МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2018
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2018
Proceedings of the Forty-seventh Spring Conference
of the Union of Bulgarian Mathematicians
Borovets, April 2–6, 2018

ON THE WORK OF MARYAM MIRZAKHANI

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We present a brief overview of results of M. Mirzakhani in the context of symplectic and complex geometry and dynamics of moduli spaces.

1. Lighting a candle. In the early 1950-ies, the German–American mathematician Ernst Strauss posed the *illumination problem*: can one illuminate a room with mirrored walls by a single point light source? Can we always find a spot where we can put a candle, so that everything is illuminated?

In 1958 Roger Penrose described a room with curved walls, which always has dark regions. In the case of polygonal rooms, G. Tokarsky in 1995 and D. Castro in 1997 gave examples of rooms, having respectively 25 and 27 walls, with the property that for every position of the light source there is always a finite number of dark points in the room. A related question is whether, given any two points in our room, we can always prevent the light emitted by the first point from reaching the second point via removing (blocking) finitely many points. While it is not possible to answer this question in all cases, it is known that in most rooms and for most pairs of points, this is impossible. These questions have natural interpretations as security problems or problems on (polygonal) billiards. Yet another question in the same vein is whether, given a polygonal billiard table, one has billiard trajectories of two types only: periodic and "table-filling".

While such questions are easily posed, they are very hard to answer, and progress on them is often made only as a by-product of developing sophisticated machinery. Instead of bouncing off the wall, we have to go through the looking glass and flip over the room. This procedure leads to the notion of "translation surface", and we shall return to it at the end of our exposition. For now we just observe that rather than considering problems on one surface, one has to consider the moduli space of Riemann surfaces, an object of interest to geometers since the middle of 19-th century.

In the focal point of our discussion here will be the work of Maryam Mirzakhani, who has greatly influenced our understanding of the geometry and dynamics of Riemann's moduli space. Maryam Mirzakhani was born on May 03, 1977 in Tehran. She was accepted in the Farzanegan school for girls in Tehran, which is part of Iran's National Organization for Development of Exceptional Talents (NODET). She won gold medals at the International Mathematical Olympiads in 1994 and 1995. After completing her BSc at Sharif University of Technology (Tehran) in 1999, she began her PhD at Harvard University, under the direction of C. McMullen. Mirzakhani defended her PhD thesis in 2004, obtaining spectacular results, among which was a new proof of the Witten conjecture. Her thesis work appeared as [15, 16, 17]. She was awarded the Clay Mathematics

Institute Research Fellowship in 2004. She worked at Princeton University until 2008, when she became a full professor at Stanford University. She was awarded the Fields Medal in 2014, at the ICM in Seoul. She also received the Clay Research Award in the same year. Mirzakhani was diagnosed with breast cancer in 2013 and died on July 14, 2017 at the age of fourty.

In this note we touch upon some key points of Mirzakhani's work and discuss the context into which her research fits.

2. Setting The Stage.

2.1. Hyperbolic Riemann Surfaces. By a *Riemann surface* X we shall mean a (connected) complex manifold of dimension one. We are going to work with Riemann surfaces of *finite type*, which means that X will be isomorphic to a *closed* Riemann surface, from which a finite (possibly empty) set of points has been removed. We are also going to encounter *bordered* Riemann surfaces, i.e., Riemann surfaces with (geodesic) boundary.

The smooth surface S underlying X always admits a Riemannian metric. In a chart $(U, z = x + iy)$ such a metric is given by

$$ds^2 = Edx^2 + 2Fdx \cdot dy + Gdy^2 \ = \lambda \left| dz + \mu d\bar{z} \right|^2 ,$$

where $E, F, G, \lambda \in C^\infty(U, \mathbb{R})$ and $\mu \in C^\infty(U, \mathbb{C})$ with $|\mu| < 1$. If $\|\mu\|_\infty < 1$, the Beltrami equation $(\partial_{\bar{z}} - \mu \partial_z)w = 0$ has a solution and hence there exist local coordinates (u, v) on U, such that $ds^2 = \rho(du^2 + dv^2)$, $\rho > 0$. The coordinate $w = u + iv$ is also called *isothermal coordinate*. Thus the choice of metric induces a complex structure on S. If two oriented Riemannian structures (S_k, g_k), induce Riemann surface structures X_k, $k = 1, 2$, an orientation-preserving diffeomorphism $f : S_1 \to S_2$ is conformal if and only if $f : X_1 \to X_2$ is a biholomorphism.

