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PROBABILITIES FOR ASYMMETRIC p-OUTSIDE VALUES

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Since 2017, Jordanova and co-authors investigate probabilities for p-outside values and determine their explicit forms in many particular cases. They show that they are closely related to the concept of heavy tails. Tukey's box-plots are very popular and useful in practice. In particular, the relative frequencies for observing a data point in different sections of the box-plot can help practitioners find the exact probability distribution of the underlying random variable. These open the door to working with distribution-sensitive estimators, which can be more accurate, especially in small sample investigations. All these methods, however, suffer from the drawback that they use the interquantile range in a symmetric way. This study gives greater influence on asymmetry in the analysis of the tails of distributions. New theoretical and empirical box-plots and characteristics of the tails of distributions are suggested. The theoretical asymmetric p-outside value functions do not depend on the center and scaling factor of the distribution and do not need existence of moments. Therefore, they are appropriate for comparing the tails of distributions and for estimating the parameters that govern their tail behaviour.

Keywords: heavy-tailed distributions, extremal index estimation.

ВЕРОЯТНОСТИ ЗА АСИМЕТРИЧНИ p-ВЪНШНИ СТОЙНОСТИ

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От 2017 г. Йорданова и съавтори изследват вероятностите за p-външни стойности и намират техния явен вид в много конкретни случаи. Те показват, че тези вероятности са тясно свързани с концепцията за тежки опашки. Кутиите с мустачки на Тюки са много популярни и полезни на практика. Относителните честоти на събитията дадено наблюдение да попадне в различни техни части допринасят за намирането

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на точното вероятностно разпределение на наблюдаваната величина. Те правят възможна работата с чувствителни към разпределението оценки, които могат да бъдат по-точни, особено при малки извадки. Всички тези методи, обаче страдат от недостатъка, че използват междуквантилния размах по симетричен начин. Резултатите от това изследване позволяват по-голямо влияние на асиметрията върху анализа на опашките на разпределенията. Предложени са нови техни теоретични и емпирични графики и характеристики. Асиметричните функции на външните стойности не зависят от центъра и коефициента на мащабиране на разпределението и нямат нужда от съществуване на моменти. Те са подходящи за сравняване на опашките и за оценяване на параметрите, които управляват тяхното поведение.

Ключови думи: разпределения с тежки опашки, оценки на екстремалния индекс

1. Introduction. Throughout this work, we consider a sample of independent identically distributed (i.i.d.) observations $X_1, \ldots, X_n$ on a random variable (r.v.) X and the corresponding increasing order statistics $X_{1:n} \leq \ldots \leq X_{n:n}$. Denote the right-continuous version of the cumulative distribution function (c.d.f.) of X by $F_X(x) = \mathbb{P}(X \leq x)$ and its quantile function (generalized left-continuous inverse) by $F_X^{\leftarrow}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$, $p \in (0, 1]$. We assume that F_X is continuous, $F_X^{\leftarrow}(0) = \sup\{x \in \mathbb{R} : F_X(x) = 0\}$, and $\sup \emptyset = -\infty$. The empirical p-quantiles can be defined in different ways. We restrict ourselves to the version defined via the empirical c.d.f.,

$$F_n(x) := \begin{cases} 0, & x < X_{1:n}; \\ \dfrac{i}{n}, & x \in [X_{i:n}; X_{i+1:n}); \\ 1, & x \geq X_{n:n}, \end{cases} \qquad x \in \mathbb{R}. \tag{1}$$

Then $F_n(X_{i:n}) = i/n$, $i = 1, 2, \ldots, n$.

Assume that $F_n^{\leftarrow}(0) = X_{1:n}$ and, for $p \in ((i-1)/n, i/n]$, $i = 1, 2, \ldots, n$, define the empirical quantile function by

$$F_n^{\leftarrow}(p) := \inf\{x \in \mathbb{R} : F_n(x) \geq p\} = X_{\lceil np \rceil : n} = X_{i:n}, \tag{2}$$

where $\lceil a \rceil$ is the ceiling of a, i.e. the smallest integer greater than or equal to a. See, e.g., [2, 5, 10]. The above can be implemented in R using the function *quantile* with parameter $Type = 1$ [9]. Then $F_n^{\leftarrow}(i/n) = X_{i:n}$, $i = 1, 2, \ldots, n$.

