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ONE PARAMETRIC REPRESENTATION OF THE GENERALIZED FOCAL CURVE OF A NON-HELICAL SPHERICAL CURVE

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In this paper, we establish relations between Frenet frames and differential-geometric invariants of a non-helical space curve in a three-dimensional Euclidean space and its closely related curve in a four-dimensional Euclidean space. The focal curve of the constructed four-dimensional curve is used to obtain a parametric representation of the generalized focal curve of a spherical curve.
Keywords: Frenet curves, Non-helical curves, Shape curvatures, Euclidean curvatures, Focal curves, Spherical curves

ЕДНО ПАРАМЕТРИЧНО ПРЕДСТАВЯНЕ НА ОБОБЩЕНАТА ФОКАЛНА КРИВА НА СФЕРИЧНА КРИВА, КОЯТО НЕ Е СПИРАЛА

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В тази статия установяваме връзки между векторите на Френе и диференциално-геометрични инварианти на неспирална пространствена крива в тримерно Евклидово пространство и нейната тясно свързана крива в четиримерно Евклидово пространство. Фокалната крива на построената четиримерна крива се използва за получаване на параметрично представяне на обобщената фокална крива на сферична крива.

Ключови думи: Криви на Френе, Неспирални криви, Шейп-кривини, Евклидови кривини, Фокални криви, Сферични криви

1 Introduction

One important problem in the differential geometry of curves is determining the focal curve of a particular Frenet curve. The concepts of a focal curve and focal curvatures of a smooth curve in n-dimensional Euclidean space $\mathbb{E}^n$, $n \in \mathbb{N}$, $n \geqq 2$ were first introduced by Uribe-Vargas [14]. The focal curve can be parametrized using the original curve's parametrization, its curvatures, and unit normal vectors. Our investigation is restricted to regular curves of order n in $\mathbb{E}^n$. In other words, we assume that every curve $\boldsymbol{\alpha} : I \subset \mathbb{R} \to \mathbb{E}^n$ (nD-curve) possesses derivatives up to order n, i.e. the curve is of class C^n, and for any $t \in I$ the derivatives $\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}''(t), \ldots, \boldsymbol{\alpha}^{(n)}(t)$ are linearly independent vectors in the n-dimensional real vector space $\mathbb{R}^n$. Those curves are named nD-Frenet curves. The differential geometry of space curves is described in more detail in [2, 11]. Any Frenet curve can be determined up to a direct similarity of $\mathbb{E}^n$ by $n-1$ functions called shape curvatures. They are introduced in [4, 5, 6]. Three important classes contain Frenet curves in $\mathbb{E}^3$: non-helical curves, general helices, and circular helices. Izumiya and Takeuchi [13] have determined relations between plane curves and general helices. Ali [1] studies parametric representations of general helices with a given curvature function and a given constant angle between the tangent vectors and a fixed unit vector. Examples of general helices with a unit speed parametrization can be found in [10, 8]. In this work, we will consider only non-helical curves that do not have a constant curvature to torsion ratio. We construct a new curve in $\mathbb{E}^4$ ($4D$-curve) that is associative to a given space curve in $\mathbb{E}^3$. Then we find the focal curve of the corresponding $4D$-curve. The orthogonal projection of the obtained curve in $\mathbb{E}^3$ is called a generalized focal curve of the initial curve. The relations between the differential-geometric invariants of the corresponding curves make it possible to study the properties of one curve via the other and vice versa. The focal curve of any spherical curve degenerates into one point, the centre of the curve's containing sphere. This is the reason why we explore the generalized focal curves of spherical curves. In [7], only the generalized focal curves of Viviani's curve and a spherical helix are considered.

2 Preliminaries

We consider the Euclidean four-space $\mathbb{E}^4$ to be an affine space with column four-dimensional vectors in its corresponding real vector space $\mathbb{R}^4$. Accordingly, the position vector $\boldsymbol{X} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$ may be used to identify any point $X \in \mathbb{E}^4$. There is a standard scalar (or dot) product of two vectors in the vector space $\mathbb{R}^4$. If $\mathbf{U} = (u_1, u_2, u_3, u_4)^T$ and $\mathbf{V} = (v_1, v_2, v_3, v_4)^T$ are four-dimensional vectors, then the scalar product of $\mathbf{U}$ and $\mathbf{V}$ is the real number $\mathbf{U}\cdot\mathbf{V} = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ and the norm of the vector $\mathbf{U}$ is $\|\mathbf{U}\| = \sqrt{\mathbf{U}\cdot\mathbf{U}} = \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$.