By the uniformisation theorem of Poincaré, Klein and Koebe, any simply-connected Riemann surfaces is biholomorphic to $\mathbb{CP}^1$, $\mathbb{C}$, or the upper half plane $\mathbb{H}$. Closed Riemann surfaces of genus $g \geq 2$ are hyperbolic: one has an isomorphism $X \simeq \mathbb{H}/\Gamma$, where $\pi_1(X) \simeq \Gamma \subset \mathrm{Aut}(H) = PSL_2(\mathbb{R})$ is a discrete group of automorphisms of the upper half plane. Then the hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$ on $\mathbb{H}$ induces a Riemannian metric with constant negative Gaussian curvature -1 on X, and that is unique within the conformal class of X.

We are going to use the term *hyperbolic surface* to mean a smooth, connected and orientable surface (almost always of genus $g \geq 2$), equipped with complete hyperbolic Riemannian metric of constant curvature -1.

By an *n-pointed curve* $(n \in \mathbb{N})$ or *n-pointed compact Riemann surface* we shall mean a compact Riemann surface X, together with a choice of n (ordered) points $(p_1, \dots, p_n)$ on X. One of our main objects is the set

$$M_{g,n} = \{n\text{-pointed compact Riemann surfaces of genus } g\} / \text{ Isomorphism}$$

Here an isomorphism $(X, p_1, \dots, p_n) \simeq (Y, q_1, \dots, q_n)$ means an isomorphism (biholomorphism) $f : X \simeq Y$, such that $f(x_i) = y_i$ for all i. In particular, we shall write $M_g := M_{g,0}$. Finally, given $L = (L_1, \dots, L_n) \in \mathbb{R}^n_{\geq 0}$, we shall write $M_{g,n}(L)$ or $M_{g,n}(L_1, \dots, L_n)$ for the set of isomorphism classes of complete bordered hyperbolic surfaces of genus g and n cusps, having geodesic boundaries of lengths $L_1, \dots, L_n$, respectively. Thus $M_{g,n} = M_{g,n}(0, \dots, 0)$.

2.2. Complex and Symplectic Geometry of the Moduli Space and the Teichmüller Space. The properties of M_g have been an object of interest for geometers since the time of Riemann. It is not merely a set, but carries plenty of additional structures – several metrics, complex and symplectic structure – and satisfies important functoriality properties.

The existence of a complex structure on M_g was in some way clear to Riemann, who writes in [23] that *a curve of genus* $g \geq 2$ *depends on* $3g-3$ *moduli.* This was later studied by O. Teichmüller in the 1940-ies. The ideas of Teichmüller, while being bold and original, were notoriously difficult to understand, in part due to matters of exposition and choice of terminology. Many of his arguments were only sketched and were later reformulated and reestablished in the works of Bers, Rauch, Ahlfors, Weil...

In particular, L. Bers elucidated and developed the ideas of O. Teichmüller, and applied to them the methods of Kodaira–Spencer theory. He identified M_g with the quotient of a bounded contractible domain in $\mathbb{C}^{3g-3}$ by the action a properly discontinuous group of biholomorphisms. This domain, known as *Teichmüller space* $\mathcal{T}_g$, has points which classify surfaces with extra structure (framing or marking). The complex structure on M_g eventually turned out to be that of a quasi-projective algebraic variety.

Teichmüller theory can be approached from many directions: analysis, hyperbolic, Riemannian or algebraic geometry, which makes it both a fascinating and a technically challenging subject. For a more detailed introduction see, e.g., the lecture notes [9] or the books [10], [19].

2.2.1. Teichmüller space. Let S be a closed oriented Riemann surface of genus $g \geq 2$, and $o \in S$ a point. We define the Teichmüller space of S as the set

$$\mathcal{T}_g(S) = \left\{ \rho : \pi_1(S, o) \hookrightarrow PSL_2(\mathbb{R}) \,\middle|\, \begin{array}{l} \mathrm{Im}(\rho) \text{ acts freely on } \mathbb{H} \\ \mathbb{H}/\mathrm{Im}(\rho) \text{ is a compact RS of genus } g \end{array} \right\} \Big/ PSL_2(\mathbb{R}).$$

This is an open subset of the representation variety and inherits from it a natural topology.

