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DIFFERENT PROBLEMS FOR "BEST" CONVEX INTERPOLATION OF $\mathbf{D^2}$ AND $\mathbf{D^3}$ CONVEX DATA*

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A non-linear approach to finding the "best" convex interpolation of convex data in $\mathbf{R^3}$ through curve networks has been proposed in [1]. However, these curve networks cannot be extended by convex functions in $C^{(1)}(\mathbf{R^2})$. We propose a new, non-linear approach, by means of which we extend the "best" convex curve network to a smooth convex function in $C^{(1)}(\mathbf{R^2})$. Let us note, that the "best" (convex) interpolation depends on the norm of the second derivative of the respective curve network.

1. Introduction. In the last 35 years, many results generalizing different aspects of "best" (constrained) interpolation of data in $\mathbf{R}^2$ by **functions** have appeared. The purpose of the current paper is not to give a detailed review of these results (this is done for example in [1]). Here we consider different approaches to extremal problems for **curve networks** in $\mathbf{R}^3$.

Let $V_j = (x_j, y_j) \in \mathbf{R}^2, j = 0, 1, 2, 3$ be four different points in $\mathbf{R}^2$. We suppose that V_1, V_2, V_3 are vertices of a triangle – $\triangle(V_1, V_2, V_3)$ in $\mathbf{R}^2$. The edges of this triangle are the segments $\mathbf{b}_1 := [V_1, V_2]$, $\mathbf{b}_2 := [V_2, V_3]$ and $\mathbf{b}_3 := [V_3, V_1]$. If we denote by $\mathbf{c}_i$ the segment $[V_0, V_i]$, $i = 1, 2, 3$, we suppose that the points $V_i = (x_i, y_i) \in \mathbf{R}^2$, $i = 1, 2, 3$ are such that $\mathbf{c}_i \cap \mathbf{c}_k = V_0$; $i, k = 1, 2, 3$; $i \neq k$ (Fig. 1). The last assumption requires that the point V_0 is strictly inside $\triangle(V_1, V_2, V_3)$. We denote by V the set of four points $V_j = (x_j, y_j) \in \mathbf{R}^2$, $j = 0, 1, 2, 3$ with these properties and notations.

Let z_j, $j = 0, 1, 2, 3$, be four different numbers in $\mathbf{R}$. We denote $Z := (z_0, z_1, z_2, z_3)$.

Definition 1.

(**i**) *We call* $\mathbf{D}^3$ ***data*** *the set* $(V, Z) := ((V_0, z_0), (V_1, z_1), (V_2, z_2), (V_3, z_3))$.

(**ii**) *We call* ***convex*** $\mathbf{D}^3$ ***data*** *the set* (V, Z) *if the point* $(V_0(x_0, y_0), z_0) \in \mathbf{R}^3$ *is* ***strictly*** *under the plane in* $\mathbf{R}^3$*, made of the points* $(V_1, z_1), (V_2, z_2), (V_3, z_3) \in \mathbf{R}^3$.

Remark 1. From here on we suppose that the set Z is such that (V, Z) is convex $\mathbf{D}^3$ data – briefly (V, Z) is convex data.

If θ is the segment $[a, b]$, we use the Hilbert space of one dimensional real functions:

$$L^2(\theta) = \{\text{all functions} f, \text{ defined on } \theta \text{ for which } \int_\theta (f)^2 < \infty\}.$$

If $f, g \in L^2(\theta)$ then the norm is $\|f\|_2 = \sqrt{\int_\theta (f)^2}$, and the inner product is $\langle f, g\rangle = \int_\theta fg$.

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Key words: triangulation, convex data in $\mathbf{R}^3$, shape preserving condition, curve network.

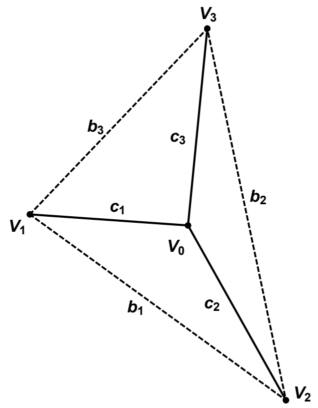

Fig. 1

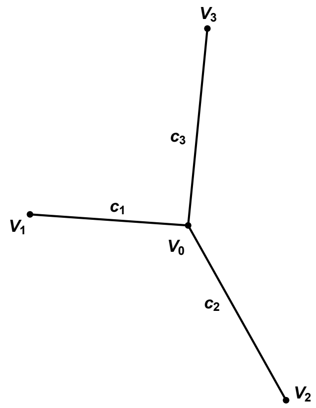

Fig. 2

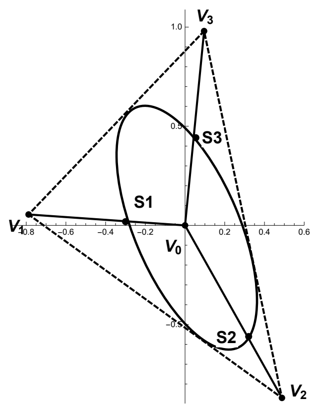

Fig. 3

In addition, we use the set of functions: $W^2(\theta) = \{\text{all functions} f, \text{for which} f, f', f'' \in L^2(\theta)\}$.