In 1977-1978, Tukey et al. [11, 8] defined the box-and-whisker plots. In [1], there is a definition for mild and extreme outliers related to similar extremal box-plots. These works define outliers as points isolated to varying degrees from the center of the distribution, without taking into account the exact distribution of the observed r.v. Given a sample of independent observations on a r.v. of a fixed probability type, the authors of [6] compute the probabilities of an observation to be a mild or extreme outlier and suggest to classify the tails of the linear probability types with respect to these probabilities. The latter, compared with Devore's plots for extreme outliers, allows one to determine approximately the distribution of the observed r.v. Later, Jordanova and co-authors [7, 4, 5] generalize these results for arbitrary $p \in (0, 0.5]$. They investigate the probabilities for p-outside values and determine them in many particular cases. All these results suffer from the drawback that they use the interquartile range (IQR) in a symmetric way. More precisely, the midpoint of the interval between the p-th and $(1-p)$-th quantile coincides with the midpoint of the interval between the corresponding left and right p-fences. It is well-known that the tail behaviour is not obligatory symmetric with respect to the center of the distribution. Here, we give greater influence to asymmetry when analyzing the tails of distributions and improve these results. We suggest new theoretical and empirical asymmetric p-box-plots and corresponding probabilities for asymmetric left and right p-outside values. We then find their explicit forms in the Pareto and Fréchet cases. This allows us to improve the existing classification of distributions in terms of their tail behaviour. Finally, we present how these characteristics can help us find the most appropriate classes of distributions for fitting the corresponding distribution of the observed r.v.

This work mainly considers the right tails of distributions in the case where there are positive probabilities for right p-outside values. There are many cases where these probabilities are zero. In these cases, we do not need to estimate the parameter which governs the tail behaviour. Therefore, these cases are not the subject of this study. Analogous results for the left tails can be easily obtained by multiplying the considered r.v. by -1.

2. Definitions and preliminaries.

Definition 1. *Fix $p \in (0, 0.5]$. We call the* **theoretical asymmetric right p-fence** *the value of*

$$\begin{aligned} R^A(F_X, p) \equiv R^A(X, p) &:= F_X^{\leftarrow}(1-p) + 2\frac{1-p}{p}(F_X^{\leftarrow}(1-p) - F_X^{\leftarrow}(0.5)) \\ &= \frac{2-p}{p}F_X^{\leftarrow}(1-p) - 2\frac{1-p}{p}F_X^{\leftarrow}(0.5) \\ &= F_X^{\leftarrow}(0.5) + \frac{2-p}{p}(F_X^{\leftarrow}(1-p) - F_X^{\leftarrow}(0.5)). \end{aligned}$$

The corresponding function of $p \in (0, 0.5]$ is called the **theoretical asymmetric right fence function**.

Definition 2. *Fix $p \in (0, 0.5]$. We call the* **theoretical asymmetric left p-fence** *the value of*

$$\begin{aligned} L^A(F_X, p) \equiv L^A(X, p) &:= F_X^{\leftarrow}(p) - 2\frac{1-p}{p}(F_X^{\leftarrow}(0.5) - F_X^{\leftarrow}(p)) \\ &= \frac{2-p}{p}F_X^{\leftarrow}(p) - 2\frac{1-p}{p}F_X^{\leftarrow}(0.5) \\ &= F_X^{\leftarrow}(0.5) - \frac{2-p}{p}(F_X^{\leftarrow}(0.5) - F_X^{\leftarrow}(p)). \end{aligned}$$

The corresponding function of $p \in (0, 0.5]$ is called the **theoretical asymmetric left fence function**.

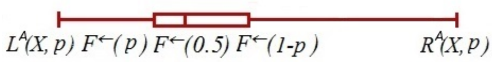

Figure 1: Scheme of asymmetric theoretical box-plot.

Figure 1 depicts an example of the placement of asymmetric fences with respect to the center of the distribution in the case when $p = 0.25$.