Another description of $\mathcal{T}_g(S)$ is via *marked Riemann surfaces.* We consider the set of pairs (X, f), consisting of a Riemann surface X and a "marking"– an orientation-preserving diffeomorphism $f : S \to X$. One then defines $\mathrm{Teich}_g(S) = \{(X, f)\}/\sim$, where $(X, f) \sim (Y, g)$ if $g \circ f^{-1} : X \to Y$ is homotopic to an isomorphism (i.e., biholomorphism). The natural bijection $\mathcal{T}_g(S) \simeq \mathrm{Teich}_g(S)$ [10, Theorem 1.4] is used to topologise $\mathrm{Teich}_g(S)$. Alternatively, one can endow $\mathrm{Teich}_g(S)$ with the topology induced by the Teichmüller distance [10, Chapter V].

There are other equivalent descriptions of $\mathcal{T}_g(S)$, see e.g., [10, Ch. I]. One can consider framed *hyperbolic* surfaces, and declare $(X, f) \sim (Y, g)$ if $g \circ f^{-1}$ is homotopic to an isometry. For a purely Riemannian description, consider $\mathrm{Diff}^0(S)$, the group of orientation-preserving diffeomorphisms, isotopic to the identity. The set $\mathcal{M}(S)$ of Riemannian metrics on S is equipped with an action of $\mathrm{Diff}^0(S) \times C^\infty(S, \mathbb{R}_+)$, via pullback and scaling. Then $\mathcal{T}_g(S) \simeq \mathcal{M}(S)/\mathrm{Diff}^0(S) \times C^\infty(S, \mathbb{R}_+)$. If instead of $\mathcal{M}(S)$ one considers the metrics of constant curvature (-1), the quotient is still identified with $\mathcal{T}_g(S)$, but is known as *the Fricke space* F_g. We recall also the classical result of [6]: two (orientation-preserving) self-maps of a closed surface are isotopic precisely when they are homotopic.

Bers [2] has shown that $\mathcal{T}_g(S)$ is homeomorphic to an open ball in $\mathbb{R}^{6g-6}$ (for $g \geq 2$), while for $g = 1$ one has $\mathcal{T}_1(S) \simeq \mathbb{H}$. This result, proved also by Fricke, is known as *Teichmüller's theorem.*

A very convenient description of $\mathcal{T}_g(S)$ in terms of hyperbolic geometry is provided by the *Fenchel–Nielsen coordinates.* One can show [10, III,Proposition 3.6] that S, being of genus $g \geq 2$, can be decomposed into $M = 2g - 2$ pants $\{P_k\}_{k=1}^M$, by cutting S along $N = 3g - 3$ disjoint circles $C_1, \ldots, C_N$. We fix a hyperbolic structure on S, and replace the $\{C_i\}_{i=1}^N$ with simple closed disjoint geodesics $\{\gamma_i\}_{i=1}^N$. The complex structure of each pair of pants P_k is uniquely determined by the lengths of its three boundary components [10, III, Theorem 3.5]. Thus, after fixing a pants decomposition on the base surface S, one obtains $3g-3$ functions $\ell_i : \mathcal{T}_g(S) \to \mathbb{R}^+$, which are in fact real-analytic. To recover the hyperbolic structure on a surface from the hyperbolic structure on each pair P_k, one needs to intorduce appropriate "twisting parameters" τ_i along each boundary geodesic γ_i. *A priori*, these take values in S^1, but, due to the simply-connectedness of $\mathcal{T}_g$, they can be lifted to real-valued functions. The functions $(\ell_i, \tau_i)_{i=1}^{3g-3}$ are known as the *Fenchel–Nielsen coordinates.* They establish a homeomorphism, and in fact, a diffeomorphism, between $\mathcal{T}_g(S)$ and $(\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3}$, see [10, III, Theorem 3.10].

The Teichmüller spaces of different surfaces of the same topological type are naturally isomorphic, so we are going to write $\mathcal{T}_g$, rather than $\mathcal{T}_g(S)$.

One can analogously define Teichmüller spaces $\mathcal{T}_{g,n}$ for Riemann surfaces of genus g with n marked points, following the methods outlined above, but that requires some extra care, e.g., if this is done via hyperbolic structures, one must restrict to hyperbolic metrics of finite type. These spaces have dimension $\dim_{\mathbb{R}} \mathcal{T}_{g,n} = 2(3g - 3 + n)$, see [2].