2. Problems for "best" interpolation of convex D^3 data by convex curve networks. Returning to Fig. 1, we denote

$$C := \bigcup_{i=1}^{3} c_i; \quad B := \bigcup_{i=1}^{3} b_i; \quad E := C \bigcup B; \quad \|c_i\| := \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2}, i = 1, 2, 3;$$

$$\|b_1\| := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}; \quad \|b_2\| := \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2};$$

$$\|b_3\| := \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}; \quad V := \{V_i\}_{i=1}^{3}.$$

Let the set Z be such that (V, Z) is convex data, and let $F(*, *)$ be an arbitrary two dimensional function, defined on E, for which $F(x_j, y_j) = z_j, \; j = 0, 1, 2, 3$ (i.e. F interpolates the data). The function $F(*, *)$ may be described by the following family of 6 univariate functions:

$$(1) \qquad \begin{cases} f_i(t) := F\left((1 - \dfrac{t}{\|c_i\|})V_0 + \dfrac{tV_i}{\|c_i\|}\right) = F\left(\overline{x}_i(t), \overline{y}_i(t)\right) \quad \text{where} \\ \overline{x}_i(t) = \left(1 - \dfrac{t}{\|c_i\|}\right)x_0 + \dfrac{tx_i}{\|c_i\|} \;\; ; \;\; \overline{y}_i(t) = \left(1 - \dfrac{t}{\|c_i\|}\right)y_0 + \dfrac{ty_i}{\|c_i\|} \;\; ; \\ \quad 0 \le t \le \|c_i\|; \quad i = 1, 2, 3 \end{cases}$$

and

$$(2) \qquad \begin{cases} g_i(t) := F\left(\widehat{x}_i(t), \widehat{y}_i(t)\right); \quad 0 \le t \le \|b_i\|; \quad i = 1, 2, 3 \;\; \text{where} \\ \widehat{x}_1(t) = \left(1 - \dfrac{t}{\|b_1\|}\right)x_1 + \dfrac{tx_2}{\|b_1\|}; \quad \widehat{x}_2(t) = \left(1 - \dfrac{t}{\|b_2\|}\right)x_2 + \dfrac{tx_3}{\|b_2\|}; \\ \widehat{x}_3(t) = \left(1 - \dfrac{t}{\|b_3\|}\right)x_3 + \dfrac{tx_1}{\|b_3\|}; \quad \widehat{y}_1(t) = \left(1 - \dfrac{t}{\|b_1\|}\right)y_1 + \dfrac{ty_2}{\|b_1\|}; \\ \widehat{y}_2(t) = \left(1 - \dfrac{t}{\|b_2\|}\right)y_2 + \dfrac{ty_3}{\|b_2\|}; \quad \widehat{y}_3(t) = \left(1 - \dfrac{t}{\|b_3\|}\right)y_3 + \dfrac{ty_1}{\|b_3\|}. \end{cases}$$

We supposed that $F(x_j, y_j) = z_j, j = 0, 1, 2, 3$. Then from (1) and (2) we have

$$(3) \qquad \begin{cases} f_i(0) = z_0, \quad i = 1, 2, 3; \\ f_1(\|c_1\| = g_1(0) = g_3(\|b_3\|) = z_1; \qquad f_2(\|c_2\| = g_2(0) = g_1(\|b_1\|) = z_2; \\ f_3(\|c_3\| = g_3(0) = g_2(\|b_2\|) = z_3. \end{cases}$$

For the sake of brevity we denote $F := \{f_i, g_i\}_{i=1}^3 := \{f_i(t), g_i(t)\}_{i=1}^3$. Let us introduce some function classes on the set E:

$L^2(E) = \{F = \{f_i, g_i\}_{i=1}^3 : f_i \in L^2([0, \|c_i\| \,]), g_i \in L^2([0, \|b_i\| \,])\}$,
$W^2(E) = \{F = \{f_i, g_i\}_{i=1}^3 : f_i, f_i', f_i'' \in L^2([0, \|c_i\| \,]), g_i, g_i', g_i'' \in L^2([0, \|b_i\| \,])\}$

and the main two classes (unconstrained and constrained ones):

$M(E) = \{F : F \in W^2(E), F = H|_E, H \in C^{(1)}(R^2), H(x_j, y_j) = z_j, j = 0, 1, 2, 3\}$,
$M^*(E) = \{F : F \in M(E), f_i$ and g_i are convex functions in $[0, \|c_i\| \,]$ and $[0, \|b_i\|]\}$.