Definition 3. *Fix $p \in (0, 0.5]$. We denote by*

$$p_{A,R,p}(X) \equiv p_{A,R,p}(F_X) := \mathbb{P}(X > R^A(F_X, p))$$

the probability that a r.v. X exceeds the theoretical asymmetric right p-fence of its distribution, and call it the **probability for asymmetric right p-outside value***. The corresponding function $p_{A,R,p}(X)$ of $p \in (0, 0.5]$ is called the* **theoretical asymmetric right outside value function***.*

Definition 4. *Fix $p \in (0, 0.5]$. We denote by*

$$p_{A,L,p}(X) \equiv p_{A,L,p}(F_X) := \mathbb{P}(X < L^A(F_X, p))$$

the probability that a r.v. X is less than the theoretical asymmetric left p-fence of its distribution, and call it the **probability for asymmetric left p-outside value***. The corresponding function $p_{A,L,p}(X)$ of $p \in (0, 0.5]$ is called the* **theoretical asymmetric left outside value function***.*

The main but easy to prove property of these probabilities is that, for any constants $a \in \mathbb{R}$ and $b > 0$,

$$p_{A,L,p}(X) = p_{A,L,p}\left(\frac{X+a}{b}\right) \quad \text{and} \quad p_{A,R,p}(X) = p_{A,R,p}\left(\frac{X+a}{b}\right). \tag{3}$$

This means that they do not depend on the center and scaling factor of the distribution. Therefore, they are very suitable for comparison of the tails of distributions and estimation of the coefficients which govern the tail behaviour. Here again we must clarify that if the scaling is negative, then we have to change our considerations, and instead of the left tail the results are for the right tail and vice versa. The most important case is $p_{A,L,p}(-X) = p_{A,R,p}(X)$, because $p_{A,L,p}(-bX) = p_{A,R,p}(X)$, for any $b > 0$.

Another interesting property is that these probabilities are invariant with respect to exceedances of the distribution. More precisely, for all $p \in (0, 0.5]$ and $t \in \mathbb{R}$ such that $\mathbb{P}(X < t) > 0$,

$$p_{A,L,p}(X - t|X > t) = p_{A,L,p}(X|X > t), \quad p_{A,R,p}(X - t|X > t) = p_{A,R,p}(X|X > t).$$

Remark. The asymmetric outside values depend on the distribution of the considered r.v. An observation can be an asymmetric p-outside value with respect to (w.r.t.) a distribution, and for the same p, not be an asymmetric p-outside value w.r.t. another probability law. In this way, one can chose the probability type that fits the data in the best way.

By plugin the estimators (1) and (2) in the above four definitions, we obtain the corresponding empirical counterparts.

Definition 5. *Fix $p \in (0, 0.5]$. The* **empirical asymmetric right p-fence** *is*

$$\begin{aligned} R_n^A(p) &:= F_n^{\leftarrow}(1-p) + 2\frac{1-p}{p}(F_n^{\leftarrow}(1-p) - F_n^{\leftarrow}(0.5)) \\ &= \frac{2-p}{p}F_n^{\leftarrow}(1-p) - 2\frac{1-p}{p}F_n^{\leftarrow}(0.5) \\ &= F_n^{\leftarrow}(0.5) + \frac{2-p}{p}(F_n^{\leftarrow}(1-p) - F_n^{\leftarrow}(0.5)). \end{aligned}$$

The corresponding function of $p \in (0, 0.5]$ is called the **empirical asymmetric right fence function***.*

Definition 6. *Fix* $p \in (0, 0.5]$. *The* **empirical asymmetric left** p**-fence** *is*

$$\begin{aligned} L_n^A(p) &:= F_n^{\leftarrow}(p) - 2\frac{1-p}{p}(F_n^{\leftarrow}(0.5) - F_n^{\leftarrow}(p)) \\ &= \frac{2-p}{p}F_n^{\leftarrow}(p) - 2\frac{1-p}{p}F_n^{\leftarrow}(0.5) \\ &= F_n^{\leftarrow}(0.5) - \frac{2-p}{p}(F_n^{\leftarrow}(0.5) - F_n^{\leftarrow}(p)). \end{aligned}$$

The corresponding function of $p \in (0, 0.5]$ *is called the* **empirical asymmetric left fence function**.