2.2.2. Complex-analytic structure of Teichmüller space. The existence of a natural complex-analytic structure on $\mathcal{T}_g(S)$ has been claimed by Teichmüller [24]. The first proof of this is due to Ahlfors [1], continuing the work of Rauch [22], who constructed local analytic coordinates away from the locus of hyperelliptic curves. Other proofs were given by Kodaira–Spencer and A. Weil [26].

L. Bers constructed, in a series of papers (see [2]) embeddings $\mathcal{T}_g \hookrightarrow \mathbb{C}^{3g-3}$ (for $g \geq 2$), $\mathcal{T}_{g,n} \hookrightarrow \mathbb{C}^{3g-3+n}$, and showed that these spaces are biholomorphic to contractible bounded ones. Bers' embedding provides a global holomorphic chart and gives a natural description of the (co)tangent spaces to Teichmüller space a point $[X, f]$. Namely,

$$T_{\mathcal{T}_g,[X,f]} = H^1(X, T_X),\ T^{\vee}_{\mathcal{T}_g,[X,f]} = H^0(X, K_X^2),$$

and the two formulae are linked by Serre duality. By Hodge theory, the elements of $H^1(X, T_X)$ can be identified with the "harmonic Beltrami differentials". These, if sufficiently small in the ∞-norm, give rise to a deformation of the complex structure of X via a solution of the corresponding inhomogeneous $\overline{\partial}$-equation.

From this viewpoint, Teichmüller metric is (the Finsler metric, arising from the dual of) the L^1-norm $\|q\| = \int_X |q(z)||dz|^2$ on $H^0(X, K_X^2) = T^{\vee}_{\mathcal{T}_g,[X]}$. Royden has identified the Teichmüller metric with the Kobayashi metric of $\mathcal{T}_g$.

While being homeomorphic to a ball, $\mathcal{T}_g$ is quite far from being *biholomorphic* to one: it is in fact a totally inhomogeneous space. Moreover, $\mathcal{T}_g$ has a discrete group of automorphisms, the *mapping class group*, known also as *the modular group*, which is defined as $\mathrm{Mod}_g = \pi_0 \mathrm{Diff}^+(S)$, where $\mathrm{Diff}^+(S)$ is considered as a topological group with the compact-open topology. The action of Mod_g on $\mathcal{T}_g$ is properly discontinuous, with finite stabilisers $\mathrm{Stab}_{[\rho]} = \mathrm{Aut}\,(\mathbb{H}/\mathrm{Im}\rho)$ for $[\rho] \in \mathcal{T}_g$.

Similarly, for a surface S with a set P of n marked points $P := \{p_1, \ldots, p_n\}$, one defines the modular group as $\mathrm{Mod}_{g,n} = \pi_0 \mathrm{Diff}^+(S, P)$, where $\mathrm{Diff}^+(S, P)$ is the group of (orientation-preserving) diffeomorphisms, fixing P. For surfaces with boundary one considers, analogously, $\mathrm{Diff}^+(S, \partial S)$.

By a theorem of McCool and Hatcher–Thurston, the group $\mathrm{Mod}_{g,n}$ is finitely presented for all g and n. For $g = 1$, $n = 0$ one has [9] that $\mathrm{Mod}_1 = \mathrm{SL}_2(\mathbb{Z})$.

2.2.3. Symplectic structure. By a beautiful result of W. Goldman [8] the spaces $\mathcal{T}_g$, $\mathcal{T}_{g,n}$, $\mathcal{T}_{g,n}(L)$ have canonical symplectic structures. These are induced by the embedding of Teichmüller space in the $PSL_2(\mathbb{R})$ representation variety. These symplectic structures are usually called *the Weil–Petersson symplectic structure* ω_{WP}, and, by a result of S. Wolpert [28], one has $\omega_{WP} = \sum_{i=1}^{3g-3} d\ell_i \wedge d\tau_i$. Since ω_{WP}^{3g-3} is a volume form on $\mathcal{T}_g$, one can now try to compute the volume of $\mathcal{T}_g$ (or $\mathcal{T}_{g,n}$). The volumes of moduli spaces of curves are important and interesting numbers for enumerative geometers and string theorists, among others.

2.2.4. Riemann's Moduli Space. Any detailed discussion of M_g, $M_{g,n}$ and $M_{g,n}(L)$ is beyond the scope of the current note. This is the subject of numerous monographs and a meeting point for many research programs.