Remark 2. The smoothness assumption on $M(E)$, that F is the restriction to E of some differentiable in $\mathbf{R^2}$ function of two variables $H \in C^{(1)}(\mathbf{R^2})$, which interpolates the data, should be understood in the following equivalent way. If we consider the set of functions $\{f_i, g_i\}_{i=1}^3$ as curves in $\mathbf{R^3}$ then in every point $V_j, j = 0, 1, 2, 3$ they have a common tangent plane. In this case (due to (1), (2), (3)), V_0 is a common point of $\{f_1, f_2, f_3\}$, V_1 is a common point of $\{g_3, f_1, g_1\}$, V_2 is a common point of $\{g_1, f_2, g_2\}$ and V_3 is a common point of $\{g_2, f_3, g_3\}$.

Definition 2. *We call a* ***curve network*** *the family of functions* $F := \{f_i, g_i\}_{i=1}^3$.

For $F \in W^2(E)$ let us consider the couple of functionals:

$$\sigma(F) = \sigma(\{f_i, g_i\}_{i=1}^3) = \sum_{i=1}^3 \int_0^{\|c_i\|} [f_i''(t)]^2 dt + \sum_{i=1}^3 \int_0^{\|b_i\|} [g_i''(t)]^2 dt =: \int_C [F'']^2 + \int_B [F'']^2,$$

$$\rho(F) = \rho(\{f_i, g_i\}_{i=1}^3) = \sum_{i=1}^3 \int_0^{\|c_i\|} [f_i''(t)]^2 dt =: \int_C [F'']^2.$$

The functionals $\sigma(*)$ and $\rho(*)$ determine two **different approaches** to extremal problems

$$(4) \quad \textbf{First approach} \quad \begin{cases} (\mathbf{P}_\sigma) \text{ Find } F \in M(E) \text{ such that } \sigma(F) = \inf\limits_{F \in M(E)} \sigma(F), \\ (\mathbf{P}_\sigma^*) \text{ Find } F \in M^*(E) \text{ such that } \sigma(F) = \inf\limits_{F \in M^*(E)} \sigma(F); \end{cases}$$

$$(5) \quad \textbf{Second approach} \quad \begin{cases} (\mathbf{P}_\rho) \text{ Find } F \in M(E) \text{ such that } \rho(F) = \inf\limits_{F \in M(E)} \rho(F), \\ (\mathbf{P}_\rho^*) \text{ Find } F \in M^*(E) \text{ such that } \rho(F) = \inf\limits_{F \in M^*(E)} \rho(F). \end{cases}$$

Remark 3. Let us consider the points $\{V_j\}_{j=0}^3$ as vertices of a triangulation of three triangles – $\triangle(V_0, V_1, V_2), \triangle(V_0, V_2, V_3), \triangle(V_0, V_1, V_3)$ (Fig. 1). The difference between the **First approach** (4) and the **Second approach** (5) is that in (4) we are minimizing over all the edges of the triangulation (the set $E = C + B = \bigcup_{i=1}^3 c_i + \bigcup_{i=1}^3 b_i$), whereas in (5) we are minimizing over the inside edges (the set $C = \bigcup_{i=1}^3 c_i$ in Fig. 2). This does not make (5) easier than (4), because in bought cases F is in the set of the smooth curve networks $M(E)$ or $M^*(E) \subset M(E)$.

The results of Nielson [5] (proved in a more general situation) imply:

Theorem A. *Problem* $(\mathbf{P}_\sigma)$ *has a unique solution* $S_\sigma \in M(E)$. *The curve network* S_σ *is a polynomial of degree* 3 *on* c_i *and* b_i, $i = 1, 2, 3$.

If $F = \{f_i, g_i\}_{i=1}^3 \in M^2(E)$ we define $F'' := \{f_i'', g_i''\}_{i=1}^3$.