Definition 7. *Fix* $p \in (0, 0.5]$. *We call the* **empirical asymmetric right (or left)** p**-outside values** *the observations* X_i, $i = 1, 2, \ldots, n$ *that are bigger (or smaller) than* $R_n^A(p)$ *(or* $L_n^A(p)$*).*

Definition 8. *Fix* $p \in (0, 0.5]$. *The relative frequency of the observations* X_i, $i = 1, 2, \ldots, n$ *that are bigger (or smaller) than* $R_n^A(p)$ *(or* $L_n^A(p)$*) is called the* **plug-in estimator of the probability for asymmetric right (or left)** p**-outside value**. *We denote them by* $\hat{p}_{A,L}(p)$ *(or* $\hat{p}_{A,R}(p)$*).*

Analogously to the results in [5, Section 3.3], it can be shown that under relatively general conditions the empirical asymmetric right(left) p-fences are asymptotically unbiased, asymptotically normal, weakly consistent, and asymptotically efficient estimators for the corresponding theoretical asymmetric right(left) p-fences. Moreover, the latter plug-in estimators are weakly consistent and, under mild conditions, are asymptotically unbiased and asymptotically efficient estimators for the corresponding probabilities for asymmetric p-outside values.

3. Particular cases. Probabilities for extreme outliers (or “0.25-outside values”) for the probability distributions considered in this section are computed in [7, 4, 5]. Here, we determine theoretical asymmetric fence functions and theoretical asymmetric outside value functions for four particular cases: exponential, Pareto, Fréchet, and log-logistic. We consider $p \in (0, 0.5]$ and, due to property (3), without lost of generality, we chose the simplest parametric form of the considered linear probability types.

Exponential l**-type**. Let X be a r.v. with c.d.f. $F_X(x) = 1 - e^{-x}$ for $x \geq 0$, and $F_X(x) = 0$ otherwise. Then $F_X^{\leftarrow}(p) = -\log(1-p)$ and $R^A(X, p) = -(2-p)\log(p)/p - 2(1-p)\log(2)/p$. In this case, $L^A(X, p) = -(2-p)\log(1-p)/p - 2(1-p)\log(2)/p$. The plots of $p_{A,R,p}(X) = p^{(2-p)/p}4^{(1-p)/p}$ and $p_{R,p}(X)$, see [3, 5], are presented in Figure 2. For $p \in (0, p_0]$, where p_0 is the solution of $(2-p)\log(1-p) + 2(1-p)\log(2) = 0$, i.e. $p_0 \approx 0,4041$, we have $p_{A,L,p}(X) = 0$. When $p \in (p_0, 0.5]$, we have $p_{A,L,p}(X) = 1 - (1-p)^{(2-p)/p}4^{(1-p)/p}$. In particular, $p_{A,R,0.25}(X) = 1/256 \approx 0.0039$.

Pareto l**-type with positive shape parameter** $\xi = \alpha^{-1} > 0$. In this case, X is a r.v. with c.d.f. $F_X(x) = 1 - x^{-1/\xi}$ for $x \geq 1$, and $F_X(x) = 0$ otherwise. Then $F_X^{\leftarrow}(p) = (1-p)^{-\xi}$ and $R^A(X, p) = (2-p)/p^{1+\xi} - (1-p)2^{\xi+1}/p$. Now, $L^A(X, p) = (2-p)/(p(1-p)^{\xi}) - (1-p)2^{\xi+1}/p$. The plots of $p_{A,R,p}(X) = ((2-p)/p^{1+\xi} - (1-p)2^{\xi+1}/p)^{-1/\xi}$ and $p_{R,p}(X)$, see [3, 5], for different values of the shape parameter ξ, are presented in Figure 3. We observe that the larger the parameter ξ, the greater these probabilities are. This corresponds to the well-known result from extreme value theory that the greater the parameter ξ, the heavier the right tail of the distribution. Analogously, for $p \in (0, p_0]$, where p_0 is the solution of $(2-p)(1-p)^{-\xi} - 2^{\xi+1}(1-p) = p$, we have $p_{A,L,p}(X) = 0$,

