for } i = 1, \ldots, 8.$$

As before, we note that

$$P(3) \supseteqq V(q(1), \ldots, q(i)) \supseteqq V(q(1), \cdots, q(i+1))$$

and

$$\dim V(q(1), \ldots, q(i)) \geq \dim P(3) - i. \tag{4.1}$$

The following remark is one of the essential ingredients of what we are going to discuss for the rest of this section.

Remark 4.1. For $i = 8$, the equality in (4.1) fails to hold if and only if either

(1) all the eight points lie on a conic; i.e. all the coordinates of the eight points satisfy a single homogeneous polynomial of degree two or

(2) five points among the eight given points lie on a line.

This seemingly non-trivial fact is rather **easy** to prove by elementary geometric configuration of points in a projective plane, whose proof may be carried out as follows. Indeed the necessary condition is almost trivial, if all the eight given points lie on a conic then the subspace of cubics vanishing at the eight points contains the cubic polynomials of the form $F \cdot L$, where F is an equation of the conic through the eight points and L is an equation of any line in the projective plane. Hence the dimension of the subspace of cubic polynomials vanishing at the eight points is at least three which is the dimension of all the linear forms L in x, y and z.

For the sufficient condition, one may first argue that if seven of eight points fail to impose independent conditions on cubics, then five among the seven points are collinear and this is left as an exercise to the readers. One then may proceed as follows. Suppose that we have eight points $q(1) \cdots q(8)$ imposing only seven or fewer conditions on cubics, and assume that any seven of the eight points impose independent conditions, then it follows that any cubic passing through any seven of the eight points contain all the eight. We choose three non-collinear points $q(1), q(2), q(3)$ and choose a conic C containing the remaining five points. Let L_{ij} be the line joining $q(i)$ and $q(j)$, $i \neq j, 1 \leq i, j \leq 3$. Since each cubic $C + L_{i,j}$ contains seven points, it must contain the remaining point. However this is only possible when the conic C contains them all, just because $q(1), q(2), q(3)$ are not collinear and hence the conclusion.

After this easy but elegant observation, we may push one step further which leads to the following well-known:

Remark 4.2. Nine given points in a projective plane fail to impose independent conditions on the space of cubics if and only if either

(i) five among the nine points are collinear,

(ii) eight among the nine are on a conic or

(iii) the set of all the nine points is a complete intersection of two distinct cubics.

Recall that the vector space of all the cubic homogeneous polynomials in x, y and z is of dimension 10 and hence there is at least one cubic polynomial vanishing at all the nine given points. We also remark that the condition for the given nine points failing to impose independent conditions on the system of cubics is equivalent to the condition:

$$\dim V(q(1), \ldots, q(8), q(9)) \geq 2.$$

However in case $\dim V(q(1), \ldots q(8)) = 2$, i.e. the eight points among the nine given points impose independent conditions on the system of cubics, then

$$\dim V(q(1), \ldots, q(8)) = \dim V(q(1), \ldots, q(8), q(9)) = 2.$$

In other words, a cubic passing through eight points among nine given points necessarily passes through the remaining point. Since we are assuming here that none of the eight points fails to impose independent conditions on cubics, the condition

$$\dim V(q(1), \ldots, q(8), q(9)) = 2$$

trivially (or tautologically) implies that all the nine points are contained in two distinct cubics (either reducible or irreducible) hence the nine given points all together form a complete intersection of two cubics. This elementary discussion may be considered as a very special case the so-called Cayley–Bacharach Theorem.

Theorem 4.3 (Cayley–Bacharach)**.** *Let F and G be two projective plane curves of degree m and n, respectively such that they meet at mn distinct points. Then any plane curve H of degree $m+n-3$ passing through all but one points of the common intersection of F and G necessarily passes through the remaining point as well.*

A proof for the above general statement requires far-reaching tools and terminologies in algebraic geometry not suitable to be presented here. Instead the readers are advised to look up [2, pp. 671–672]. However, we may prove the very special case $m = n = 3$ based on our preceding discussion as follows.

