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TEACHING MATHEMATICS WITH COMPUTER SYSTEMS*

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Notions lie at the foundation of the scientific knowledge systems and their development is a very important educational objective. This paper presents our pedagogical approach towards a better understanding of the mathematical notions at university level using computer software. In problem solving we use the power of Wolfram Mathematica for calculation, visualization and investigation. This allows the students to demonstrate and compare the properties of similar notions and to modify the parameters of the object under exploration. Thus the mathematical software assists students to understand the mathematical notions more deeply. The notions development is a long process, following to the process of a scientific discovery.

I. Introduction. The notions are the base of scientific knowledge. Notions in science are permanently evolving. The existing notions are enriched, concretized and summarized. New notions appear. This goes on in a spiral and reflects the development of science.

The importance of the notions in the scientific knowledge determines the attention to the notions development and puts it as one of the main educational objectives. This process is long and complicated and has been studied by many authors from different perspectives – psychology-cognitive, pedagogical, and in our case – mathematical aspect [2, 3, 4, 5, 9, 10], and others. Teachers have to know the specifics of the notions development and its steps, how to manage this process, criteria for understanding notions, typical mistakes and how to prevent them [1]. A. V. Usova accepts the following steps for the notions development [2, 3]:

1) Perception: object observation, teacher demonstrations, schemes, modeling, simulations. Attention is directed to the object properties, the relationships between the objects observed. Graphics, and animations are used for visualization.

2) Common essential properties for the class of objects observed; conclusion about the class properties.

3) Abstraction: from concrete examples to generalization and conclusion.

4) Definition: if possible to underline the properties of the whole class of objects and to identify the specifics of the studied object.

5) Training the essential properties trough simple tasks.

6) Studying the relationship of the notion using other notions.

*Key words: Mathematical software in education, mathematical notions.

7) Application of the notion in elementary situations. Students are learning to apply the notion in simple situations - through simple tasks. Relationships with other notions are also practiced.

8) Classification of the notions. The aim is to see the relationship between the notions in a common system. This gives an understanding of the role of the classification and the systematization of knowledge.

9) Application of the notion for solving creative tasks. Tasks with higher complexity take place in this step. It creates links to other systems of notions (in the same discipline, in other disciplines).

10) Enriching the notion. The notion is complemented by new essential properties and in this way the knowledge becomes fuller and more complete.

11) The given notion is used as a base for the creation of new notions and new relations with other notions. In this way the given notion is under perpetual development and is further included in new relations and new conceptual systems.

The notions that were introduced to the students using the steps of the above process stay longer in their memory and the students build deeper, conscious understanding.

The use of mathematical software provides rich opportunities to develop notions in various fields of mathematical knowledge. So far, there is no research that reveals these opportunities and compares the results of developing notions by using software and in the traditional way.

This article does not aim at exhaustive research on how to use computer algebra systems to develop mathematical notions. The goal is simpler. We present examples from our experience that show how to use Mathematica system of symbolic calculations at university level, using its features for doing symbolic, algebraic and graphical manipulations with computers. Our experience is based on the steps given in [2, 3]. In the examples below we have used the features of Mathematica for fast calculations and visualization of the properties of the notions when solving problems:

- Graphically illustrating the properties of the studied object;
- Tracking changes in the properties by changing the parameters of the object studied through animation, modeling and simulation;
- Fast calculations;
- Finding limits of the notion trough counter-example.

II. An example of teaching mathematical notions with Mathematica. We will follow the steps of developing the notion "tangent to a curve".

First, we involve the notion "tangent" when γ is a curve of the differentiable function $y = f(x)$ defined on $[a; b]$.

1) Discovering (perceiving) properties leading to the notion. We draw a curve γ of the function $y = 6\sin(x)$ on the interval $[0; \pi]$ and investigate how the lines are located in the plane relative to the curve γ.

The following cases are possible (Fig. 1):

1) The line has no common points with the curve γ (for example the line q).