What is important for us here is that $M_g = \mathcal{T}_g / \mathrm{Mod}_g$, and hence it inherits a topology, and in fact, the structure of a complex space. This space is singular, but has finite quotient singularities. The morphism of spaces $\mathcal{T}_g \to M_g$ is a Mod_g-cover away from the locus of curves with (excessive) automorphisms.

It should be noted, however, that for many purposes such a description is inadequate. For instance, one would like M_g to have a universal property (represent a certain functor). In particular, any holomorphic family of Riemann surfaces, parametrised by a complex space T should induce a "classifying map" $T \to M_g$. Thus it is more appropriate to treat M_g as an *orbifold* (or, in an algebro-geometric setting, a *stack*). Then $\mathcal{T}_g$ is identified as the *orbifold universal covering* of M_g, while the *orbifold fundamental group* $\pi_1^{orb}(M_g, [X]) = \mathrm{Mod}_g$, if $\mathrm{Aut} X = \{e\}$. Similarly, the orbifold fundamental group of $M_{g,n}$, based at $([X], p_1, \ldots, p_n)$, is identified with $\mathrm{Mod}_{g,n}$, provided the base point has no automorphisms (as a pointed curve).

In the exceptional case $g = 1$, we can identify $M_1 \simeq \mathbb{C}$ (the j-line), but it is more satisfying to think of this space as $\mathbb{H}/SL_2(\mathbb{Z})$, with its two orbifold points corresponding to elliptic curves with automorphisms $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$.

For a gentle introduction to these matters we suggest R. Hain's lectures [9], W. Thurston's unpublished book [25], or, in an algebro-geometric setting, §2 of [18].

The space $M_{g,n}$ has a compactification $\overline{M}_{g,n}$, which is an irreducible algebraic variety and a coarse moduli space for stable curves [3].

3. Simple Closed Geodesics, Weil–Petersson volumes and the Witten conjecture. The first result of Mirzakhani we are going to discuss here concerns the number of simple closed geodesics on a complete hyperbolic surface X.

Primitive (or prime) geodesics – i.e., geodesics, covering precisely once their image – are Riemannian geometers' prime numbers. The number $c_X(L)$ of primitive closed geodesics on X of length $\leq L$ has been studied classically. In particular, we know, by the work of Delsarte, Huber and Selberg from the 1940 that $c_X(L) \sim e^L/L$ as $L \to \infty$. This asymptotic formula is *universal*, i.e., it is independent of the Riemann surface X.

Mirzakhani addressed in her Harvard thesis [14] the much more subtle question of determining the asymptotic behaviour of the number $s_X(L)$ of *simple* (i.e., non-intersecting) closed geodesics of length $\leq L$ on X. As shown in [17], one has $s_X(L) \sim C_X L^{6g-6}$. More generally, for $X \in M_{g,n}$ the asymptotics is $s_X(L) \sim C_X L^{6g-6+2n}$. Not only do we have a power law here (rather than an exponential), but, more surprisingly – this dependence is not universal, and the behaviour depends on X – via the proportionality constant C_X.

Even more unexpectedly, the answer to this question – which is a question about geodesics on an individual surface X – came as a by-product of computing Weil–Petersson volumes of moduli spaces. This circle of ideas is the subject of [15], [16] and [17].

3.1. Weil–Petersson volumes. The key result of [15] is the derivation of the *Mirzakhani volume formulas*, see Theorem 1.1, *ibid.* There it is shown that the numbers $V_{g,n}(L_1, \ldots, L_n)$, defined as the Weil–Petersson volumes $\mathrm{Vol}_{WP}(M_{g,n}(L_1, \ldots, L_n))$ are in fact *polynomials* of degree $\leq 3g-3+n$ in L_i, with coefficients in $\mathbb{Q}[\pi]$. These polynomials can be computed by a recursive procedure, *the Mirzakhani volume relations.* For example (see Table 1.1, *ibid.*) $V_{1,1}(L) = \frac{1}{24}(L^2 + 4\pi^2)$. In fact, the theorem contains a more precise statement about the coefficients of these polynomials. Prior to this work, only a few cases were known: $V_{0,n}$ [30], $V_{1,2}$ [21], and $V_{1,1}(L)$ [20]. It was also known, by work of S. Wolpert, that $V_{g,n}$ are rational multiples of $\pi^{6g-6+2n}$. Here we write, as usual, $V_{g,n} := V_{g,n}(0)$.