Let $\boldsymbol{\gamma} : I \to \mathbb{E}^4$ be a unit-speed curve of class C^4 defined on an interval $I \subseteq \mathbb{R}$ by a vector-parametric equation

$$\boldsymbol{\gamma}(s) = \big(\gamma_1(s),\, \gamma_2(s),\, \gamma_3(s), \gamma_4(s)\big)^T, \quad s \in I. \tag{1}$$

This suggests that $\boldsymbol{\gamma}'(s) = \dfrac{d}{ds}\boldsymbol{\gamma}(s)$ is a unit vector for each $s \in I$ and that the coordinate functions $\gamma_i(s)$, $i = 1, 2, 3, 4$ have continuous derivatives up to order 4. The curve $\boldsymbol{\gamma}$ given by (1) is a unit-speed Frenet curve if the vectors $\boldsymbol{\gamma}'(s)$, $\boldsymbol{\gamma}''(s) = \dfrac{d}{ds}\boldsymbol{\gamma}'(s)$, $\boldsymbol{\gamma}'''(s) = \dfrac{d}{ds}\boldsymbol{\gamma}''(s)$ and $\boldsymbol{\gamma}^{(4)}(s) = \dfrac{d}{ds}\boldsymbol{\gamma}'''(s)$ are linearly independent for all $s \in I$ or, equivalently, $\det(\boldsymbol{\gamma}^{'}(s), \boldsymbol{\gamma}^{''}(s), \boldsymbol{\gamma}^{'''}(s), \boldsymbol{\gamma}^{(4)}(s)) \neq 0$ for all $s \in I$. According to [12], we can examine four unit vector functions $\mathbf{T}(s) = \boldsymbol{\gamma}'(s)$, $\mathbf{N}_1(s)$, $\mathbf{N}_2(s)$, $\mathbf{N}_3(s)$ and three real-valued curvature functions $\kappa_1(s)$, $\kappa_2(s)$, $\kappa_3(s)$. Keep in mind that the requirement that the curve $\boldsymbol{\gamma}$ is a Frenet curve in $\mathbb{E}^4$ is equivalent to the condition that $\boldsymbol{\gamma}$ is a curve of class C^4 with curvatures $\kappa_1(s) > 0$, $\kappa_2(s) > 0$, $\kappa_3(s) \neq 0$.

The unit-speed Frenet curve $\boldsymbol{\gamma}$ defined by (1) has a unit tangent vector $\mathbf{T}(s) = \boldsymbol{\gamma}'(s)$, and the first, second, and third unit normal vectors of $\boldsymbol{\gamma}$ can be expressed as $\mathbf{N}_1(s) = \boldsymbol{\gamma}''(s)/\|\boldsymbol{\gamma}''(s)\|$, $\mathbf{N}_2(s) = (\mathbf{N}_1'(s) + \kappa_1(s)\mathbf{T}(s))/\kappa_2(s)$, $\mathbf{N}_3(s) = (\mathbf{N}_2'(s) + \kappa_2(s)\mathbf{N}_1(s))/\kappa_3(s)$.

Analogously to the three-dimensional case, there is a unique osculating hypersphere at any point of a Frenet curve $\boldsymbol{\gamma}$ in $\mathbb{E}^4$. The centers of these osculating hyperspheres form a new curve, called a focal curve of $\boldsymbol{\gamma}$. Uribe-Vargas [14] investigated focal curves in arbitrary dimension. It follows from his main result that the unit-speed Frenet curve $\boldsymbol{\gamma}$ given by (1) possesses a focal curve $\boldsymbol{foc}_{\boldsymbol{\gamma}}$ with parametrization

$$\boldsymbol{foc}_{\boldsymbol{\gamma}}(s) = \boldsymbol{\gamma}(s) + C_1(s)\mathbf{N}_1(s) + C_2(s)\mathbf{N}_2(s) + C_3(s)\mathbf{N}_3(s), \tag{2}$$