Proof of Cayley–Bacharach for the case $m = n = 3$. Since the nine points form the intersection of two distinct cubics C_1 and C_2 without common component, we see that

$$\dim V(q(1), \ldots, q(9)) > \dim P(3) - 9 = 10 - 9 = 1.$$

Assume that a cubic C containing eight among these nine points is not a linear combination of the defining equations of C_1 and C_2, then C, C_1, C_2 are linearly independent in $P(3)$ and hence

$$\dim V(q(1), \ldots q(8)) \geq 3 > \dim P(3) - 8.$$

Then it follows from the **Super Abundance** result in Remark 4.1, that either the eight points $q(1), \ldots q(8)$ all lie on a conic or five of the eight points are collinear, which is impossible, since either one of the cubics C_1 or C_2 intersects with a conic in at most $6 = 3 \cdot 2$ points by the Bézout's theorem; for example, union of three lines and a union of two lines intersect in at most 6 points. Likewise, C_1 or C_2 intersect with a line in at most three points.

Hence, we conclude that any cubic C containing eight points is of the form

$$C = aC_1 + bC_2,$$

and C must contain all the nine given points, just because both C_1 and C_2 do. □

5. Pappus' Theorem and Pascal's Theorem. As an easy application of the discussion in the preceding sections, we start this section with Pappus' Theorem which has been known to people from ancient times.

Theorem 5.1 (Pappus of Alexandria)**.** *Given lines l and m in the plane, three distinct points A_1, A_2, A_3 on l (but not on m), and three distinct points B_1, B_2, B_3 on m (but not on l), we consider the intersections $C_{12} = A_1B_2 \cap A_2B_1, C_{31} = A_3B_1 \cap A_1B_3$ and $C_{23} = A_2B_3 \cap A_3B_2$. Then C_{12}, C_{31}, C_{23} must be collinear.*

Proof. Consider the cubic F which is the union of three lines A_1B_2, A_3B_1, A_2B_3 and the other cubic G which is the union of the remaining three lines B_2A_3, B_1A_2, B_3A_1. Then the nine points A_1, A_2, A_3, B_1, B_2, B_3, C_{12}, C_{31}, C_{23} are all the intersection points of two cubics F and G. The union of the lines l, m and the line joining C_{12}, C_{31} passes through eight points of the complete intersection of F and G and thus must pass through the remaining point C_{23} by the Cayley–Bacharach Theorem. Hence the points C_{12}, C_{31}, C_{23} are collinear. □

Another similar well-known result is Pascal's Theorem, which can be also prove by

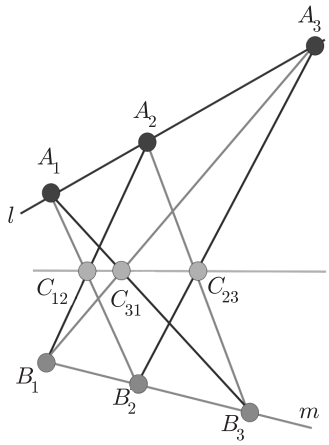

Fig. 2. The three points C_{12}, C_{31}, C_{23} are collinear

using Cayley–Bacharch Theorem. The proof is almost identical to the proof of Pappus' Theorem.

Theorem 5.2 (Pascal)**.** *If six points in a plane can be inscribed in a circle(or a conic), then the opposite sides of the hexagon meets in collinear points.*

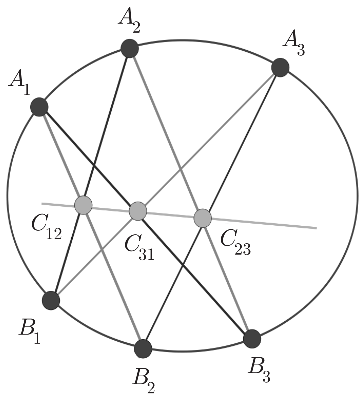

Fig. 3. Pascal's theorem

Remark 5.3. (1) One regards a hexagon with six vertices of any order together with corresponding six edges. In the picture in the next page, the order of the vertices of the hexagon is $A_1, B_2, A_3, B_1, A_2, B_3$. Hence the opposite side of the edge A_1B_2 is A_3B_1, etc.