2) The line has one common point with the curve γ (for example lines t_A and s).

3) The line has more than one common point with the curve γ (for example the line $p(AB)$ crosses the given curve γ at points A and B).

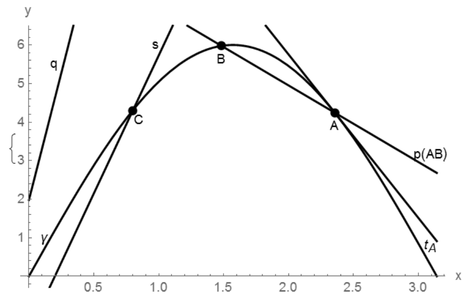

Fig. 1. Locations of the lines relative to a given curve in the plane

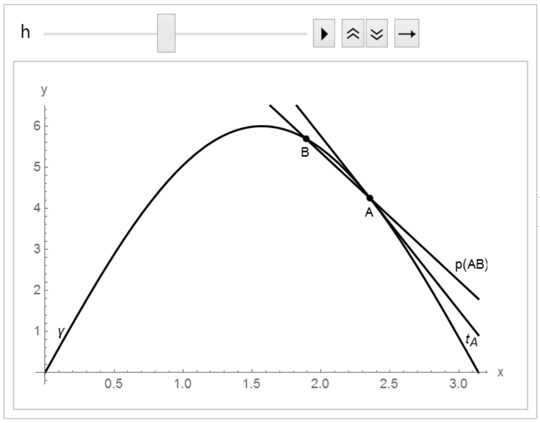

Fig. 2. The consequtive lines $p(AB)$ move to the line t_A

We ask the question: What happens with the line $p(AB)$ when the point B is moved towards the point A without leaving the curve γ? With animation (Fig. 2), we can see the property: the lines $p(AB)$ move to the line t_A. The code of the animation is given below.

```
In[3]:= f[x_] = 6 * Sin[x];
f1[x_] = D[f[x], x];
x0 = (3 * π)/4;
b[h_] = {x0 + h, f[x0 + h]};
a0 = {x0, f[x0]};
g[h_, x_] = (x * f[x0] - b[h][[1]] * f[x0] - x * b[h][[2]] + x0 * b[h][[2]])/(x0 - b[h][[1]]);
s[h_] := Plot[{f[x], g[h, x], f1[x0] * (x - x0) + f[x0]}, {x, 0, π}, PlotRange → {-0.3, 6.5},
  AxesLabel → {"x ", "y"},
  Epilog → {PointSize[0.015], Point[a0], Point[b[h]], Text["p(AB)", {3.1, g[h, 3.1] + 0.7}],
    Text["tA", {3.1, f1[x0] * (3.1 - x0) + f[x0] - 0.3}], Text["A", a0 + {0, -0.35}],
    Text["γ", {0.12, 1}], Text["B", b[h] + {0, -0.35}]}, PlotStyle → Directive[Black]]
Animate[s[h], {h, -5/9 * π/2, 0}, AnimationRunning → False]
```

The animation helps the students to understand that the property does not depend on which side of A the point B moves along γ to it.

Instead of point A, we fix another point D. It is obviously that through point $D \in \gamma$ there is a line t_D having the same property.

We demonstrate that the lines s and t_C are different lines.

2) We can summarize the common property of the class of lines t_D: The lines $p(DF)$, which pass through the points D and F belonging to γ, move to the line t_D, when F moves to D along γ.

