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ON A PROBLEM FROM THE BOOK "APPROXIMATION OF FUNCTIONS" BY G. G. LORENTZ

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We generalize and give a new solution to one famous problem in approximation theory from the book "Approximation of Functions" by G.G. Lorentz.

Keywords: Finite difference, Moduli of smoothnes, K-functional, Algebraic polynomials, Linear operator

ВЪРХУ ЕДНА ЗАДАЧА ОТ КНИГАТА "АПРОКСИМАЦИЯ НА ФУНКЦИИ" НА ДЖ. Г. ЛОРЕНЦ

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Обобщаваме и предлагаме ново решение на една известна задача в теорията на апроксимациите от книгата "Апроксимация на функции" на Дж.Г. Лоренц.

Ключови думи: Крайни разлики, Модули на гладкост, K-функционали, Алгебрични полиноми, Линеен оператор

1. Introduction. Theorems concerning equivalence between K-functionals and moduli of smoothness are well-known results in approximation theory (see, e.g., [1, p. 177], [2, p. 11], [3]). If $D^r g = g^{(r)}$, $X = L_p = L_p[a,b]$ with the usual L_p-norm denoted by $\|\cdot\|_p$, for $1 \le p < \infty$, or $X = C[a,b]$ with the uniform norm denoted by $\|\cdot\|_\infty$, and $Y \subseteq (D^r)^{-1}(X) = \{g \in X : D^r g \in X\}$, the K-functional has the form

$$K(f,t;X,Y,D^r) = \inf_{g\in Y}\big\{\|f-g\|_X + t\|D^r g\|_X\big\}.$$

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The classical moduli of smoothness are defined by

$$\omega^r(f,t)_p = \sup_{0<h\le t} \|\Delta_h^r f\|_p \qquad \text{for } r=1,2,\dots \text{ and } 0\le t\le (b-a)/r,$$

and the finite difference with a fixed step h is given by

$$\Delta_h^r f(x) = \begin{cases} \sum_{k=0}^{r} (-1)^{r+k}\binom{r}{k} f(x+kh) & \text{if } x, x+rh\in[a,b],\\ 0 & \text{otherwise.}\end{cases}$$

The space AC_{loc}^k is given by

$$AC_{loc}^k = AC_{loc}^k(a,b) = \{g : g^{(1)},\dots,g^{(k)}\in AC[c,d] \text{ for all } a<c<d<b\},$$

where $AC[c,d]$ is the set of absolutely continuous functions on $[c,d]$. As already mentioned, the moduli $\omega^r(f,t)_p$ and $K(f,t^r;L_p,AC_{loc}^{r-1},D^r)$ are equivalent. For example, Theorem 2.1.1 in [2, Chapter 2] states that (in the case where the weight function $\varphi\equiv 1$), there exist positive constants C and t_0 such that

$$C^{-1}\omega^r(f,t)_p \le K(f,t^r;L_p,AC_{loc}^{r-1},D^r)\le C\omega^r(f,t)_p,$$

for $f\in L_p(a,b)$, $1\le p\le\infty$, $0<t\le t_0$.

Using the equivalence between moduli of smoothness and appropriate K-functionals, we present a solution to the following:

Problem 1. *Let $f\in C[a,b]$ and $r\in\mathbb{N}$. Prove that $\omega^r(f,t)_\infty=0$ for some real number $t>0$ if and only if f is a polynomial of degree $r-1$.*

Problem 1 is a generalization of the following problem in [4, p. 52]: *Prove that a function $f\in C[a,b]$ is linear if and only if $\omega^2(f,t)_\infty=0$ for some $t>0$.*

We give a solution to Problem 1 in the general case and a new solution in the particular case $r=2$.

2. Solution to Problem 1. In what follows, we shall use the notation $\omega^r(f,t)=\omega^r(f,t)_\infty$ and $\|\cdot\|$ for the uniform norm on the space $C[a,b]$.

If f is a polynomial of degree $r-1$, then

$$\begin{aligned}&\Delta_h f \text{ is a polynomial of degree } r-2,\\ &\Delta_h^2 f = \Delta_h(\Delta_h f) \text{ is a polynomial of degree } r-3,\\ &\dots\\ &\Delta_h^{r-1} f = \Delta_h(\Delta_h^{r-2} f) \text{ is a polynomial of degree } 0, \text{ i.e. a constant},\\ &\Delta_h^r f = \Delta_h(\Delta_h^{r-1} f)\equiv 0;\end{aligned}$$

hence for all $t>0$, we have $\omega^r(f,t)=0$.