(2) If one thinks of an ordinary regular hexagon in the Euclidean plane $\mathbb{R}^2$, the opposite sides are parallel and does not meet inside the Euclidean plane. However if we extend the Euclidean plane to the projective plane, three pairs of parallel lines meet in the line at infinity and hence they are collinear.

(3) The usual Pappus' Theorem is just the situation where the conic in Pascal's

Theorem degenerates into a pair of lines.

The inverse of Pascal's Theorem also holds as follows.

Theorem 5.4 (Inverse Pascal's Theorem)**.** *If opposite sides of a hexagon meet in 3 collinear points then the hexagon can be inscribed in a conic.*

Proof. With the same figure above as in Pascal's Theorem (without assuming that all the six vertices of a hexagon lie on a conic), set

$$C_1 = A_1B_2 \cup A_3B_1 \cup A_2B_3,$$

which is the union of three lines extending the edges such that none of the two are opposite to the other or adjacent to another, and set

$$C_2 = B_2A_3 \cup B_1A_2 \cup B_3A_1,$$

the union of three lines extending the remaining three edges. We let C_3 be the unique conic through the five points A_1, A_2, A_3, B_1, B_2 and let L be the line passing through C_{12}, C_{31}, C_{23} by the assumption. Then we see that

$$C_1 \cap C_2 = \{A_1, A_2, A_3, B_1, B_2, B_3, C_{12}, C_{31}, C_{23}\}$$

consists of nine points which is the complete intersection of two cubics. Note that $C_3 \cup L$ contains all the nine points except possibly B_3. By Cayley–Bacharach property, $C_3 \cup L$ contains B_3 and indeed $B_3 \in C$. □

We conclude this section with the following remark which gives a synthetic way to construct the conic through any five given points in general position in a projective plane.

Remark 5.5 (Synthetic construction of a conic through five points)**.**

(1) As we have seen at the beginning, there is a unique conic C through five given points in general position, i.e. no three points are collinear.

(2) Given five points A, B, C, D, E in the plane in sufficiently general position, we let $N = AE \cap BD$.

(3) We then pick a point P along the edge CD.

(4) Let $Q = NP \cap CE$, then three points N, P, Q become collinear.

(5) Let $X = AP \cap BQ$. Then the hexagon with six vertices in the order A, E, C, D, B, X has the property described in the Inverse Pascal's Theorem, opposite sides of a hexagon meet in three collinear points.

(6) By the Inverse Pascal's Theorem, the last vertex X of the hexagon lies on the conic C through five given points.

(7) Finally we vary P on CD for different points X which will eventually sweep out all the points on the conic: here wee see that the conic is parametrized by the line CD.

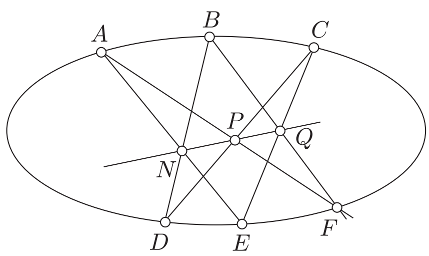

Fig. 4. Synthetic construction of conic through 5 points

6. Epilogue: an odd end with elementary Brill–Noether problem on an algebraic curve. Given a compact Riemann surface X with g handles or a smooth complex projective algebraic curve of genus g, it is well known that there is no non-constant holomorphic functions on X. However there may exist some meromorphic functions on X. A basic problem in Brill-Noether theory of algebraic curves can be addressed as follows:

Question 6.1. Given a finite set of points on X, does there exists a meromorphic function f on X with poles only at the given finite set of points? If so, how many?

In order to rephrase this naive but natural question more precisely, we recall and make a note of the following notations and terminologies together with several well-known results.