Consider first geodesics on a once-punctured hyperbolic torus X. Every line with rational slope determines a simple closed loop γ on X. This loop can be pushed towards the puncture, but within $[\gamma] \in \pi_1(X)$ there is unique simple closed geodesic of minimal length, $\ell(\gamma)$. This is very different from the case of the (non-punctured) torus, which is foliated by such geodesics. McShane proved [13] that summing over all geodesics one has $\sum_\gamma (1 + e^{-\ell(\gamma)}) = 1/2$. Mirzakhani showed [15] that this identity can be used to compute $V_{1,1} = \mathrm{Vol}(M_{1,1}) = \frac{\pi^2}{6}$. This is done by integrating both sides of the identity over $M_{1,1}$, and converting the left side into an integral over the covering space $M^*_{1,1} \to M_{1,1}$, consisting of pairs (X, γ), where γ is a simple closed geodesic on $X \in M_{1,1}$. Then one identifies $M^*_{1,1}$ with the quotient of $M_{1,1}$ by the group of Dehn twists of a simple closed curve, and uses the simple form of ω_{WP} in Fenchel–Nielsen coordinates to compute the integral.

The rest of [15] is occupied with 1) generalising McShane's identity to punctured surfaces with geodesic boundary and 2) developing a method for integration over $M_{g,n}(L)$. The method of integration involves passing to appropriate covering spaces of $M_{g,n}(L)$. For $L = 0$ one considers a simple closed curve γ and its $M^\gamma_{g,n} \to M_{g,n}$ consists of pairs (X, α), $\alpha \in \mathcal{O}_\gamma$. The Mirzakhani relations are obtained by a detailed analysis of these covering spaces, their symplectic geometry and their decompositions into simpler pieces.

3.2. Simple Closed Geodesics. The integration techniques on $M_{g,n}$ allowed Mirzakhani to obtain results on the asymptotics of $s_X(L)$. For a fixed simple closed geodesic γ she introduces a counting function

$$s_X(L, \gamma) = \#\{\alpha \in \mathrm{Mod}_{g,n} \cdot \gamma | \ell_\alpha(X) \leq L\},$$

and studies *its* asymptotics. The relation of the two problems is via the identity $s_X(L) = \sum_\gamma s_X(L, \gamma)$, where the sum is over the finitely many $\mathrm{Mod}_{g,n}$ cosets ("types of geodesics").

Here the key role is played by the space of measured geodesic laminations $\mathcal{ML}_{g,n}$ on a surface $X \in M_{g,n}$ [25]. This PL-space carries unique (up to scaling) $\mathrm{Mod}_{g,n}$-invariant measure, the Thurston measure μ_{Th}. After quotienting by scalars, this space gives *the*

Thurston boundary of $\mathcal{T}_{g,n}$. Let us denote by B_X the unit ball (with respect to the length function on X) in the space of measured geodesic laminations.

The function $s_X(L, \gamma)$ is in fact a polynomial in L. Mirzakhani proved ([14], [17]) that, as $L \to \infty$,

$$\frac{s_X(L,\gamma)}{L^{6g-6+2n}} \sim \frac{c(\gamma)B(X)}{b_{g,n}},$$

where $c(\gamma) \in \mathbb{Q}_{>0}$, $B(X) = \mu_{Th}(B_X)$ and $b_{g,n} = \int_{M_{g,n}} B(X)dX < \infty$. Here γ can be, more generally, not just a single closed geodesic, but a (rational) multi-curve.

She then used these results to obtain relative frequences of different types of simple closed geodesics. For example, the probability that a simple closed geodesic on a genus 2 surface will be separating turned out to be 1/7.

3.3. Tautological Classes and Witten's conjecture. The rational numbers $c(\gamma)$ arise as volumes of moduli spaces of *bordered* hyperbolic surfaces. More precisely, they are given as certain intersection numbers of line bundles over the moduli space of hyperbolic structures of the surface, obtained by cutting X along γ.

We recall that $\overline{M}_{g,n}$ carries tautological (orbifold) line bundles $(\mathcal{L}_i)_1^n$. The fibre of $\mathcal{L}_i$ over $(X, c_1, \ldots, c_n) \in M_{g,n}$ is $T^\vee_{X,c_i}$. Edward Witten [27] introduced a generating function $F(t_0, t_1, \ldots)$ encoding the intersection numbers of the various $\mathcal{L}_i$, for all g and n. This is a formal series in infinitely many variables. Witten conjectured that F is determined by two conditions: that $\partial^2_{t_0} F$ satisfies the KdV equation and that F satisfies the string equation. This is a conjecture about a recursive formula for the intersection numbers of the $\mathcal{L}_i$. The conjecture implied that $Z = e^F$ is annihilated by a certain collection of differential operators $\{L_n\}_{n\in\mathbb{Z}}$ satisfying the Virasoro relations. Witten's conjecture was based on the premise that since "gravity is unique", partition functions arising from different approaches to quantisation of two-dimensional gravity should coincide.

