APPENDIX

A GLANCE AT THE BEST RESULTS OF BLAGOVEST SENDOV AND HRISTO SENDOV

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Our collaboration started in the Winter of 2012 and was in the area known as Geometric Theory of Polynomials. In this note, we give a brief overview of some of our dearest results. Let us begin with several definitions. A point $\zeta \in \mathcal{C}$ is called *critical* for the complex polynomial $p(z)$ if $p'(\zeta) = 0$.

Definition 1. A domain Θ_n is called a *Rolle's domain*, if every complex polynomial p of degree n, satisfying $p(i) = p(-i)$, has at least one critical point in it. A Rolle's domain Θ_n^X is *stronger* than the Rolle's domain Θ_n^Y, if $\Theta_n^X \subsetneqq \Theta_n^Y$. A Rolle's domain Θ_n^X is *sharp* if Θ_n^X is minimal with respect to inclusion.

A result asserting that Θ_n is a Rolle's domain is called a *Rolle's Theorem.* There are several known Rolle's Theorems for complex polynomials. The most famous one are the following.

Theorem 2 (Grace-Heawood, 1902–1907)**.** *Let $p(z)$ be a polynomial of degree $n \geq 2$ satisfying $p(i) = p(-i)$. The disk*

$$\Theta_n^{GH} := D[0; \cot(\pi/n)] \tag{1}$$

is a Rolle's domain.

Theorem 3 (Fekete, 1925)**.** *Let $p(z)$ be a polynomial of degree $n \geq 3$, satisfying $p(i) = p(-i)$. The double disk $\Theta_n^F := D[-c; r] \cup D[c; r]$, where*

$$c = \cot(\pi/(n-1)), \quad r = 1/\sin(\pi/(n-1))$$

is a Rolle's domain.

Neither one of the above two domains is stronger than the other when $n \geq 5$. The results in Theorem 2 and 3 have not been improved essentially in almost a hundred years. Our first main result produces a domain that is stronger than both of them.

Theorem 4 (Sendov-Sendov, 2018)**.** *Let $p(z)$ be a polynomial of degree $n \geq 3$, satisfying $p(i) = p(-i)$.*

(a) *If $n = 3$, then $\Theta_3^{GH} = D[0; 1/\sqrt{3}]$ is a sharp Rolle's domain.*

(b) *If $n = 4$, then $\Theta_4^{SS} := D[-1/3; 2/3] \cup D[1/3; 2/3]$ is a sharp Rolle's domain.*

(c) *If $n \geq 5$, then $\Theta_n^{SS} := D[-c; r] \cup D[c; r]$, where*

$$c = \cot(2\pi/n), \quad r = 1/\sin(2\pi/n) \tag{2}$$

is a Rolle's domain.

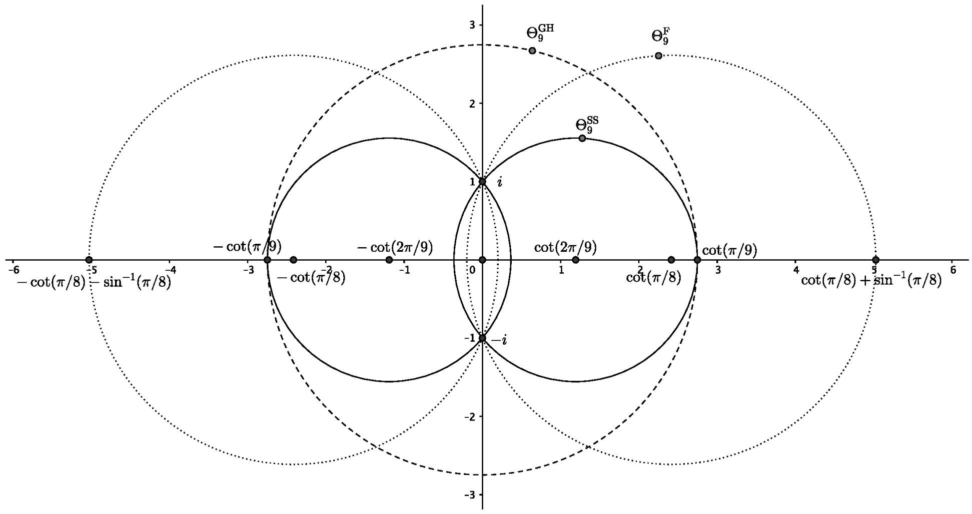

Fig. 1. The Rolle's domains $\Theta_{12}^{GH}, \Theta_{12}^{F}$, and Θ_{12}^{SS}

The novelty, in the case $n = 3$ is the sharpness. When $n = 4$, we not only have $\Theta_4^{SS} \subset \Theta_4^{GH} \subset \Theta_4^{F}$, but also that the domain Θ_4^{SS} is sharp. Finally, when $n \geq 5$, we have $\Theta_n^{SS} \subset \Theta_n^{GH} \cap \Theta_n^{F}$ as illustrated in Figure for the case $n = 9$.