We give in [1] a new proof of the result in [5] and by means of some technics from [3], [4] and [2], we prove (in a more general situation):

Theorem B. *Problem* $(\mathbf{P}_\sigma^*)$ *has a unique solution* $S_\sigma^* \in M(E)$. *The curve network* S_σ^* *is such that* $(S_\sigma^*)''$ *is a positive part of a linear function on* c_i *and* b_i, $i = 1, 2, 3$.

There is a strong reason to change here the problem $(\mathbf{P}_\sigma^*)$ by $(\mathbf{P}_\rho^*)$. This reason will be explained in the next section. In order to prove the main result, we need some additional notations and results.

Let us consider the following 4 linear systems:

$$(6)\quad \left\{\begin{array}{l} \lambda_1 \dfrac{x_1 - x_0}{\|c_1\|} + \lambda_2 \dfrac{x_2 - x_0}{\|c_2\|} + \lambda_3 \dfrac{x_3 - x_0}{\|c_3\|} \\ = 0 \\ \lambda_1 \dfrac{y_1 - y_0}{\|c_1\|} + \lambda_2 \dfrac{y_2 - y_0}{\|c_2\|} + \lambda_3 \dfrac{y_3 - y_0}{\|c_3\|} \\ = 0 \\ \lambda_1 + \lambda_2 + \lambda_3 = 1 \end{array}\right\}, \left\{\begin{array}{l} \mu_1^1 \dfrac{x_2 - x_1}{\|b_1\|} + \mu_2^1 \dfrac{x_0 - x_1}{\|c_1\|} + \mu_3^1 \dfrac{x_3 - x_1}{\|b_3\|} \\ = 0 \\ \mu_1^1 \dfrac{y_2 - y_1}{\|b_1\|} + \mu_2^1 \dfrac{y_0 - y_1}{\|c_1\|} + \mu_3^1 \dfrac{y_n - y_1}{\|b_3\|} \\ = 0 \\ \mu_1^1 + \mu_2^1 + \mu_3^1 = 1 \end{array}\right\},$$

$$(7)\quad \left\{\begin{array}{l} \mu_1^2 \dfrac{x_3 - x_2}{\|b_2\|} + \mu_2^2 \dfrac{x_0 - x_2}{\|c_2\|} + \mu_3^2 \dfrac{x_1 - x_2}{\|b_1\|} \\ = 0 \\ \mu_1^2 \dfrac{y_3 - y_2}{\|b_2\|} + \mu_2^2 \dfrac{y_0 - y_2}{\|c_2\|} + \mu_3^2 \dfrac{y_1 - y_2}{\|b_1\|} \\ = 0 \\ \mu_1^2 + \mu_2^2 + \mu_3^2 = 1 \end{array}\right\}, \left\{\begin{array}{l} \mu_1^3 \dfrac{x_1 - x_3}{\|b_3\|} + \mu_2^3 \dfrac{x_0 - x_3}{\|c_3\|} + \mu_3^3 \dfrac{x_2 - x_3}{\|b_2\|} \\ = 0 \\ \mu_1^3 \dfrac{y_1 - y_3}{\|b_3\|} + \mu_2^3 \dfrac{y_0 - y_3}{\|c_3\|} + \mu_3^3 \dfrac{y_2 - y_3}{\|b_2\|} \\ = 0 \\ \mu_1^3 + \mu_2^3 + \mu_3^3 = 1 \end{array}\right\}.$$

The determinants of (6) and (7) are not equal to 0, since V_1, V_2, V_3 are vertices of a triangle and V_0 is strictly inside this triangle. Then the solutions $\{\lambda_i\}_{i=1}^3$; $\{\mu_i^s\}_{i=1}^3$, $s = 1, 2, 3$, always exist and it is easy to see that:

$$(8)\qquad \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \quad \text{and} \quad \mu_1^s > 0, \mu_2^s < 0, \mu_3^s > 0, \ s = 1, 2, 3.$$

We define four numbers (generalized second divided differences) by:

$$(9)\quad \left\{\begin{array}{l} w := \dfrac{\lambda_1}{\|c_1\|}(z_1 - z_0) + \dfrac{\lambda_2}{\|c_2\|}(z_2 - z_0) + \dfrac{\lambda_3}{\|c_3\|}(z_3 - z_0); \\ d_1 := \dfrac{\mu_1^{(1)}}{\|b_1\|}(z_2 - z_1) + \dfrac{\mu_2^{(1)}}{\|c_1\|}(z_0 - z_1) + \dfrac{\mu_3^{(1)}}{\|b_3\|}(z_3 - z_1); \\ d_2 := \dfrac{\mu_1^{(2)}}{\|b_2\|}(z_3 - z_2) + \dfrac{\mu_2^{(2)}}{\|c_2\|}(z_0 - z_2) + \dfrac{\mu_3^{(2)}}{\|b_1\|}(z_1 - z_2); \\ d_3 := \dfrac{\mu_1^{(3)}}{\|b_3\|}(z_1 - z_3) + \dfrac{\mu_2^{(3)}}{\|c_3\|}(z_0 - z_3) + \dfrac{\mu_3^{(3)}}{\|b_2\|}(z_2 - z_3). \end{array}\right.$$