Using specific examples, we illustrate that the property discovered for the class of line

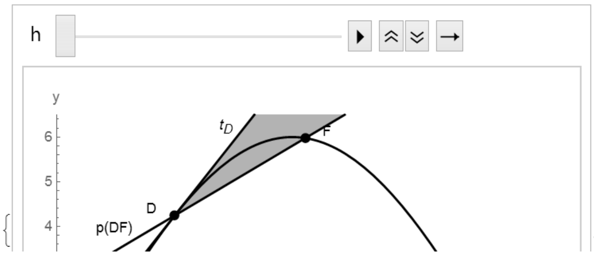

Fig. 3. There is a line t_D through point $D \in \gamma$ having the same property

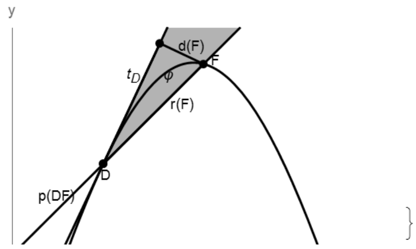

Fig. 4. The line $p(DF)$ moves to the line t_D, when the angle φ moves to zero

t_D has a local character. This stimulates the students to reach an abstraction.

3) Abstraction. The examples show that the line $p(DF)$ (Fig. 4) moves towards the line t_D, when the angle φ between the lines moves to zero. We can write this as follows:

$$\lim_{F \to D} \frac{d\,(F)}{r(F)} = 0, \tag{1}$$

where $d(F)$ is distance from F to the line t_D and $r(F)$ is length of the line segment DF. (We use that $\dfrac{d\,(F)}{r(F)} = \sin(\varphi)$ and $\sin(\varphi)$ moves to zero if and only if φ tends to zero.)

4) Definition 1. The line t_D is called “tangent” to the curve γ at the point$D \in \gamma$, when it contains the point D and equality (1) is achieved.

5) To train the property leading to the notion “tangent” we investigate the curves of known functions and some special points of theirs.

6) Relation of the given notion with other notions. The relation with the derivative of a function is important for the notion “tangent”.

We prove that for each point $A(x_0; f(x_0))$, $a < x_0 < b$, where the derivative $f'(x_0)$ exists, there is a tangent. The tangent has the equation:

$$t_A:\ y = f'\,(x_0)\,(x - x_0) + f\,(x_0)$$

Moreover, if $\lim\limits_{x \to x_0} \dfrac{f\,(x) - f\,(x_0)}{x - x_0} = +\infty$ or $\lim\limits_{x \to x_0} \dfrac{f\,(x) - f\,(x_0)}{x - x_0} = -\infty$, then the curve of the $y = f(x)$ has a tangent at the point $A(x_0; f(x_0))$ and this is the vertical line $t_A:\ x = x_0$ (Fig. 5).

This allows finding tangents to the curves of different differentiable functions.

We emphacize that when $y = f(x)$ is continuous only in $(a;\ b)$ and has a tangent at the point $A(x_0; f(x_0))$, $a < x_0 < b$ with an equation$y = kx + b$, then $f'\,(x_0)$ exists and $f'\,(x_0) = k$.

7)–9) Finding the tangents simple. More complicated cases leads to the notion “tangent” when γ is:

- smooth curve defined by parametric equations $x = x(t)$ and $y = y(t)$ for $t \in [\alpha; \beta]$
- smooth curve defined by Cartesian equation $g\,(x,\ y) = 0$.

10) Enriching the notion. The relation found between the notions *derivative* and *tangent* leads to further questions: Is it possible to define a tangent at the endpoints? What happens if $f(x)$ is continuous at x_0, but is not differentiable? Experimenting with

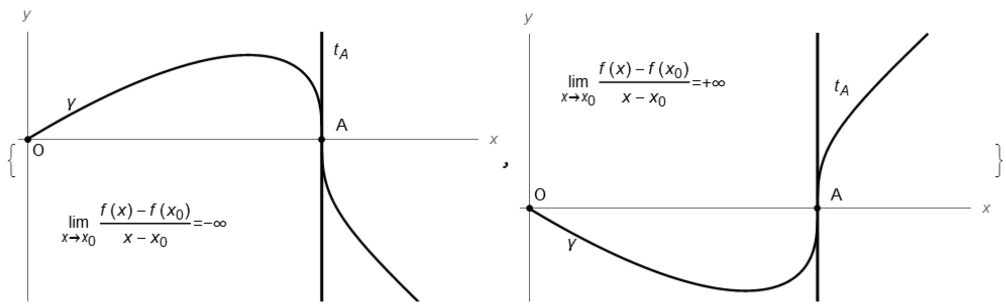

Fig. 5. Vertical tangents

animation (Fig. 6) leads to notion "one-sided semitangent".