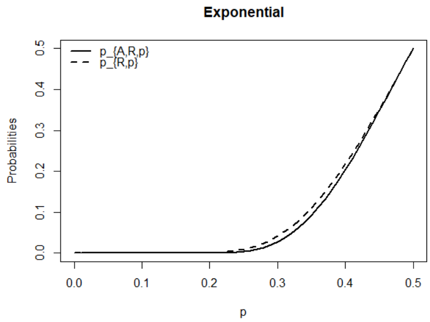

Figure 2: $p_{A,R,p}(X)$ (solid line) and $p_{R,p}(X)$ (dashed line) in the exponential case.

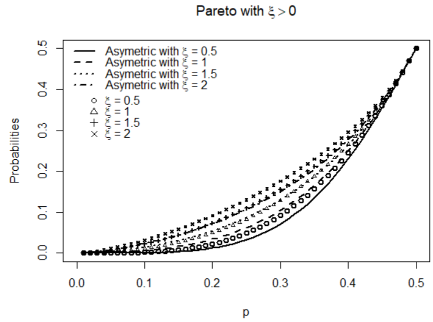

Figure 3: $p_{A,R,p}(X)$ and $p_{R,p}(X)$ in the Pareto cases.

and $p_{A,L,p}(X) = 1 - ((2-p)/(p(1-p)^{\xi}) - (1-p)2^{\xi+1}/p)^{-1/\xi}$, for $p \in (p_0, 0.5]$.

Fréchet l-type with parameter $\alpha > 0$. Let us now consider a r.v X with c.d.f. $F_X(x) = \exp\{-x^{-\alpha}\}$ for $x \geq 0$, and $F_X(x) = 0$ otherwise. In this case, the quantile function is $F_X^{\leftarrow}(p) = (-\log(p))^{-1/\alpha}$. Therefore, Definitions 1 and 2 entail

$$L^A(X,p) = \frac{(2-p)}{p}(-\log(p))^{-\frac{1}{\alpha}} - 2\frac{1-p}{p}(\log(2))^{-\frac{1}{\alpha}},$$
$$R^A(X,p) = \frac{(2-p)}{p}(-\log(1-p))^{-\frac{1}{\alpha}} - 2\frac{1-p}{p}(\log(2))^{-\frac{1}{\alpha}}.$$

In this case, $L^A(X,p) \geq 0$ if and only if $(2-p)(-\log(p))^{-1/\alpha} \geq 2(1-p)(\log(2))^{-1/\alpha}$ if and only if $(-\log(2)/\log(p))^{1/\alpha} \geq 2(1-p)/(2-p)$ if and only if $\log(-\log(2)/\log(p)) \geq \alpha\log(2(1-p)/(2-p))$ if and only if $\alpha \geq \log(-\log(2)/\log(p))/\log(2(1-p)/(2-p))$. The last consideration means that if we denote by $\alpha_0 := \log(-\log(2)/\log(p))/\log(2(1-p)/(2-p))$, then for $\alpha \in (0, \alpha_0]$, we have $p_{A,L,p}(X) = 0$, and for $\alpha > \alpha_0$, letting p_0 be the solution of $(2-p)(-\log(p))^{-1/\alpha} = 2(1-p)(\log(2))^{-1/\alpha}$, we have, for $p \in (0, p_0]$, that $p_{A,L,p}(X) = 0$ and, for $p \in (p_0, 0.5]$,

$$p_{A,L,p}(X) = \exp\left\{-\left(\frac{1}{p}(-\log(p))^{-\frac{1}{\alpha}} - \frac{1-p}{p}(\log(2))^{-\frac{1}{\alpha}}\right)^{-\alpha}\right\}.$$

Analogously, we obtain $R^A(X,p) \geq 0$, for all $\alpha > 0$ and $p \in (0, 0.5]$. Therefore,

$$p_{A,R,p}(X) = 1 - \exp\left\{-\left(\frac{(2-p)}{p}(-\log(1-p))^{-\frac{1}{\alpha}} - 2\frac{1-p}{p}(\log(2))^{-\frac{1}{\alpha}}\right)^{-\alpha}\right\}.$$

In this case, the plots of $p_{A,R,p}(X)$ and $p_{R,p}(X)$ (the latter is defined in [3, 5]) as functions of p and for different values of the parameter α could be seen in Figure 4. We observe that the smaller the parameter α, the greater these probabilities are and the heavier the right tail of the distribution.