Now we define four linear networks $\widehat{B}$, $\widehat{D}_1$, $\widehat{D}_2$, and $\widehat{D}_3$ on E called in [1] **basic**

networks:

$$\widehat{B} := \begin{cases} \lambda_1\left(1 - \dfrac{t}{\|c_1\|}\right), & 0 \le t \le \|c_s\| \quad \text{i.e. on} \quad c_1; \\ \lambda_2\left(1 - \dfrac{t}{\|c_2\|}\right), & 0 \le t \le \|c_2\| \quad \text{i.e. on} \quad c_2; \\ \lambda_3\left(1 - \dfrac{t}{\|c_3\|}\right), & 0 \le t \le \|c_3\| \quad \text{i.e. on} \quad c_3; \\ 0 \quad \text{otherwise}, \end{cases}$$

$$\widehat{D}_1 := \begin{cases} \mu_1^1\left(1 - \dfrac{t}{\|b_1\|}\right), & 0 \le t \le \|b_1\| \quad \text{i.e. on} \quad b_1; \\ \mu_2^1\left(1 - \dfrac{t}{\|c_1\|}\right), & 0 \le t \le \|c_1\| \quad \text{i.e. on} \quad c_1; \\ \mu_3^1\left(1 - \dfrac{t}{\|b_3\|}\right), & 0 \le t \le \|b_3\| \quad \text{i.e. on} \quad b_3; \\ 0 \quad \text{otherwise}, \end{cases}$$

and for $s = 2, 3$:

$$\widehat{D}_s := \begin{cases} \mu_1^s\left(1 - \dfrac{t}{\|b_s\|}\right), & 0 \le t \le \|b_s\| \quad \text{i.e. on} \quad b_s; \\ \mu_2^s\left(1 - \dfrac{t}{\|c_s\|}\right), & 0 \le t \le \|c_s\| \quad \text{i.e. on} \quad c_s; \\ \mu_3^s\left(1 - \dfrac{t}{\|b_{s-1}\|}\right), & 0 \le t \le \|b_{s-1}\| \quad \text{i.e. on} \quad b_{s-1}; \\ 0 \quad \text{otherwise}. \end{cases}$$

If $F = \{f_i, g_i\}_{i=1}^3 \in M(E)$ and $\widehat{F} = \{\widehat{f}_i, \widehat{g}_i\}_{i=1}^3 \in L^2(E)$ let us define the number $\langle F'', \widehat{F} \rangle := \int_E F''\widehat{F}$. This number is an inner product in $W(E)$.

We proved in [1] the following two lemmas:

Lemma A (Peano-type)**.** *Let $F = \{f_i, g_i\}_{i=1}^3 \in W(E)$. Then $F \in M(E)$ if and only if:*

(10) $$\langle \widehat{B}, F'' \rangle_E = w; \quad \textit{and} \quad \langle \widehat{D}_s, F'' \rangle_E = d_s \quad \textit{for} \quad s = 1, 2, 3,$$

*where $\widehat{B}, \widehat{D}_1, \widehat{D}_2, \widehat{D}_3$ are the **basic networks** and w, d_1, d_2, d_3 are the numbers defined in* (9)*.*

Lemma B. *The data (V, Z) is convex if and only if for the numbers defined in* (9) *we have $w > 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$.*

Let us state the following theorem, which is the main result of the paper. Here $(x)_+$ denotes the positive part of x.

Theorem 1. *Assume that (V, Z) is convex data. Then there exists a unique curve network $S^* \in W(E)$, such that $(S^*)''$ is of the form*

(11) $$S'' = \left(\alpha\widehat{B} + \sum_{i=1}^3 \beta_i \widehat{D}_i\right)_+ ,$$

where the coefficients $\alpha, \beta_1, \beta_2, \beta_3$ are determined as the unique solution $\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*$ of *the nonlinear system of equations*

$$(12)\qquad \begin{cases} \displaystyle\int_C \widehat{B}\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ = w \\ \displaystyle\int_C \widehat{D}_1\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ = \frac{-\lambda_1}{2\mu_2^1} \\ \displaystyle\int_C \widehat{D}_2\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ = \frac{-\lambda_2}{2\mu_2^2} \\ \displaystyle\int_C \widehat{D}_3\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ = \frac{-\lambda_3}{2\mu_2^3}. \end{cases}$$

Proof. We are using some technics from [3] and [7] for the proof. These technics are useful for constructing numerical methods for solving systems of nonlinear equations.

