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INTEGER CODES AND THEIR APPLICATIONS: AN OVERVIEW [1]

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Abstract

The paper is a short overview of the known applications of codes over a ring of integers modulo a given positive integer to error controlling in digital communications and storage. The presented successful applications demonstrate that integer codes with their simple encoding and decoding are at least commensurable with other codes with similar parameters. Moreover in many situations they are preferable.

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Key words: integer codes, QAM, flash memory, optical channels.

1 Introduction. Integer codes is the name used to refer to codes over a ring $\mathbb{Z}_q$ of integers modulo a given positive integer q (not necessarily a power of a prime). They are defined as the dual space of an $m \times n$ parity-check matrix $\mathbf{H}$ with elements from $\mathbb{Z}_q$, namely

$$\mathcal{C}(\mathbf{H}) = \{\mathbf{c} \in \mathbb{Z}_q^n \mid \mathbf{c}\mathbf{H}^T = \mathbf{0}\}.$$

We will write only $\mathcal{C}$ if there is no possibility for ambiguity.

$\mathcal{C}$ is said to be $[n, n-m]_q$ integer code although it is a linear space only in the case q prime. In general so defined codes are algebraic modules over a ring. Their properties, although very close, differ from ones of linear spaces. In particular the size of $\mathcal{C}$ may differ from q^{n-m}.

Maybe the first application of integer codes was proposed by Varshamov and Tenengolz in [23, 22] as codes correcting single asymmetric errors. Since that time integer codes have been successfully used in many applications. Vinck and Morita [5] have applied such codes to magnetic recording and frame synchronization. Kostadinov, Morita and Manev proposed in [12, 13] integer codes to be applied to coded modulation. The papers [11, 15, 10, 2] present investigations in the same direction.

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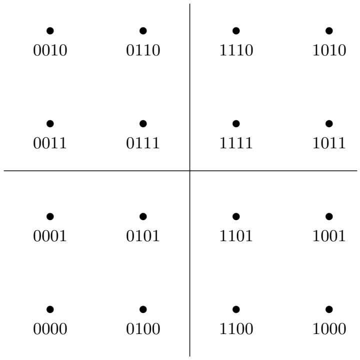

Figure 1: 16-SQAM constellation with Gray mapping

Other areas of applications of integer codes are flash memory [3, 6, 7, 8] and several specific uses given in [19, 20, 17, 21, 18, 16, 14].

The successful application of integer codes to many different areas in communications and storage is due to not only their simple encoding and decoding but the fact that integer codes can be constructed to correct the dominant in the channel type of errors. By choosing a proper parity-check matrix one can construct a code correcting specific types of errors.

Definition 1.1. *Let $k_i, l_j,\ i = 1, \ldots, s,\ j = 1, \ldots, m$ be positive integers. A code $\mathcal{C}(\mathbf{H})$ is said to be w $(\pm k_1, \pm k_2, \ldots, \pm k_s, l_1, l_2, \ldots, l_m)$-**error correctable** if it can correct any error vector with at most w nonzero entries with values l_j or $\pm k_i$.*

Requirements for error-correcting properties of integer codes lead to a lower bound for q. For example, if the code $\mathcal{C}(\mathbf{H})$ is a single $(\pm k_1, \pm k_2, \ldots, \pm k_s, l_1, l_2, \ldots, l_m)$-error correctable then

$$q \geq (2s + m)n + 1.$$

The goal of this paper is to present in short the research results on integer codes with an accent on construction of such codes for practical applications in digital communication and storage.

2 Application to coded Quadrature Amplitude Modulation.

The name coded modulation schemes is commonly used for communication schemes that integrate modulation and error-correcting techniques. Quadrature Amplitude Modulation (QAM) is a communication scheme for transmission of data by signals that are sum of two orthogonal signals with prescribed amplitudes. Signals are represented as points in the plane and form a finite lattice called constellation. The most implemented constellation scheme is Square Quadrature Amplitude Modulation (SQAM) whose constellation of signal points is a square or rectangular lattice with sides parallel to coordinate axes. However Triangular Quadrature Amplitude Modulation (TQAM) and several specific types of constellations are also studied.

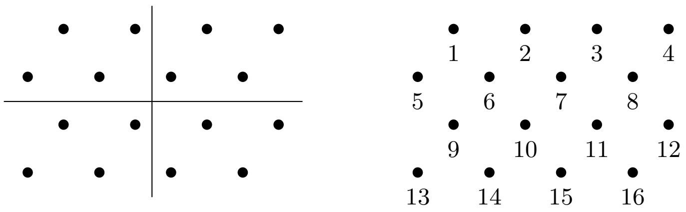

Figure 2: 16-TQAM constellations and a possible numbering of its points by elements of $\mathbb{Z}_{17}$

Signal points of TQAM form a lattice of equilateral triangles. They are placed in rows parallel to the horizontal axis and the constellation is symmetric with respect to the origin. If the side of triangles has length $2d$, the distance between horizontal rows is $h = d\sqrt{3}$ (which is the altitude of the equilateral triangles). An example of 16-TQAM is given in Figure 2.