where $\mathbf{N}_i(s)$, $i = 1, 2, 3$ are the three unit normal vectors of $\boldsymbol{\gamma}$, and the coefficients $C_i(s)$, $i = 1, 2, 3$, called focal curvatures, can be expressed by the curvatures $\kappa_1(s)$, $\kappa_2(s)$, $\kappa_3(s)$ of $\boldsymbol{\gamma}$. If $\kappa_i(s)$ are non-constant functions, then the focal curvatures are

$$\begin{gathered} C_1(s) = \frac{1}{\kappa_1(s)}, \qquad C_2(s) = \left(\frac{1}{\kappa_1(s)}\right)' \frac{1}{\kappa_2(s)}, \\ C_3(s) = \left(\frac{C_1(s)}{C_2(s)}C_1'(s) + C_2'(s)\right)\frac{1}{\kappa_3(s)} = \left(\frac{\kappa_2(s)}{\kappa_1(s)} + \left[\left(\frac{1}{\kappa_1(s)}\right)' \frac{1}{\kappa_2(s)}\right]'\right)\frac{1}{\kappa_3(s)}. \end{gathered} \tag{3}$$

Let $I \subseteq \mathbb{R}$ be a zero-containing interval, and let $\boldsymbol{\alpha} : I \to \mathbb{E}^3$ be a Frenet curve of class C^4 with an arc-length parametrization and a parametric equation

$$\boldsymbol{\alpha}(s) = \big(x(s),\, y(s),\, z(s)\big)^T, \quad s \in I. \tag{4}$$

According to the above assumptions, for any $s \in I$, the curvature $\varkappa(s)$ of $\boldsymbol{\alpha}$ is a positive real number, the torsion $\tau(s)$ of $\boldsymbol{\alpha}$ is a nonzero real number, and the Frenet frame $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$ is well-defined. The unit-speed curve $\boldsymbol{\gamma}$ that is closely related to $\boldsymbol{\alpha}$ is studied in [9]. If $\vec{e}_4$ is the fourth unit coordinate vector in $\mathbb{E}^4$, then the curve $\boldsymbol{\gamma}$ can be expressed by the equality

$$\gamma(s) = \mathcal{A}\boldsymbol{\alpha}(s) + \frac{s}{\sqrt{2}}\vec{e}_4, \quad s \in I, \text{ where } \mathcal{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{pmatrix}. \tag{5}$$

The suggested curve $\boldsymbol{\gamma}$ is situated on a generalized cylinder with rulings parallel to $\vec{e}_4$ and a base curve $\mathcal{A}\boldsymbol{\alpha}$. Using the considered curve is motivated by the fact that the curve $\boldsymbol{\alpha}$ is a non-helical Frenet curve, from which it follows that the curve $\boldsymbol{\gamma}$ has a well-defined regular focal curve. The generalized focal curve of $\boldsymbol{\alpha}$ is determined by the orthogonal projection of this focal curve onto the hyperplane in $\mathbb{E}^4$ that contains the curve $\boldsymbol{\alpha}$. When the curve $\boldsymbol{\alpha}$ is a general helix, this is not the case. Then it is appropriate to apply other related curves.

According to [9, Theorem 3.1], the relations between the Euclidean curvatures $\varkappa$ and τ of a non-helical Frenet curve $\boldsymbol{\alpha}$ in $\mathbb{E}^3$ and the Euclidean curvatures κ_1, κ_2, κ_3 of its associated Frenet curve $\boldsymbol{\gamma}$ in $\mathbb{E}^4$ are

$$\kappa_1(s) = \frac{\varkappa(s)}{\sqrt{2}}, \quad \kappa_2(s) = \frac{\varkappa(s)}{\sqrt{2}}\sqrt{1 + 2\left(\frac{\tau(s)}{\varkappa(s)}\right)^2}, \quad \kappa_3(s) = -\frac{\sqrt{2}\left(\frac{\tau(s)}{\varkappa(s)}\right)'}{1 + 2\left(\frac{\tau(s)}{\varkappa(s)}\right)^2}. \tag{6}$$