Log-logistic l-type with parameter $\alpha > 0$. Let X be a r.v. with c.d.f. $F_X(x) = 1/(1 + x^{-\alpha})$ for $x \geq 0$, and $F_X(x) = 0$ otherwise. This c.d.f. corresponds to $F_X^{\leftarrow}(p) = (p/(1-p))^{1/\alpha}$, $R^A(X,p) = (2-p)((1-p)/p)^{1/\alpha}/p - 2(1-p)/p$ and $L^A(X,p) = (2 -$

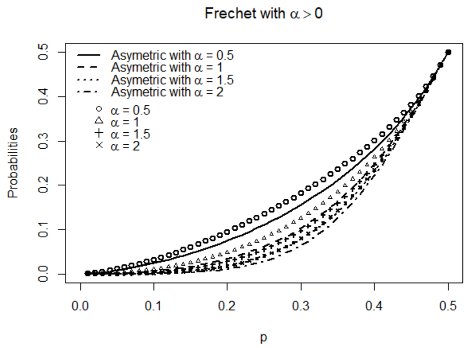

Figure 4: $p_{A,R,p}(X)$ and $p_{R,p}(X)$ in the Fréchet cases.

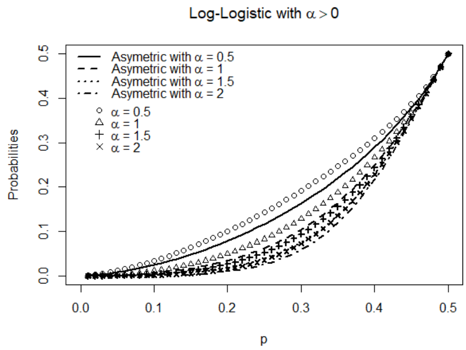

Figure 5: $p_{A,R,p}(X)$ and $p_{R,p}(X)$ in the Log-Logistic cases.

$p)(p/(1-p))^{1/\alpha}/p - 2(1-p)/p$. The latter means that $L^A(X,p) \geq 0$ if and only if $\alpha \geq \alpha_0 \geq 0$, where $\alpha_0 := \log(p/(1-p))/\log(2(1-p)/(2-p))$. Thus, $p_{A,L,p}(X) = 0$, for all $p \in (0, 0.5]$, when $\alpha \leq \alpha_0$. If $\alpha \geq \alpha_0$ and p_0 is the solution of the equation $(p/(1-p))^{1/\alpha} = 2(1-p)/(2-p)$, then $p_{A,L,p}(X) = 0$ when $p \in (0, p_0]$, and

$$p_{A,L,p}(X) = \frac{1}{1 + \left(\frac{2-p}{p}\left(\frac{p}{1-p}\right)^{\frac{1}{\alpha}} - 2\frac{1-p}{p}\right)^{-\alpha}} \qquad \text{when } p \geq p_0.$$

Analogously, we obtain $R^A(X,p) \geq 0$, for all $\alpha > 0$. Therefore, for all $p \in (0, 0.5]$,

$$p_{A,R,p}(X) = \frac{1}{1 + \left(\frac{2-p}{p}\left(\frac{1-p}{p}\right)^{\frac{1}{\alpha}} - 2\frac{1-p}{p}\right)^{\alpha}}.$$

The last probability, together with $p_{R,p}(X)$, are plotted in Figure 5. They confirm that the tail behaviour of the log-logistic distribution is very similar to the one of Pareto and Fréchet distributions.

4. A new classification of linear probability types with respect to their tail behaviour. It is well-known that in the case where it exists, the tail index governs the tail behaviour of the corresponding probability c.d.f. independently of the location and scale parameters. The same is true for the probabilities for asymmetric right p-outside values. However, these probabilities allow us to insert a larger class of distributions in this partial ordering for any fixed p. Here, we fix $p = 0.25$. To simplify our considerations, we discuss only absolutely continuous distributions with infinite supports.