Due to the noise in the communication channel the sent point can be erroneously detected as another point, most probably as one of the nearest neighbor points. In the case of SQAM there are 4 such points (see Figure 1) while in the case of TQAM there are 6 nearest points (see Figure 2). Figure 2 shows that the nearest points correspond to differences $E = \{\pm 1, \pm 3, \pm 4, \pm 5\}$ between their labels. It is easy to see (the numbers of next row increase by t) that in the general case of t^2-TQAM constellation labeled by the nonzero elements of $\mathbb{Z}_{t^2+1}$ the set of possible differences to the nearest neighbors' labels is

$$E = \{\pm 1, \pm(t-1), \pm t, \pm(t+1)\}.$$

In [2] single $(\pm 1, \pm(t-1), \pm t, \pm(t+1))$-error correctable codes with maximum possible length are described. Namely, the following is proven

Theorem 2.1 ([2]). *Let*

$$H_{a_0} = \{a_0 t + a_1 \mid 3a_0 + 1 \le a_1 \le t - 2a_0 - 2\}.$$

For a given even integer $t \ge 6$, the integer code $\mathcal{C}(\mathbf{H})$ over $\mathbb{Z}_{t^2+1}$ with parity-check matrix $\mathbf{H} = (H_0, H_1, H_2, \ldots, H_m)$, where $m \le \frac{t-2}{5}$, is single $(\pm 1, \pm(t-1), \pm t, \pm(t+1))$-error correctable.

Table 1 contains the parity-check matrices $\mathbf{H}$ with maximum possible length of single $(\pm 1, \pm(t-1), \pm t, \pm(t+1))$-error correctable integer codes for even $t:\ 6 \le t \le 20$.

The graphs in Figure 3 present the probability of error per bit (BER) in the cases when the code defined by $\mathbf{H} = (12, 6, 5, 4, 3, 2, 1)$ and Gray encoding are used in a channel with Gaussian noise.

Let us remark that any shorten matrix of the ones given in Table 1 also defines a code with the same error-correcting property but with a lower coding rate. However,

t	q	**H**
6	37	(1,2,3,4)
8	65	(1,2,3,4,5,6,12)
10	101	(1,2,3,4,5,6,7,8,14,15,16)
12	145	(1,2,...,10,16,17,18,19,20)
14	197	(1,2,...,12,18,...,24,35,36)
16	257	(1,2,...,14,20,...,28,39,...,42,56)
18	325	(1,2...,16,22,...,32,43,...,48,64)
20	401	(1,2...,18,24,...,36,47,...,54,70,71,72)

Table 1: Parity-check matrices H of single $(\pm 1, \pm(t-1), \pm t, \pm(t+1))$-error correctable integer codes over $\mathbb{Z}_{t^2+1}$ for even $t: \ 6 \leq t \leq 20$. [2]

depending on noise characteristics of the communication channel shortened codes can give better performance.

3 Application to Flash Memory. Flash memory requires error control coding like other devices for communication or storage. However due to the used technology of vertically stacked cells, in contrast with the classical case, the error vectors with the same weight are not equally probable.

A mathematical model that corresponds well to the writing and reading is as follows. If the cells have q states then one can consider pages and blocks as vectors over the ring $\mathbb{Z}_q$ and errors can be considered as changes of the values of vectors' entries. Changes of the entries of the vectors are mainly in an upward direction with limited-magnitude values but decreasing with 1 can be observed, too. Hence errors are of type ± 1 and $(2, 3, \dots l)$ for a small l.

Asymmetric limited-magnitude error-correcting codes were proposed by Varshamov and Tenengolz [23, 22] and in a more general form by Ahlswede, Aydinian, and Khachatrian [1]. Another general method for construction of asymmetric limited-magnitude error-correcting codes is provided by Cassuto, Schwartz, Bohossian and Burck [3]. In 2011, Klove and Bose [6] proposed systematic codes that correct single limited-magnitude systematic asymmetric errors and achieve a higher rate than the ones given in [3]. They also showed how their code construction can be slightly modified to give codes correcting symmetric errors of limited magnitude. Later Klove, Lou, Naydenova and Yari [7] extended their results and gave a necessary and sufficient condition for the existence of a code over $GF(p)$ correcting a single asymmetric error.

In [8] we propose constructions of codes over $\mathbb{Z}_{2^n+1}$ correcting single asymmetric errors of type $(1,2)$, $(\pm 1, \pm 2)$ or $(1,2,3)$. The argument $q = 2^n + 1$ is chosen due to the fact that it is the smallest integer greater than a power of two. The choice of q is essential since the use of the inappropriate size of the alphabet can generate problems in implementation of encoding and decoding procedures.