As per [3, Theorem 1.4.2], the shape curvature $\widetilde{\varkappa}(s) = \left(\frac{1}{\varkappa(s)}\right)'$ and the shape torsion $\widetilde{\tau}(s) = \frac{\tau(s)}{\varkappa(s)}$ of a non-helical Frenet curve $\boldsymbol{\alpha}$ in $\mathbb{E}^3$, and the shape curvatures $\widetilde{\kappa}_1(s) = \left(\frac{1}{\kappa(s)}\right)'$, $\widetilde{\kappa}_2(s) = \frac{\kappa_2(s)}{\kappa_1(s)}$, $\widetilde{\kappa}_3(s) = \frac{\kappa_3(s)}{\kappa_1(s)}$ of its associated Frenet curve $\boldsymbol{\gamma}$ in $\mathbb{E}^4$ are related in the following ways:

$$\widetilde{\kappa}_1(s) = \sqrt{2}\widetilde{\varkappa}(s), \quad \widetilde{\kappa}_2(s) = \sqrt{1 + 2\widetilde{\tau}^2(s)}, \quad \widetilde{\kappa}_3(s) = -\frac{2\widetilde{\tau}'(s)}{\widetilde{\varkappa}(1 + 2\widetilde{\tau}^2(s))}. \tag{7}$$

3 Main results

Based on the results of the previous section, the relations between the Frenet-Seret frame of $4D$-curve $\boldsymbol{\gamma}$ and the differential-geometric invariants of a non-helical $3D$-Frenet curve $\boldsymbol{\alpha}$, parametrized about an arc-length parameter s, are given in the next theorem.

Theorem 1. *Let* (4) *be a parametrization of a unit-speed non-helical Frenet curve* $\boldsymbol{\alpha} : I \to \mathbb{E}^3$ *of class* C^4*, and let* $\widetilde{\varkappa}(s) \neq 0$ *and* $\widetilde{\tau}(s)$ *be the shape curvature and the shape torsion of* $\boldsymbol{\alpha}$*, respectively. Suppose that the curve* $\boldsymbol{\gamma} : I \to \mathbb{E}^4$ *is defined by (5). Then the vector invariants* $\mathbf{T}(s)$*,* $\mathbf{N}_1(s)$*,* $\mathbf{N}_2(s)$*,* $\mathbf{N}_3(s)$ *of the unit-speed* $4D$*-Frenet curve* $\boldsymbol{\gamma}$ *can be expressed by the differential-geometric invariants of* $\boldsymbol{\alpha}$ *with the following equations:*

$$\mathbf{T}(s)=\mathcal{A}\boldsymbol{t}+\frac{\vec{e}_4}{\sqrt{2}},\quad \mathbf{N}_1(s)=\sqrt{2}\mathcal{A}\boldsymbol{n},\quad \mathbf{N}_2(s)=\frac{-\mathcal{A}\boldsymbol{t}+2\widetilde{\tau}(s)\mathcal{A}\boldsymbol{b}+\vec{e}_4/\sqrt{2}}{\sqrt{1+2\widetilde{\tau}^2(s)}},$$

$$\mathbf{N}_3(s)=-\frac{\sqrt{2}\widetilde{\tau}(s)\mathcal{A}\boldsymbol{t}+\sqrt{2}\mathcal{A}\boldsymbol{b}-\widetilde{\tau}(s)\vec{e}_4}{\sqrt{1+2\widetilde{\tau}^2(s)}}, \tag{8}$$

where $(\boldsymbol{t},\boldsymbol{n},\boldsymbol{b})$ *is the Frenet frame of* $\boldsymbol{\alpha}$ *and* $\vec{e}_4$ *is the fourth coordinate unit vector in* $\mathbb{E}^4$*.*

Proof. The derivatives of $\boldsymbol{\gamma}$ up to fourth order are $\boldsymbol{\gamma}'(s)=\frac{1}{\sqrt{2}}(x'(s),y'(s),z'(s),1)^T$, $\boldsymbol{\gamma}''(s)=\frac{1}{\sqrt{2}}(x''(s),y''(s),z''(s),0)^T$, $\boldsymbol{\gamma}'''(s)=\frac{1}{\sqrt{2}}(x'''(s),y'''(s),z'''(s),0)^T$, $\boldsymbol{\gamma}^{(4)}(s)=\frac{1}{\sqrt{2}}(x^{(4)}(s),y^{(4)}(s),z^{(4)}(s),0)^T$. They are linearly independent whenever