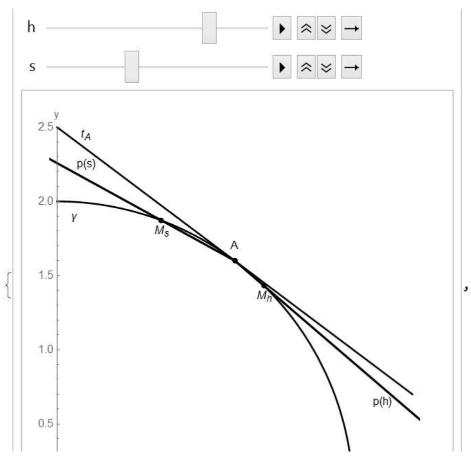

Fig. 6. Semitangents exist

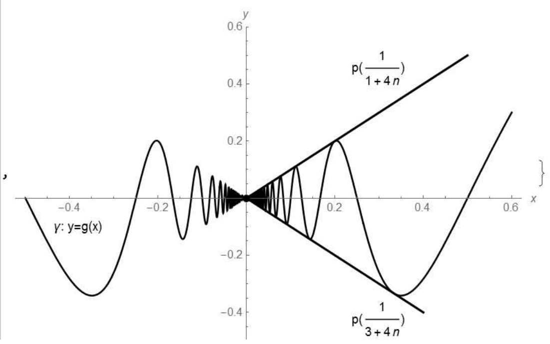

Fig. 7. Semitangents do not exist at the point $A(0;0)$

Definition 2. Semiline t_+ with initial point $A\,(x_0, f(x_0))$, $a \leq x_0 < b$ is called ***semitangent*** at the point A (right semitangent), if the angle between t_+ and semi-line $p\,(h)$, with vertex A and passing through the point $M_h\,(x_0 + h, f(x_0 + h))$, tends to zero.

The semiline t_- with initial point $A\,(x_0, f(x_0))$, $a < x_0 \leq b$ is called ***semitangent*** at the point A (left-semitangent), if the angle between t_- and semiline $p\,(s)$, with vertex A and passing through the point $M_s\,(x_0 + s, f(x_0 + s))$, tends to zero, when $s < 0$ and tendss to zero.

Similarly to step 6, a relation is established between the notion "semitangent" at the point $A\,(x_0, f(x_0))$ and the existing derivatives $f'_+\,(x_0) = \lim\limits_{h \to 0+} \dfrac{f\,(x_0 + h) - f\,(x_0)}{h}$ and $f'_-\,(x_0) = \lim\limits_{s \to 0-} \dfrac{f\,(x_0 + s) - f\,(x_0)}{s}$. We prove that if $f'_+(x_0)$ exists, then the semitangent t_+ has an equation $t_+ :\ y = f'_+(x_0)(x - x_0) + f(x_0), \ \ x \geq x_0$.

(If $f'_-(x_0)$ exists, then $t_- :\ y = f'_-(x_0)(x - x_0) + f(x_0), \ \ x \leq x_0$.)

The semitangent is vertical when: (a) $f'_+(x_0) = +\infty$; (b) $f'_+(x_0) = -\infty$; (c) $f'_-(x_0) = +\infty$; (d) $f'_-(x_0) = -\infty$.

Thus we can give another definition to the notion "tangent":

Definition 3. The line t_0 is a tangent at the point$A\,(x_0, f(x_0))$, $a < x_0 < b$, when semitangents t_+ and t_- at the point $A\,(x_0, f(x_0))$ exist and the angle between them is equal to π.