In Figure 1, the Rolle's domain Θ_9^{GH} is the area bounded by the dashed circle; Θ_9^{F} is the area bounded by the two dotted circles; and Θ_9^{SS} is the area bounded by the two solid circles.

It is important to notice that when one substitutes n with 3 or 4 in formula (2) one does not obtain the domains in parts (a) and (b) in Theorem 4. This shows that parts (a), (b), and (c) belong to "different families". Further investigations rectified this deficiency.

Theorem 5 (Sendov-Sendov, 2019)**.** *Let $p(z)$ be a polynomial of degree $n \geq 3$, satisfying $p(i) = p(-i)$. Then, $\Theta_n^* := D[-c'; \rho] \cup D[c'; \rho]$, with*

$$c' := \frac{\cot(\pi/n)\cos(\tau(\kappa_n))}{2\cos^2(\tau(\kappa_n)/2)} \quad \textit{and} \quad \rho := \frac{\cot(\pi/n)}{2\cos^2(\tau(\kappa_n)/2)} \tag{3}$$

is the smallest Rolle's domain made up of two disks that is symmetric with respect to the coordinate axes. In the formula

$$\tau(\kappa_n) := 2\arctan(t_{n-1,1} \cdot \tan(\pi/n)),$$

where $t_{n-1,1}$ is the unique positive zero of the polynomial

$$f_{n-1,1}(t) := (t+1)^{n-1}(t(n+1) - (n-1)) - (t-1)^{n-1}(t(n+1) + (n-1)).$$

The next table lists the first several values of $t_{n-1,1}, c'$, and ρ.

n	$t_{n-1,1}$	c'	ρ
3	$1/\sqrt{3}$	0	$1/\sqrt{3}$
4	$1/\sqrt{3}$	1/3	2/3
5	$1/\sqrt{\sqrt{5}}$	0.2763932024	0.8506508084

The table shows that $\Theta_3^* = \Theta_3^{GH}$, $\Theta_4^* = \Theta_4^{SS}$, and we know from Theorem 4 that these are sharp Rolle's domains. Moreover, for any $n \geq 5$, we have $\Theta_n^* \subset \Theta_n^{SS}$. Theorem 5 improves on Theorem 4 by placing the three cases of the latter into one family, producing a smaller Rolle's domain, but also stating that one cannot do better using only two disks symmetric with respect to the axes. The trade off is that Theorem 4 is easier to apply.

The road to Theorems 4 and 5 was long and sinuous. We worked hard for six years to reach them. On the way we started the development of the theory of loci of complex polynomials and as a by-product we obtained the following non-convex analogue of the classical Gauss-Lucas theorem—the first of its kind, so far, to our knowledge.

Theorem 6 (Gauss-Lucas, 1879). *The critical points of a complex polynomial are contained in the convex hull of its zeros.*

For any $\phi \in [0, 2\pi]$, define the sector

$$S(\phi) := \{te^{i\psi} : \psi \in [0, \phi], t \geq 0\}.$$

Let $P_n(\phi)$ be the set of all polynomials of degree n, i.e. $a_n \neq 0$,

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

with coefficients in the sector $S(\phi)$ and zeros not in the interior of $S(\phi)$.

Theorem 7 (Sendov-Sendov, 2020). *If $p(z) \in P_n(\phi)$, then $p'(z) \in P_{n-1}(\phi)$.*

The theorem is trivial when $\phi \in [\pi, 2\pi]$ since then it is a direct consequence of the Gauss-Lucas theorem. For $\phi \in [0, \pi)$ the theorem makes a statement about the location of the critical points in a non-convex domain. Surprisingly, one can show that in fact Theorem 7 is stronger than the Gauss-Lucas theorem, meaning that the latter is a corollary of the former.

In the bibliography, we give a complete list of our publications. The results presented here are from [7], [10], and [12]. The authors are in alphabetical order.

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