$$\begin{aligned}\det\Big(\boldsymbol{\gamma}'(s),\boldsymbol{\gamma}''(s),\boldsymbol{\gamma}'''(s),\boldsymbol{\gamma}^{(4)}(s)\Big)&=-\frac{1}{4}\det\big(\boldsymbol{\alpha}''(s),\boldsymbol{\alpha}'''(s),\boldsymbol{\alpha}^{(4)}(s)\big)\\&=-\frac{1}{4}(\varkappa(s))^5(\widetilde{\tau}(s))'\neq 0,\end{aligned}$$

which is true when the Frenet curve $\boldsymbol{\alpha}$ is a non-helical. It is easy to see that $\boldsymbol{\gamma}$ is a unit-speed Frenet curve with a unit tangential vector $\mathbf{T}(s)=\boldsymbol{\gamma}'(s)=\mathcal{A}\boldsymbol{t}+\vec{e}_4/\sqrt{2}$. Then, by applying Frenet-Serret formulas for the curves $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ and replacing the curvatures of $\boldsymbol{\gamma}$ from (6), we obtain (8), where $\widetilde{\tau}(s)=\tau(s)/\varkappa(s)$. □

The next statement gives us the relations between the focal curvatures of the 4*D*-Frenet curve $\boldsymbol{\gamma}$, and the Euclidean and the shape curvatures of $\boldsymbol{\alpha}$, parametrized about an arc-length parameter s.

Theorem 2. *Let* (4) *be a parametrization of a unit-speed Frenet curve* $\boldsymbol{\alpha}: I\to\mathbb{E}^3$ *of class* C^4*. Then the focal curvatures* $C_1(s),C_2(s),C_3(s)$ *of the unit-speed* 4*D-Frenet curve* $\boldsymbol{\gamma}$ *can be expressed by the Euclidean curvatures* $\varkappa(s)>0$, $\tau(s)$*, the shape curvatures* $\widetilde{\varkappa}(s)\neq 0$, $\widetilde{\tau}(s)$ *of the non-helical* 3*D-Frenet curve* $\boldsymbol{\alpha}$*, and their derivatives by the following equations:*

$$C_1(s)=\frac{\sqrt{2}}{\varkappa(s)},\qquad C_2(s)=\frac{2\widetilde{\varkappa}(s)\widetilde{\tau}(s)}{\tau(s)\sqrt{1+2\widetilde{\tau}^2(s)}},$$

$$C_3(s)=-\frac{(1+2\widetilde{\tau}^2(s))^2+2\left(\left(\frac{\widetilde{\varkappa}(s)}{\tau(s)}\right)'\widetilde{\tau}(s)(1+2\widetilde{\tau}^2(s))+\frac{\widetilde{\varkappa}(s)}{\tau(s)}\widetilde{\tau}'(s)\right)}{\sqrt{2}\widetilde{\tau}'(s)\sqrt{1+2\widetilde{\tau}^2(s)}}. \tag{9}$$

Proof. Using that $\widetilde{\kappa}_1(s)=(1/\kappa_1(s))'$ and $\widetilde{\kappa}_2(s)=\kappa_2(s)/\kappa_1(s)$, from (3) we get $C_1(s)=1/\kappa_1(s)$, $C_2(s)=C_1'(s)/\kappa_2(s)=\widetilde{\kappa}_1(s)/\kappa_2(s)$, and

$$C_3(s)=\left(\frac{C_1(s)}{C_2(s)}C_1'(s)+C_2'(s)\right)\frac{1}{\kappa_3(s)}=\left(\frac{\kappa_2(s)}{\kappa_1(s)}+C_2'(s)\right)\frac{1}{\kappa_3(s)}=\frac{\widetilde{\kappa}_2(s)}{\kappa_3(s)}+\frac{C_2'(s)}{\kappa_3(s)}.$$

After some simplifications and rearrangements, we reach (9) by applying (6) and (7). □