Remark 6.2. (1) A divisor D on X is a formal finite sum of points in X with integer coefficient. In other words, a divisor is a member of the free abelian group with the generating set X. Hence a divisor D is written as $D = n_1p_1 + \cdots + n_sp_s$ where $n_i \in \mathbb{Z}$ and $p_i \in X$. We set $\deg D := \sum n_i$. A divisor D is called *effective* (written as $D \geq 0$) if $n_i \geq 0$ for all i. Given two divisors D and E, we write $D \geq E$ if and only if $D - E$ is effective.

(2) A meromorphic function f on a smooth algebraic curve has the same number of zeros and poles if one counts their multiplicities properly; $(f)_0$ and $(f)_\infty$ denote the locus of zeros and poles of f, respectively. The divisor of f which is denoted by (f) is the divisor $(f)_0 - (f)_\infty$. Hence $\deg(f) = 0$.

(3) The collection of all the meromorphic functions on X forms a field which is denoted by $\mathcal{M}(X)$. Given a divisor $D = n_1p_1 + \cdots + n_sp_s$ (usually with $n_i > 0$ for all i), we consider the collection of meromorphic functions f on X such that the divisor $(f) + D$ is effective. Since

$$(f) + D = (f)_0 - (f)_\infty + D,$$

this amounts to say that $(f)_0 + D \geq (f)_\infty$. We note that $(f)_0$ and $(f)_\infty$ has no common support and hence we have $D \geq (f)_\infty$. Therefore such a meromorphic function has poles at worst at D.

(4) Let $\mathcal{L}(D)$ be the collection

$$\mathcal{L}(D) = \{f \in \mathcal{M}(X) | (f) + D \geq 0\} \cup 0.$$

It is known that $\mathcal{L}(D)$ is a finite dimensional vector space over $\mathbb{C}$ and we denote the vector space dimension of $\mathcal{L}(D)$ by $l(D)$.

(5) Measuring the dimension $l(D)$ of the vector space $\mathcal{L}(D)$ has been a long standing problem in algebraic curve theory and the well-known Riemann–Roch formula gives an answer:

$$l(D) = \deg D - g + 1 + l(K - D),$$

where K is a canonical divisor on X which is a zero locus of first order holomorphic differentials on X.

(6) Since $\mathcal{L}(D)$ is a vector space, its associated projective space $\mathbb{P}(\mathcal{L}(D))$ is the space parametrizing all the effective divisors which are linearly equivalent to the given divisor D; two divisors D and E are said to be *linearly equivalent* if and only if they differ by a divisor of a meromorphic function $f \in \mathcal{M}(X)$, i.e. $D - E = (f)$ for some $f \in \mathcal{M}(X)$.

(7) Given a divisor D on X and $\mathcal{L}(D)$ as well, the effective divisors $(f) + D$ with

$f \in \mathcal{L}(D)$ constitute all the members of $\mathbb{P}(\mathcal{L}(D))$, which one may call an *effective divisor* class (under linear equivalence) corresponding to the given divisor D, which is denoted by $|D|$. Obviously, the dimension of $\mathbb{P}(\mathcal{L}(D))$ as a projective space is denoted by $r(D)$ which is just $l(D) - 1$.

(8) Since a divisor of a meromorphic function has always degree zero, all the divisors in the effective divisor class $|D|$ have the same degree. Following classical notation, we write

$$|D| \text{ is a } g_d^r \quad \text{if} \quad r(D) = r \text{ and } \deg D = d$$

or verbally – **a complete linear system(series) of dimension r and degree d.**

In order to illustrate and utilize whatever we have carried out in previous sections, we finally consider the following example and count what kind and how many linear systems of given degree and dimension may exist on a specific curve.

Example 6.3. Let Γ be a projective plane curve of degree six with four nodal singularities (Fig. 5), a plane curve defined by a single homogeneous polynomial of degree six with four isolated singularities whose tangent lines at the singularities are a union of two distinct lines. Let X be a non-singular model of Γ. Since the arithmetic genus of Γ is ten, the geometric genus of Γ and hence the genus $g = l(K)$ of X is $10 - 4 = 6$ by the Clebsch formula.