We are going to prove that the nonlinear system (12) has always a unique solution $\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*$. To prove that we need to apply the following

Statement. If $\Phi : \mathbf{R}^m \Rightarrow \overline{\mathbf{R}} \subset \mathbf{R}$ is a convex and differentiable function of m unknowns and if for some non-zero vector $\mathbf{R}^m \ni \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ we have

$$\lim_{t\to\infty}\Phi(t\gamma_1, t\gamma_2, \ldots, t\gamma_m) = \infty \quad\text{ and }\quad \lim_{t\to-\infty}\Phi(t\gamma_1, t\gamma_2, \ldots, t\gamma_m) = \infty,$$

then Φ reaches a minimum in $\gamma \in \mathbf{R}^m$.

This statement is often used for extremal problems in convex analysis.

Let us define $a_i := -\dfrac{\lambda_i}{\mu_2^i}; i = 1, 2, 3$ and Φ as

$$\begin{aligned}\Phi(\alpha,\beta_1,\beta_2,\beta_3) &:= \int_C \Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+^2 - 2w\alpha - \sum_{i=1}^{3}a_i\beta_i \\ &= \Psi(\alpha,\beta_1,\beta_2,\beta_3) + L(\alpha,\beta_1,\beta_2,\beta_3),\end{aligned}$$

where $\Psi(\alpha,\beta_1,\beta_2,\beta_3)$ is the function $\displaystyle\int_C \Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+^2$ and $L(\alpha,\beta_1,\beta_2,\beta_3)$ is the hyperplane $-2w\alpha - \displaystyle\sum_{i=1}^{3}a_i\beta_i$.

Obviously Φ is a differentiable function of 4 unknowns. Furthermore Φ is a strictly convex function, because Ψ is strictly convex and L is a hyperplane. We have

$$(13)\qquad \begin{cases} \dfrac{\partial\Phi}{\partial\alpha} = 2\displaystyle\int_C \widehat{B}\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ - 2w, \\ \dfrac{\partial\Phi}{\partial\beta_1} = 2\displaystyle\int_C \widehat{D_1}\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ - a_1, \\ \dfrac{\partial\Phi}{\partial\beta_2} = 2\displaystyle\int_C \widehat{D_2}\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ - a_2, \\ \dfrac{\partial\Phi}{\partial\beta_3} = 2\displaystyle\int_C \widehat{D_3}\Big(\alpha\widehat{B}+\sum_{i=1}^{3}\beta_i\widehat{D}_i\Big)_+ - a_3. \end{cases}$$

In addition, we have $w > 0$ from Lemma B since (V, Z) is convex data, and we obtain $a_1 > 0, a_2 > 0, a_3 > 0$ from (8). Then it is easy to see that

$$\lim_{t\to\infty} \Phi(t\alpha, t\beta_1, t\beta_2, t\beta_3) = \infty \quad \text{and} \quad \lim_{t\to-\infty} \Phi(t\alpha, t\beta_1, t\beta_2, t\beta_3) = \infty.$$

Finally, it follows from the **Statement** that Φ reaches a minimum at the point $(\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*) \in \mathbf{R}^4$. Since Φ is a strictly convex function everywhere in $\mathbf{R}^4$ this minimum is a unique local minimum. But then for the partial derivatives of Φ we have

$$\begin{aligned} &\frac{\partial\Phi(\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*)}{\partial\alpha} = 0, \quad \frac{\partial\Phi(\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*)}{\partial\beta_1} = 0,\\ &\frac{\partial\Phi(\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*)}{\partial\beta_2} = 0, \quad \frac{\partial\Phi(\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*)}{\partial\beta_3} = 0. \end{aligned} \tag{14}$$

From (13) and (14) we obtain that $\alpha^*, \beta_1^*, \beta_2^*, \beta_3^*$ is the unique solution of the nonlinear system of equations:

$$\begin{cases} \int_C \widehat{B}\left(\alpha\widehat{B} + \sum_{i=1}^{3} \beta_i \widehat{D}_i\right)_+ = w, \\ \int_C \widehat{D_1}\left(\alpha\widehat{B} + \sum_{i=1}^{3} \beta_i \widehat{D}_i\right)_+ = \frac{1}{2}\, a_1, \\ \int_C \widehat{D_2}\left(\alpha\widehat{B} + \sum_{i=1}^{3} \beta_i \widehat{D}_i\right)_+ = \frac{1}{2}\, a_2, \\ \int_C \widehat{D_3}\left(\alpha\widehat{B} + \sum_{i=1}^{3} \beta_i \widehat{D}_i\right)_+ = \frac{1}{2}\, a_3, \end{cases}$$

which is the same as (12) since $a_i = -\dfrac{\lambda_i}{\mu_2^i}; i = 1, 2, 3$.