Theorem 3. *Let* (4) *be a parametrization of a unit-speed Frenet curve* $\boldsymbol{\alpha}: I\to\mathbb{E}^3$ *of class* C^4*, and let* $\varkappa(s)>0$, $\tau(s)$, $\widetilde{\varkappa}(s)\neq 0$ *and* $\widetilde{\tau}(s)$ *be the Euclidean curvatures and the* *shape curvatures of $\boldsymbol{\alpha}$, respectively. Suppose that the curve $\boldsymbol{\gamma} : I \to \mathbb{E}^4$ is defined by* (5)*. Then the focal curve of $\boldsymbol{\gamma}$ has parametric representation*

$$\begin{aligned} \boldsymbol{foc_\gamma}(s) &= \mathcal{A}\boldsymbol{\beta}(s) + \frac{s - \widetilde{\tau}(s)U(s)}{\sqrt{2}} \cdot \vec{e}_4, \\ \boldsymbol{\beta}(s) &= \boldsymbol{\alpha}(s) + \widetilde{\tau}(s)U(s)\boldsymbol{t} + \frac{2}{\varkappa(s)}\boldsymbol{n} + \left(U(s) + \frac{2\widetilde{\varkappa}(s)}{\tau(s)}\right)\boldsymbol{b}, \end{aligned} \tag{10}$$

where $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ is the Frenet frame of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ is the generalized focal curve of $\boldsymbol{\alpha}$, and

$$U(s) = \frac{1 + 2\widetilde{\tau}^2(s) + 2\left(\frac{\widetilde{\varkappa}(s)}{\tau(s)}\right)'\widetilde{\tau}(s)}{\widetilde{\tau}'(s)}. \tag{11}$$

Proof. From (2) and Theorems 1 and 2, we get to (10). □

Corollary 1. *Let* (4) *be a parametrization of a unit-speed Frenet curve $\boldsymbol{\alpha} : I \to \mathbb{E}^3$ of class C^4, and let $\varkappa(s) > 0$, $\tau(s)$, $\widetilde{\varkappa}(s) \neq 0$ and $\widetilde{\tau}(s)$ be the Euclidean curvatures and the shape curvatures of $\boldsymbol{\alpha}$, respectively. Then the generalized focal curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ can be represent as $\boldsymbol{\beta}(s) = \boldsymbol{foc_\alpha}(s) + \boldsymbol{F_\alpha}(s)$, where $\boldsymbol{F_\alpha}(s) = \widetilde{\tau}(s)U(s)\boldsymbol{t} + \boldsymbol{n}/\varkappa(s) + (U(s) + \widetilde{\varkappa}(s)/\tau(s))\boldsymbol{b}$, $U(s)$ is defined by* (11)*, and $\boldsymbol{foc_\alpha}$ is the focal curve of $\boldsymbol{\alpha}$.*

Proof. The proof follows from $\boldsymbol{foc_\alpha}(s) = \boldsymbol{\alpha}(s) + \boldsymbol{n}/\varkappa(s) + (\widetilde{\varkappa}(s)/\tau(s))\boldsymbol{b}$ and (10). □

We can assume that the sphere is centred at the origin, without losing the generality. With all that has been discussed so far, and with the well-known equality $0 = \tau/\varkappa + ((1/\varkappa)'/\tau)' = \widetilde{\tau} + (\widetilde{\varkappa}/\tau)'$ which characterizes the spherical curves in $\mathbb{E}^3$, we can arrive at the following theorem:

Theorem 4. *Let* (4) *be a parametrization of a unit-speed non-helical spherical curve $\boldsymbol{\alpha} : I \to \mathbb{E}^3$ of class C^4 with a Frenet frame $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$, and let $\varkappa(s) > 0$, $\tau(s)$, $\widetilde{\varkappa}(s) \neq 0$ and $\widetilde{\tau}(s)$ be the Euclidean curvatures and the shape curvatures of $\boldsymbol{\alpha}$, respectively. Then the generalized focal curve $\boldsymbol{\beta}$ of $\boldsymbol{\alpha}$ has parametric equation*

$$\boldsymbol{\beta}(s) = \frac{\widetilde{\tau}(s)}{\widetilde{\tau}'(s)}\boldsymbol{t} + \frac{1}{\varkappa(s)}\boldsymbol{n} + \left(\frac{1}{\widetilde{\tau}'(s)} + \frac{\widetilde{\varkappa}(s)}{\tau(s)}\right)\boldsymbol{b}.$$

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