The relation between the definitions 1 and 3 is shown through an animation (Fig. 6).

The Definition 3 allows to demonstrate the cases when the tangent to the curve $y = f(x)$ does not exist at point $A\,(x_0, f(x_0))$. (We suppose the function $y = f(x)$ is continuous at the point x_0.) This leads towards two new notions – "corner point" and "cusp":

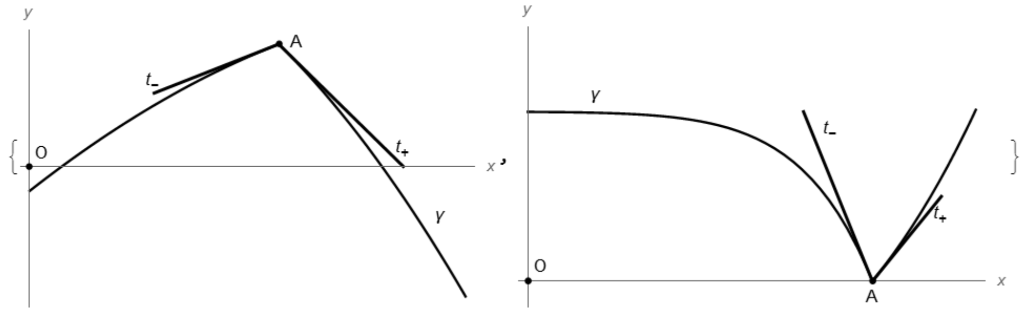

Fig. 8. Corner point

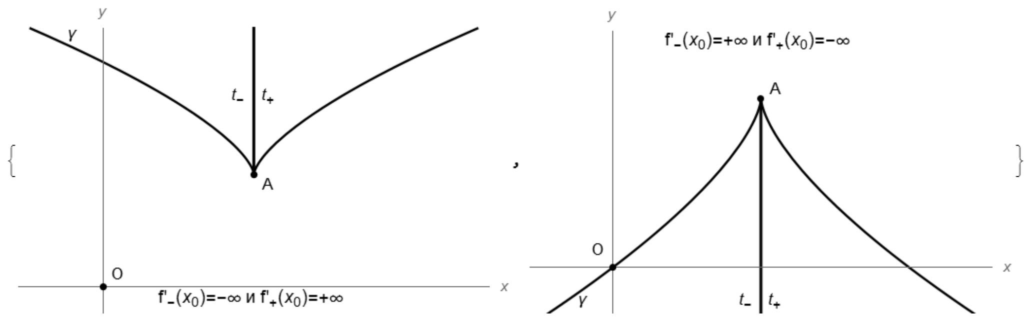

Fig. 9. Cusp

When the derivatives $f'_+(x_0)$ and $f'_-(x_0)$ exist but $f'_+(x_0) \neq f'_-(x_0)$ then we have a corner point. When $f'_+(x_0) = +\infty$ and$f'_-(x_0) = -\infty$, and when $f'_+(x_0) = -\infty$ and $f'_-(x_0) = +\infty$ then we have a cusp.

We demonstrate another case of a missing tangent trough the function $g(x) = x \sin\left(\dfrac{\pi}{2x}\right)$, $x \neq 0$ and $g\,(0) = 0$ (Fig. 7). There is no tangent to the curve $y = g(x)$ at the point $A\,(0;\ 0)$, but the point A is neither a corner point nor a cusp.

When the curve γ is defined by a parametric equations through $x = x(t)$ and $y = y(t)$ for $t \in [\alpha; \beta]$ the last notions lead to new notions such as a *tangent vector*.

III. Some results. We made a survey among the students of Informatics at the New Bulgarian University about the use of computer tools in mathematics education. Thirty eight students participated in a course Linear Algebra and Geometry – 28 boys and 10 girls. The course envisages the use of computer. The survey was anonymous. The results are shown below.