Now, let $\omega^r(f,t)=0$ for some $t>0$. If

$$K(f,t^r) = \inf_{g\in AC_{loc}^r}\left\{\|f-g\| + t^r\|g^{(r)}\|\right\},$$

then

$$C^{-1}\omega^r(f,t)\le K(f,t^r)\le C\omega^r(f,t),$$

for some positive constant C, i.e. $K(f,t^r)=0$ for some $t>0$. Hence, if n is a sufficiently large natural number, then there exists a function $g=g_n$ such that

$$\|f-g\|\le 1/n \qquad \text{and} \qquad \|g^{(r)}\|\le 1/n, \tag{1}$$

where $g\in C^{r-1}[a,b]$ and $g^{(r-1)}\in AC_{loc}[a,b]$.

Expanding g by Taylor's formula,

$$g(x) = g(x_0) + g^{(1)}(x_0)(x - x_0) + \cdots + g^{(r-1)}(x_0)\frac{(x - x_0)^{r-1}}{(r-1)!} + g^{(r)}(\xi)\frac{(x - x_0)^r}{r!},$$

where $x_0 \in (a, b)$, and denoting

$$P_{r-1,n}(x) = g(x_0) + g^{(1)}(x_0)(x - x_0) + \cdots + g^{(r-1)}(x_0)\frac{(x - x_0)^{r-1}}{(r-1)!},$$

we have, using (1) with $g = g_n$, for all $x \in [a, b]$,

$$|g_n(x) - P_{r-1,n}(x)| \le \frac{\|g_n^{(r)}\|(b - a)^r}{r!} \le \frac{c}{n}. \tag{2}$$

Now, from (2) and the first inequality in (1), we get, for all $x \in [a, b]$,

$$|f(x) - P_{r-1,n}(x)| \le |f(x) - g_n(x)| + |g_n(x) - P_{r-1,n}(x)| \le \frac{1 + c}{n}. \tag{3}$$

Since $\lim\limits_{n\to\infty}(1 + c)/n = 0$ and $f \in C[a, b]$, we obtain from (3),

$$\lim_{n\to\infty} \|f - P_{r-1,n}\| = 0. \tag{4}$$

Since the degree of all $P_{r-1,n}$ polynomials is at most $r - 1$, using (4), we will prove that f is also a polynomial of degree at most $r - 1$. Indeed, let

$$P_{r-1,n}(x) = a_0^{(n)} + a_1^{(n)}x + \cdots + a_{r-1}^{(n)}x^{r-1},$$

where the coefficients $a_i^{(n)}$, $i = 0, 1, \ldots, r-1$ depend on n, and let us fix x_i, $i = 1, 2, \ldots, r$ such that

$$a < x_1 < x_2 < \ldots < x_r < b. \tag{5}$$

Then

$$\begin{cases} a_0^{(n)} + a_1^{(n)}x_1 + \cdots + a_{r-1}^{(n)}x_1^{r-1} = f(x_1) + \varepsilon_1^{(n)} \\ a_0^{(n)} + a_1^{(n)}x_2 + \cdots + a_{r-1}^{(n)}x_2^{r-1} = f(x_2) + \varepsilon_2^{(n)} \\ \qquad\vdots \\ a_0^{(n)} + a_1^{(n)}x_r + \cdots + a_{r-1}^{(n)}x_r^{r-1} = f(x_r) + \varepsilon_r^{(n)}, \end{cases}$$

for some real numbers $\varepsilon_i^{(n)}$. Since $|\varepsilon_i^{(n)}| = |P_{r-1,n}(x_i) - f(x_i)| \le \|f - P_{r-1,n}\|$, it follows from (4) that

$$\lim_{n\to\infty} \varepsilon_i^{(n)} = 0, \qquad i = 1, 2, \ldots, r. \tag{6}$$

From (5), the Vandermonde determinant,

$$\Delta = \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{r-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_r & x_r^2 & \ldots & x_r^{r-1} \end{vmatrix} \ne 0,$$

does not depend on n. Then

$$a_i^{(n)} = \frac{1}{\Delta}\begin{vmatrix} 1 & \ldots & x_1^{i-1} & f(x_1) + \varepsilon_1^{(n)} & x_1^{i+1} & \ldots & x_1^{r-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ldots & x_r^{i-1} & f(x_r) + \varepsilon_r^{(n)} & x_r^{i+1} & \ldots & x_r^{r-1} \end{vmatrix}, \qquad i = 0, 1, \ldots, r-1.$$

From (6), it is clear that the following limits exist

(7)
$$\lim_{n\to\infty} a_i^{(n)} = a_i := \frac{1}{\Delta}\begin{vmatrix} 1 & \dots & x_1^{i-1} & f(x_1) & x_1^{i+1} & \dots & x_1^{r-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & x_r^{i-1} & f(x_r) & x_r^{i+1} & \dots & x_r^{r-1} \end{vmatrix}, \qquad i = 0, 1, \dots, r-1.$$