Using the results in the previous section, we obtain that, for all distributions in the exponential linear probability type, the probability of asymmetric 0.25-outside values is $p_{A,R,p}(X) \approx 0.0039$. Analogously, we could compute that for the Gumbel linear type it is ≈ 0.0015, and for the logistic case it is ≈ 0.0005. The dependence of the probabilities for asymmetric 0.25-outside values on the parameter $\alpha = 1/\xi$ within the set of positive-Weibull, Hill-Horror (see, e.g., [5] for the definition), and the considered distributions is presented in Figure 6. In this plot, the exponential distribution appears as positive-

Weibill with parameter $\alpha = 1$.

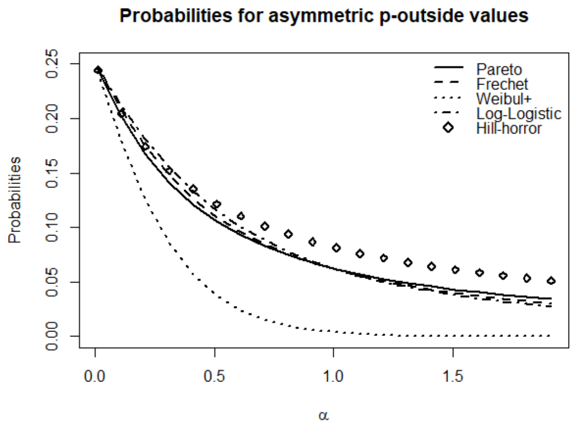

Figure 6: Dependence of $p_{A,R,p}(X)$ on α.

The classification, based on the probabilities for asymmetric 0.25-outside values, confirms the corresponding well-known results based on the concepts of light- and heavy-tailed distributions and the extremal index. Its advantage is that it is more detailed, and therefore, more useful in practice. It outperforms the popular concept of kurtosis insofar as in many of the considered cases the kurtosis does not exist.

5. Two distribution-sensitive estimators of the parameters which govern the tail behaviour. Simulation study. Here, we assume that we have a sample of i.i.d observations, that there is at least one observation which is larger than the empirical asymmetric right p-fence, and that $R_n^A(p) > 1$. Thus, we are working given $X_{n:n} > R_n^A(p) > 1$. Depending on the values of the plug-in estimators of $p_{A,R,p}(X)$ and $p_{A,L,p}(X)$, we choose the most appropriate probability types for modelling the observed r.v. Then, analogously to the method of moments and by using the expected equality between the large sample estimates and the theoretical values of $p_{A,R,p}(X)$, we express the unknown parameter. In this way, we obtain different formulae for the estimators. In order to simplify the computations, empirical asymmetric right p-fence can be used if needed. Finally, we can use some goodness of fit test in order to chose the best model. It is clear that in order to apply this algorithm, we need the explicit form of $p_{A,R,p}(X)$.

We consider two cases and obtain the following distribution-sensitive estimators.

Case 1. If the observed r.v. X is $Pareto(\alpha)$, $\alpha > 0$ distributed, then we apply

$$\widehat{\alpha}_{Par,n}(p) = -\frac{\log\left(\widehat{p}_{A,R}(p)\right)}{\log\left(\widehat{R}_n^A(p)\right)}.$$

Case 2. For $X \in Fr\acute{e}chet(\alpha)$, $\alpha > 0$ the corresponding estimator is

$$\widehat{\alpha}_{Fr,n}(p) = -\frac{\log\left(-\log\left(1 - \widehat{p}_{A,R}(p)\right)\right)}{\log\left(\widehat{R}_n^A(p)\right)}.$$

The best value of p seems to be around the middle of the interval $(0, 0.5]$, because in this case the sensitivity of $p_{A,R,p}(X)$ with respect to the different probability laws is the

Table 1: Empirical results.