In the celebrated work [11], M. Kontsevich introduced an appropriate matrix model and proved the Witten conjecture by relating intersection numbers to enumeration of ribbon graphs. There exist another proof of the conjecture, due to A. Okounkov and R. Pandharipande, using Hurwitz theory and Gromow–Witten theory on $\mathbb{P}^1$.

In [16] M. Mirzakhani used her volume formulas to show that F satisfies the Virasoro constrants and to give a new proof of the Witten conjecture.

That stimulated a flurry of activity in the physics community as well, in relation to a subject known as "topological recursion". One can consult the works of B. Eynard and N. Orantin for these developments.

4. Complex Dynamics. The results discussed in the previous section are rooted in the symplectic geometry of $\mathcal{T}_{g,n}$ and $M_{g,n}$. We now discuss briefly some results of Mirzakhani which are related to the complex geometry of these spaces.

An extremely important question in the study of dynamical systems is the question of the behaviour of trajectories and their closures. In the best-case scenario, one is able to make statements about *generic* trajectories. In most cases, however, there exist trajectories exhibiting fractal behaviour of arbitrary complexity.

There are some notable exceptions to this. If we consider the straight-line flow on $S^1 \times S^1$, we know that the closure of a trajectory will be either a copy of S^1 (if the slope of the line is rational), or all of $S^1 \times S^1$, if the slope is irrational. A torus acts on itself by translation, i.e., it has a large group of automorphisms, and the fact that we can classify all orbits was considered a consequence of the presence of continuous symmetries. The subject of dynamics on homogeneous spaces is full of beautiful results, but undoubtedly one of the most interesting ones is M. Ratner's theorem. It states that if Γ is a lattice in a Lie group G, and if $H \subset G$ is a closed subgroup, generated by unipotents (e.g., $SL_2(\mathbb{R})$) then the closure $\overline{H \cdot p}$ of any H-orbit in G/Γ is itself an orbit $K \cdot p$, for some subgroup $H \subset K \subset G$.

It came then as a surprise that a similar result holds in the world of complex dynamics on moduli space. This is, as A. Zorich put it, *the magic wand of Eskin and Mirzakhani* [31].

Infinite geodesics in the moduli space (or in Teichmüller space) i.e., local isometric immersions $\gamma : \mathbb{R} \to M_g$ have been studied by differential geometers. By work of Masur and Veech it is known that the Teichmüller geodesic flow is ergodic, so $\overline{\gamma(\mathbb{R})} = M_g$ for most geodesics.

The complex analogues of geodesics are known as *complex geodesics* or *Teichmüller disks*, and these are *holomorphic* isometric immersions $\gamma : \mathbb{H} \to M_g$. One could call this setup *studying geodesic flow in complex time.* Already O. Teichmüller considered these objects and showed that they are ubiquitous: through each point and in each direction of M_g there passes a complex geodesic. A very bold and ambitious question is to determine the closures $\overline{\gamma(\mathbb{H})} \subset M_g$. One would naturally expect the presence of arbitrarily complicated fractal-like orbit closures. On the other hand, C. McMullen [12] showed that for $g = 2$ all closures are in fact algebraic subvarieties, and there are only three possibilities for them: an algebraic curve, a Hilbert modular surface or all of M_2 (a complex 3-fold).

It was thus a huge breakthrough when the work of Eskin–Mirzakhani [4], Eskin–Mirzakhani–Mohammadi [5] and Filip [7] demonstrated that all orbit closures $\overline{\gamma(\mathbb{H})} \subset M_g$ are algebraic subvarieties!

While M_g is totally inhomogeneous and Ratner's theorem is not applicable, there is still an $SL_2(\mathbb{R})$-action behind this result – but an action on a space mapping to M_g, rather than on M_g itself.