Since the interval $[a, b]$ is bounded, we obtain from (7),

(8)
$$\lim_{n\to\infty} \left(a_0^{(n)} + a_1^{(n)}x + \dots + a_{r-1}^{(n)}x^{r-1}\right) = a_0 + a_1 x + \dots + a_{r-1}x^{r-1}.$$

If
$$P_{r-1}(x) = a_0 + a_1 x + \dots + a_{r-1}x^{r-1},$$
from (4) and (8) we get
$$\lim_{n\to\infty} P_{r-1,n} = f \qquad \text{and} \qquad \lim_{n\to\infty} P_{r-1,n} = P_{r-1}.$$
From the uniqueness of the limit, it follows that
$$f(x) = P_{r-1}(x) = a_0 + a_1 x + \dots + a_{r-1}x^{r-1},$$
i.e. f is a polynomial of degree at most $r-1$. The proof is completed.

Remark 1. *The conclusion that* $\lim_{n\to\infty} P_{r-1,n}$ *is a polynomial of degree* $r-1$ *can be obtained by using the fact that every finite linear subspace of a complex topological vector space is closed, see [6, Theorem 1.21].*

We can formulate the following generalization:

Problem 2. *Prove that if* $f \in C[a, b]$, $r \in \mathbb{N}$ *and* $\omega^r(f, t) = o(t^r)$, *as* $t \to 0+$, *then* f *is a polynomial of degree* $r-1$.

Proof. As in the proof above, we use that moduli of smoothness are equivalent to proper K-functionals:
$$K(f, t^r) = \inf_{g\in AC_{loc}^r} \left\{\|f - g\| + t^r\|g^{(r)}\|\right\} \sim \omega^r(f, t) = o(t^r),$$
so putting $t = 1/n$, we obtain that, as $n \to \infty$,
$$K(f, n^{-r}) = o(n^{-r}).$$
Then for any natural number n, there exists a function $g_n \in C[a, b]$ such that $g_n \in AC_{loc}^r$, $\|f - g_n\| = o(n^{-r})$ and $n^{-r}\|g_n^{(r)}\| = o(n^{-r})$. Hence
$$\lim_{n\to\infty} \|f - g_n\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|g_n^{(r)}\| = 0,$$
which is analogous to the inequalities in (1). As in the proof above, we conclude that f is a polynomial of degree no greater than $r-1$. □

3. New solution to Problem 1 in the case $r = 2$. We will prove that if $f \in C[a, b]$ and $\omega^2(f, t) = 0$ for some $t > 0$, then f is a linear function. Without loss of generality, we can assume that $f \in C[0, 1]$. For any natural number n, the Bernstein operator is defined by
$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \qquad \text{for } x \in [0, 1].$$
Theorem 3.2 in [5] (see also [1]) states that the following inequality holds true:
$$|f(x) - B_n(f; x)| \le C\omega^2(f, \sqrt{X/n}),$$

for all $x \in (0,1)$ and some constant $C > 0$, where $X = x(1-x)$. In particular, when $n = 1$,

$$|f(x) - B_1(f;x)| \leq C\omega^2(f, 1/2)$$

for all $x \in (0,1)$. But from the properties of moduli, see [1, p. 45, (7.7)],

$$\omega^2(f, 1/2) = \omega^2\left(f, N\frac{1}{2N}\right) \leq N^2\omega^2\left(f, \frac{1}{2N}\right) = 0,$$

for a sufficiently large natural number N such that $1/2N < t$. Then $\omega^2(f, 1/2) = 0$ and $|f(x) - B_1(f;x)| = 0$, for $x \in [0,1]$, i.e.

$$f(x) = B_1(f;x) = f(0)(1-x) + f(1)x = (f(1) - f(0))x + f(0),$$

is a linear function.

References

[1] R.A. DeVore, G.G. Lorentz. Constructive Approximation. Springer-Verlag, Berlin, 1993.

[2] Z. Ditzian, V. Totik. Moduli of Smoothness. Springer-Verlag, New-York, 1987.

[3] H. Johnen, K. Scherer. On the equivalence of K-functionals and moduli of continuity and some applications. *Lecture Notes in Math.*, **571** (1976), 119-140.

[4] G.G. Lorentz. Approximation of Functions. Chelsea Publishing Company, New York, 1986.

[5] T. Popoviciu. Sur l'approximation des fonctions convexes d'ordre supérieur. *Mathematica Cluj*, **10** (1935), 49-54, (in French).

[6] W. Rudin. Functional Analysis, 2nd edn. McGraw-Hill, Inc., New York, 1991.