(1) We first ask if there may exists a g_3^1 on X and ask how many. Assuming the existence of a $g_3^1 = |D|$ on X, let $D = q_1 + q_2 + q_3$. Recall that by the Riemann–Roch formula,

$$l(D) = \deg D - g + 1 + l(K - D) = 3 - 6 + 1 + l(K - D).$$

Hence

$$l(D) \geq 2 \text{ if and only if } l(K - D) \geq 4 > l(K) - \deg D = g - 3 = 3.$$

Therefore it follows that $l(D) \geq 2$ if and only if the divisor D imposes fewer conditions than expected on the canonical system. Recalling that the canonical system is cut out on X by cubics passing through four nodal points, we easily see that points q_1, q_2, q_3 together with four nodal points $p_1, \ldots, p_4$ also fail to impose independent conditions on the space of cubics. As we have seen in the discussion of Remark 4.1, this is only possible when five among the seven points are collinear. However this is not possible since none of the five points among seven points, say q_1, q_2, q_3, p_i, p_j $(i \neq j)$, are collinear on a singular sextic with four nodal points; note that a line through two singular points p_i and p_j meets the curve Γ (Fig. 5) in at most $6 - 2 \cdot 2 = 2$ further points, whereas we have three

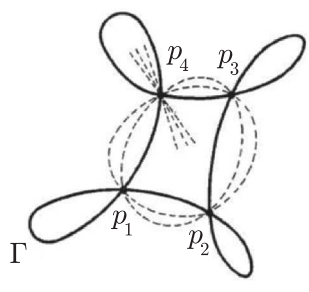

Fig. 5. Plane curve of degree 6 with 4 simple nods

points q_1, q_2, q_3. We therefore conclude that there is no g_3^1's on X and hence our X can never be realized as a triple covering of a projective line.

(2) We go further to see if there exists a g_4^1 on X and would like to specify them all if any. Let $g_4^1 = |D|$ on X, we may assume that the support of D consists of four distinct points; $D = q_1 + q_2 + q_3 + q_4$. By the Riemann–Roch formula again,

$$l(D) = \deg D - g + 1 + l(K - D) = 4 - 6 + 1 + l(K - D).$$

Hence

$$l(D) \geq 2 \text{ if and only if } l(K - D) \geq 3 > l(K) - \deg D = 6 - 4.$$

Therefore it follows that $l(D) \geq 2$ if and only if the divisor D fails to impose independent conditions on the canonical system, which is cut out on X by those cubics passing through four nodal points. This in turn implies that four points q_1, q_2, q_3, q_4 together with four nodal points p_1, p_2, p_3, p_4 also fail to impose independent conditions on the space of cubics. Then by the super abundance result in Remark 4.1, we deduce that either five among the eight points are collinear or all the eight points lie on a conic. The first case is indeed the case where five points are the support of D plus one singular point p_i. In other words, the support of D is in the line through one of the four nodal points. Therefore we have four different pencils such as g_4^1 of such kind. When all the eight points lie on a conic, i.e. our $q_1 \cdots q_4$ lie on a conic through $p_1, \cdots p_4$, we also have a linear system of degree four which is cut out on X by conics through the four singularities. These five exhaust all the g_4^1 on X we are looking for.

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ИСТОРИЧЕСКИ ЕТЮД ВЪРХУ СВОЙСТВОТО НА КЕЙЛИ–БАКАРАХ – СТАРИ И НОВИ ПРОБЛЕМИ

Чангхо Ким

В работата се разглеждат основни проблеми относно конфигурациите от точки в равнината. С минимално използване на понятия като *свръх обилност* (*super abundance*) е доказан частен случай на теоремата на Кейли–Бакарах. На тази основа получаваме добре известните класически теореми на Пап и Паскал като следствие от теоремата на Кейли–Бакарах. Накрая демонстрираме конкретен пример, свързан с проблема на Брил–Ньотер от теорията на алгебричните криви.

Ключови думи: Свръх обилност (super abundance), равнинни алгебрични криви, свойство на Кейли-Бакарах.