Distribution of X.		n	$\hat{\alpha}_{Par,n}$		$\hat{\alpha}_{Frech,n}$		Better estimator
			Mean	St. Dev.	Mean	St. Dev.	
$Pareto(\alpha)$	$\alpha = 0.5$	30	0.5434	0.1318	**0.5318**	0.1341	$\widehat{\alpha}_{Fr,n}$
		100	0.5147	0.0692	**0.5024**	0.0707	$\widehat{\alpha}_{Fr,n}$
		1000	**0.5011**	0.0205	0.4887	0.0210	$\widehat{\alpha}_{Par,n}$
		10000	**0.5002**	0.0065	0.4877	0.0066	$\widehat{\alpha}_{Par,n}$
	$\alpha = 1$	30	1.0339	0.2339	**1.0209**	0.2366	$\widehat{\alpha}_{Fr,n}$
		100	1.0345	0.1544	**1.0229**	0.1571	$\widehat{\alpha}_{Fr,n}$
		1000	**1.0033**	0.0445	0.9917	0.0453	$\widehat{\alpha}_{Par,n}$
		10000	**1.0004**	0.0141	0.9889	0.0144	$\widehat{\alpha}_{Par,n}$
	$\alpha = 2$	30	**1.8935**	0.3993	1.8764	0.4005	$\widehat{\alpha}_{Par,n}$
		100	2.0760	0.3316	**2.0655**	0.3352	$\widehat{\alpha}_{Fr,n}$
		1000	**2.0109**	0.1029	2.0012	0.1042	$\widehat{\alpha}_{Par,n}$
		10000	**2.0009**	0.0317	1.9912	0.0321	$\widehat{\alpha}_{Par,n}$
$Frechet(\alpha)$	$\alpha = 0.5$	30	0.5621	0.1469	**0.5492**	0.1485	$\widehat{\alpha}_{Fr,n}$
		100	0.5293	0.0747	**0.5159**	0.0760	$\widehat{\alpha}_{Fr,n}$
		1000	0.5151	0.0218	**0.5014**	0.0223	$\widehat{\alpha}_{Fr,n}$
		10000	0.5138	0.0069	**0.5001**	0.0070	$\widehat{\alpha}_{Fr,n}$
	$\alpha = 1$	30	1.0482	0.2376	**1.0351**	0.2399	$\widehat{\alpha}_{Fr,n}$
		100	1.0522	0.1645	**1.0407**	0.1672	$\widehat{\alpha}_{Fr,n}$
		1000	1.0147	0.0462	**1.0032**	0.0471	$\widehat{\alpha}_{Fr,n}$
		10000	1.0117	0.0143	**1.0002**	0.0146	$\widehat{\alpha}_{Fr,n}$
	$\alpha = 2$	30	**1.8662**	0.3918	1.8507	0.3923	$\widehat{\alpha}_{Par,n}$
		100	2.0789	0.3330	**2.0699**	0.3362	$\widehat{\alpha}_{Fr,n}$
		1000	2.0192	0.1068	**2.0109**	0.1080	$\widehat{\alpha}_{Fr,n}$
		10000	2.0095	0.0332	**2.0012**	0.0335	$\widehat{\alpha}_{Fr,n}$

best one. Therefore, this section ends with $p = 0.25$. First, we made n independent observations from a $Pareto(\alpha)$ distribution and, separately, n independent observations from a $Frechet(\alpha)$ distribution, for different values of $\alpha \in \{0.5, 1, 2\}$ and $n = 30, 10^2, 10^3, 10^4$. We then repeated these experiments $m = 10^4$ times. In this way, we calculated 10^4 values of $\hat{\alpha}_{Par,n}$ and $\hat{\alpha}_{Frech,n}$, their means and standard deviations. The results are given in Table 1. Analogously to [6], we observe that for distributions with heavier tails, our approach gives better results.

6. Conclusive remarks and open problems. In this work, we have shown that the suggested classification of linear probability types with respect to their tail behaviour refines the well-known classification based on the extremal index in the cases where it exists. It outperforms the role of kurtosis. The corresponding estimators have a relatively fast rate of convergence. However, their biggest drawback is that they are distribution-sensitive. Therefore, choosing the most appropriate probability type is an important preliminary step before estimating the parameters that govern the tail behaviour.

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