We can take the unique curve network S^* (from the formulation of Theorem 1) to be such that $(S^*)'' = \left(\alpha^*\widehat{B} + \sum_{i=1}^{3} \beta_i^* \widehat{D}_i\right)_+$ $\square$

3. Problems of blending. A method for interpolation and blending of scattered data (V_j, z_j) in $\mathbf{R^3}$ is presented in [5]. The method consists of three separate steps:

(i) *Triangulation.* The points V_j are used as vertices of a triangulation of a domain in $\mathbf{R^2}$.

(ii) *Construction of the minimum norm network S.*

(iii) *Blending.* S is extended to the entire domain by means of a blending method. A detailed description of this step can be found in [5] and [6].

We called $S_\sigma^* \in M(E)$ "an edge convex network" in [1] and we made the remark that: "The important problem of extending the edge convex network into $C^{(1)}(\mathbf{R^2})$-function which is convex everywhere is very hard for general data".

In fact it is possible to construct examples such that S_σ^* cannot be extended into a $C^{(1)}(\mathbf{R^2})$-function G, convex everywhere and such that $S_\sigma^* = G|_C$. This is the main reason to change the functional $\sigma(*)$ to $\rho(*)$ and to consider the second approach (5). In this case we have:

Theorem 2. *(a) It is the unique curve network S^* from Theorem* 1 *which solves problem* $(\mathbf{P}_\rho^*)$. *(b) It is possible to extend the curve network S^* into $C^{(1)}(\mathbf{R^2})$-function*

G, convex everywhere and such that $S^* = G|_C$.

Proof. Theorem 2 is a constructive theorem – we can give an algorithm to construct the function G. If (V, Z) is convex data then a sketch of this construction is:

First stage. Using (6), (7) and (9) we find the numbers $\lambda_i, \mu_i^s, w > 0, d_i > 0$ and the basic curve networks $\widehat{B}, \widehat{D}_i \in L^2(E)$.

Second stage. Using Theorem 1 we find $(S^*)''$ – the main part of the construction. We find $(S^*)'' \in L^2(E)$ applying the Newton's method to solve the nonlinear system (12). If $\varepsilon > 0$ is a positive number the scheme of the Newton's method is:

FIRST STEP (initialization). We find the solution $\left(\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}, \beta_3^{(0)}\right)$ of the following **linear system** of equations:

$$\begin{cases} \displaystyle\int_C \widehat{B}\left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = w, \\ \displaystyle\int_C \widehat{D}_1\left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_1}{2\mu_2^1}, \\ \displaystyle\int_C \widehat{D}_2\left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_2}{2\mu_2^2}, \\ \displaystyle\int_C \widehat{D}_3\left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_3}{2\mu_2^3}. \end{cases}$$

SECOND STEP (for $\mathbf{k = 1, 2, \ldots}$) We find the solution $\left(\alpha^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \beta_3^{(k)}\right)$ of the following **linear system** of equations:

$$\begin{cases} \displaystyle\int_C \widehat{B}\left(\alpha^{(k-1)}\widehat{B} + \sum_{i=1}^{3}\beta_i^{(k-1)}\widehat{D}_i\right)_+^0 \left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = w, \\ \displaystyle\int_C \widehat{D}_1\left(\alpha^{(k-1)}\widehat{B} + \sum_{i=1}^{3}\beta_i^{(k-1)}\widehat{D}_i\right)_+^0 \left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_1}{2\mu_2^1}, \\ \displaystyle\int_C \widehat{D}_2\left(\alpha^{(k-1)}\widehat{B} + \sum_{i=1}^{3}\beta_i^{(k-1)}\widehat{D}_i\right)_+^0 \left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_2}{2\mu_2^2}, \\ \displaystyle\int_C \widehat{D}_3\left(\alpha^{(k-1)}\widehat{B} + \sum_{i=1}^{3}\beta_i^{(k-1)}\widehat{D}_i\right)_+^0 \left(\alpha\widehat{B} + \sum_{i=1}^{3}\beta_i\widehat{D}_i\right) = \frac{-\lambda_3}{2\mu_2^3}. \end{cases}$$

Here $(x)_+^0$ denotes 0 if $x \leqq 0$ and 1 if $x > 0$.