All respondents (38) use computer in this course. Among them 25 students point that they use Wolfram Mathematica system, and the others did not specify any software. Everyone reports use computers in class, and 22 students report that they use computers for self-training. All students use a computer in this course every week.

Figure 10 presents the students' answers related to the benefits of using a computer as a tool during the course: to overcome technical problems in calculations; to understand notions and methods, to understand mathematical applications in problem solving. The chart shows that the most frequent answer is "in great degree".

The influence of computer usage on student's motivation is given on the Fig. 11. The results show a strong positive effect of the computer usage on student motivation to learn mathematics.

More than 92 % students enjoy Mathematics when using a software and they prefer to

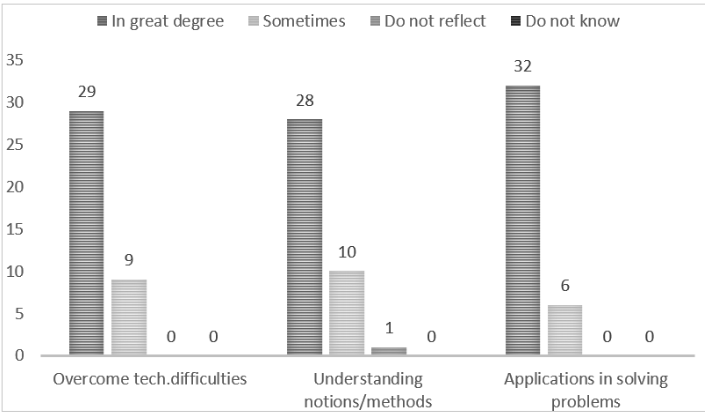

Fig. 10. Benefits of computer usage on technical problems, understanding notions/methods, application for problem solving

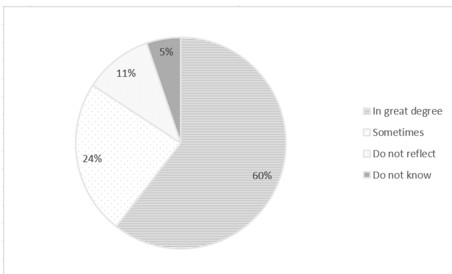

Fig. 11. Computer usage and students' motivation

learn mathematics with computers. No student has reported difficulties to use a software when learning mathematics.

There are no statistically significant differences in the answers of the boys and the girls.

IV. Conclusions. The use of systems of symbolic calculus in teaching mathematical notions increases the efficiency of the educational process. Here are some reasons:

1) Students are actively involved in exploring objects and in this way the students can find their properties and relations leading to deeper understanding of the notions.

2) The features of the mathematical software for graphical visualization, animation, symbolic calculations and programming provide a wide range of tools for creating the appropriate examples for developing and understanding notions. Their use contributes to the motivation of the students to learn mathematics.

3) The use of mathematical software practically eliminates the technical problems of the students in more complex calculations.

4) The use of mathematical software makes the process of building notions shorter and the understanding deeper.

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ПРЕПОДАВАНЕ НА МАТЕМАТИКА ЧРЕЗ КОМПЮТЪРНИ СИСТЕМИ

Петя Асенова, Марин Маринов

Понятията са в основата на системите от научни знания и тяхното формиране е важна образователна цел. Настоящият доклад представя нашия педагогически подход за разбирането на математическите понятия на университетско ниво с използване на софтуер. За решаването на задачи използваме възможностите на Wolfram Mathematica за изчисляване, визуализация и експериментиране. Това позволява на студентите да откриват и сравняват свойствата на подобни понятия, да променят параметрите на разглеждания обект и да го изследват, за да го разберат по-добре. По този начин математическият софтуер подпомага учениците да разбират по-дълбоко математическите понятия. Така развитието на понятийната система е спираловиден процес и следва логиката на научните открития.