For starters, let us notice that while a compact Riemann surface X has no non-constant holomorphic functions, it has the next best thing: holomorphic 1-forms. Locally, a non-zero 1-form ω is exact, i.e., equals $f(z)dz$, and the holomorphic function f, wherever nonzero, determines a coordinate chart on X. Thus we can cover $X \setminus \{\text{zeros}(\omega)\}$ by an atlas, having translations as transition functions. Thus, the hyperbolic surface X becomes equipped with a singular flat metric.

Conversely, given non-zero vectors $v_1, \ldots, v_n \in \mathbb{R}^2$ and a permutation $\sigma \in S_n$, one forms the line segments, joining 0, v_1, v_1+v_2,..., $v_1+\cdots+v_n$ and 0, $v_{\sigma(1)}$, $v_{\sigma(1)}+v_{\sigma(2)}$,..., $v_{\sigma(1)}+\cdots+v_{\sigma(n)}$. These segments form the boundary of $2n$-gon, having pairs of parallel sides, and we can glue these by translation to obtain a closed topological surface. For technical reasons, one often includes the vertical direction as part of the structure and calls such a surface *translation surface.* We suggest [31] and [29] as two very pleasant introductions to translation surfaces.

A translation surface inherits a flat (singular) metric from $\mathbb{R}^2$, but has trivial holonomy, as the gluing is via translations. Thus the conical singularities (of the metric) have cone angles, which are integer multiples of 2π, say $2\pi(\alpha_1+1), 2\pi(\alpha_2+1), \ldots, 2\pi(\alpha_m+1)$, satisfying a Gauss–Bonet constraint $\alpha_1 + \cdots + \alpha_m = 2g - 2$, where g is the genus of the surface. The translation surface also inherits from $\mathbb{R}^2 \simeq \mathbb{C}$ a complex structure and a holomorphic 1-form (descending from dz). The one-form has zeroes of multiplicities $\alpha_1, \ldots, \alpha_m$ at the singular points of the metric.

Once the combinatorial data is fixed, one can vary the vectors $v_1, \ldots, v_n$ to obtain a family of translation surfaces over (a subset of) $\mathbb{C}^n = (\mathbb{R}^2)^n$.

The space of pairs $([X], \omega \in H^0(X, \Omega^1_X))$ is a holomorphic rank g vector bundle $\mathcal{H}_g$ over M_g. The multiplicities $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\sum_k \alpha_k = 2g - 2$ of the zeros of the 1-forms determine stratification $\mathcal{H}(\alpha)$ of $\mathcal{H}_g$. The strata are not vector bundles, but just varieties (orbifolds). The above vectors $v_1, \ldots v_n$ give coordinates (*period coordinates*) on the respective stratum, and one can use these to determine a volume form.

The standard $\mathbb{R}$-linear action of $GL_2(\mathbb{R})$ and $SL_2(\mathbb{R})$ on $\mathbb{C}$ gives rise to an action on $\mathcal{H}_g$. Let $\mathcal{H}_1(\alpha)$ be the subset of $\mathcal{H}(\alpha)$, consisting of translational surfaces with unit area, i.e., the "unit sphere bundle".

An important result of Masur and Veech from the 1980-ies says that the strata $\mathcal{H}_1(\alpha)$ have finite volume and that $SL_2(\mathbb{R})$ acts ergodically on each connected component of $\mathcal{H}_1(\alpha)$.

The crucial part of the rigidity proof is a measure classification, given in [4]. There it is shown that every ergodic, $SL_2(\mathbb{R})$-invariant probability measure on $\mathcal{H}(\alpha)$ is supported on a complex-analytic subvariety (suborbifold). This is a monumental work, which is, as of now, 214 pages long. The second part of the argument ([5], [4, Theorem 1.5]) is topological, and claims that any orbit closure is an affine invariant subvariety of $\mathcal{H}_1(\alpha)$. This subvariety is in fact algebraic, and moreover, it is *linear* in period coordinates.

These results open new horizons in the study of moduli spaces and undoubtedly there will be numerous applications in the years to come.

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ВЪРХУ ТВОРЧЕСТВОТО НА МАРИАМ МИРЗАХАНИ

Петър Далаков

Представяме кратък обзор на резултати на М. Мирзахани* в контекста на симплектичната и комплексна геометрия и динамика на пространства от модули.

*Мариам Мирзахани е първата и засега единствена жена, удостоена с най-престижната награда за математици – медала на Филдс. Ранната ѝ кончина (юли 2017) е тежка загуба за математиката, но животът и творчеството ѝ са вдъхновение за математиците от цял свят (бел. ред.)