THIRD STEP (stop criterium). If

$$\left(\alpha^{(k-1)} - \alpha^{(k)}\right)^2 + \sum_{i=1}^{3}\left(\beta_i^{(k-1)} - \beta_i^{(k)}\right)^2 \leqq \varepsilon$$

then GoTo **END** else GoTo **SECOND STEP**.

We can prove that (i) for every $k = 0, 1, 2, \ldots$ the linear system has a unique solution and (ii) for these solutions we have the following positive and monotone properties: $0 < \alpha^{(k)} < \alpha^{(k+1)}$; $0 < \beta_i^{(k)} < \beta_i^{(k+1)}$, $i = 1, 2, 3$. Then, using the positive property we can prove that $S^* \in M(E)$. Furthermore, using the inequality of Cauchy–Buniakowski–

Schwarz and using Lemma A we can prove that S^* solves problem $(\mathbf{P}^*_\rho)$. This proves Theorem 2(a).

Third stage. Let $\overline{(S^*)''}$ be the support of $(S^*)''$ on C and the points $S1, S2, S3$ be the endpoints of $\overline{(S^*)''}$. Then we can prove that there exists an ellipse with the properties: (i) it is inscribed in $\triangle(V_1, V_2, V_3)$; (ii) the point V_0 is the very center of the ellipse and (iii) the ellipse passes through the points $S1, S2, S3$ (Fig. 3). We can compute the equation of this ellipse and then find a function $G(*, *)$, convex and smooth everywhere in $\mathbf{R^2}$ and such that $S^* = G|_C$. This proves the Theorem 2(b). $\square$

4. Conclusion. Let us note two important aspects of this paper.

First, we can obtain $\mathbf{D^2}$ case as a limit of $\mathbf{D^3}$ case by the following reasoning.

Let us suppose that the point V_0 is *very close* to the segment $[V_1, V_2]$, but still strictly inside the $\triangle(V_1, V_2, V_3)$ and that the point V_3 is *very close* to the point V_0, but $\triangle(V_1, V_2, V_3)$ is still non-degenerate one. This situation is *very close* to the $\mathbf{D^2}$ case if the points $V_1(x_1, y_1), V_2(x_2, y_2)$ are such that $y_1 = y_2 = 0$. In this case the following holds: $0 < \lambda_3 \approx 0; 0 < \lambda_1 \approx \dfrac{1}{2}; 0 < \lambda_2 \approx \dfrac{1}{2}$ and the ellipse degenerates to the segment $[S1, S2] \subset [V_1, V_2]$.

Second. An interesting and important generalization is to consider the case when we have: (i) scattered convex data $(x_i, y_i, z_i) \in \mathbf{R^3}, i = 1, 2, \ldots, n > 4$; (ii) the points $V_i(x_i, y_i) \in \mathbf{R^2}$ are used as vertices of a triangulation T of a domain in $\mathbf{R^2}$; (iii) the triangulation T is such that the boundary edges form a closed convex polygon, and (iv) the number of the edges of T is $3(n-2)-(k-3)$, where k denotes the number of boundary edges. We have considered in this paper the case when $n = 4, k = 3$. For $n = 5$, we have two cases $k = 4$ and $k = 3$.

The case $(n = 5, k = 3)$ is of a special importance in terms of applications. However, the fact that there are two points inside the polygon (instead of just one) leads to more sophisticated computations and is a matter of future work.

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РАЗЛИЧНИ ПРОБЛЕМИ ЗА „НАЙ-ДОБРО“ ИЗПЪКНАЛО ИНТЕРПОЛИРАНЕ НА $\mathbf{D^2}$ И $\mathbf{D^3}$ ИЗПЪКНАЛИ ДАННИ

Георги Илиев

Известен е нелинеен подход за намиране на „най-добра“ изпъкнала интерполация на изпъкнали данни в $\mathbf{R^3}$ чрез мрежи от криви [1]. За съжаление тези мрежи от криви не могат да бъдат продължени с изпъкнали функции от $C^{(1)}(\mathbf{R^2})$. В настоящата работа предлагаме нов, нелинеен подход, чрез който успяваме да продължим „най-добрата“ изпъкнала мрежа от криви до гладка изпъкнала функция от $C^{(1)}(\mathbf{R^2})$. Да отбележим, че „най-добра (изпъкнала) интерполация“ зависи от нормата на втората производна на съответната мрежа от криви.