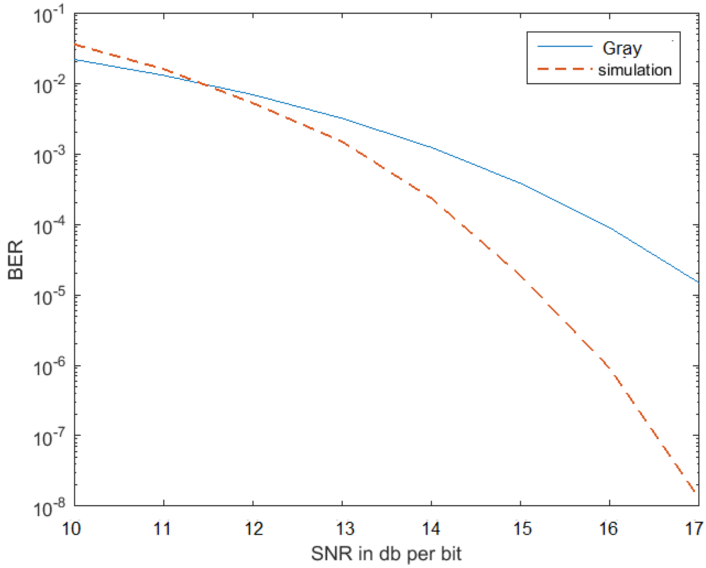

Figure 3: BER of 64-TQAM vs. signal to noise ratio [2]

Theorem 3.1 ([8]). *Let $q = 2^n + 1$ and $A_s = \{s2^i \pmod q \mid i = 0, 1, \ldots\}$ be the cyclotomic coset of 2 modulo q with leader s. Let $A_s^0 = \{s2^{2i} \pmod q \mid i = 0, 1, \ldots\}$, $A_s^1 = \{s2^{2i+1} \pmod q \mid i = 0, 1, \ldots\}$. Then $|A_s|$ is even with $|A_s| = 2n$ for $(s, q) = 1$ and both the integer codes with $\mathbf{H} = (A_1^0)$ and $\mathbf{H} = (A_1^1)$ are single $(1, 2)$ asymmetric error correctable.*

Corollary 3.2 ([8]). *Under the notations of Theorem 3.1, let $s_1, s_2, \ldots, s_t$ be all different leaders of cyclotomic cosets of 2 modulo A. Then*

$$\mathbf{H} = (A_{s_1}^0 \;\; A_{s_2}^0 \;\; \cdots \;\; A_{s_t}^0)$$

is a parity-check matrix of a single $(1, 2)$ asymmetric error correctable $[2^{n-1}, 2^{n-1} - 1]$ code over $\mathbb{Z}_q$.

Corollary 3.3. *[8] If $n > 2$ the code with $\mathbf{H} = (A_1^0)$ is a single $(1, 2, 3)$ asymmetric error correctable.*

Theorem 3.4 ([8]). *Under the notations of Theorem 3.1 let A_s^{00} be the set of first $\lfloor (|A_s^0|)/2 \rfloor$ elements of A_s^0. If $|A_s| \geq 8$ then A_s^{00} is a parity-check matrix of a single $(\pm 1, \pm 2)$ error correctable code.*

Corollary 3.5 ([8]). *Using notation and conditions of Theorem 3.1 the code with*

$$\mathbf{H} = (A_{s_1}^{00} \;\; A_{s_2}^{00} \;\; \cdots \;\; A_{s_t}^{00})$$

is a single $(\pm 1, \pm 2)$ error correctable code over A when $|A_{s_i}| \geq 4$ and s_i are distinct leaders.

4 Application to optical channels. In this section we describe integer codes and their use for error-correction in a binary channel with specific dominant errors. The channel is a theoretical model of wireless optical channels (see [4]). However our approach is applicable to any binary channel.

Any binary sequence can be considered as a sequence of blocks of bits with the same length b. (We call such blocks *b-bytes*, but b is not necessarily equal to 8). Any block can be interpreted as a nonnegative integer less than 2^b, that is, as an element of $\mathbb{Z}_q$ with $q \geq 2^b$. Hence the sequence can be considered as a sequence of elements of $\mathbb{Z}_q$ and encoding by integer codes over $\mathbb{Z}_q$ can be applied to.

It seems that the natural choice is $q = 2^b$, but it is not the right choice in general. Indeed the situation is more complex. Note that changing one bit in a b-byte is equivalent to adding or subtracting 2^i, $i = 0, 1, \ldots, b-1$, to the integer corresponding to the b-byte. However 2^i divides 2^b that generates complications. If $q > 2^b$ there are elements of $\mathbb{Z}_q$ which cannot be transmitted over the channel, thus, they must not appear as coordinates of the codewords. Hence, the encoding procedure has to comply with this requirement.

Let $B \in \mathbb{Z}_q$ be the integer corresponding to a given b-byte sent through the communication channel. The received b-byte is presented by number $V = B + E$, where $-2^b < E < 2^b$.

Single bit errors in a b-byte are presented in [9] by:

E1 The $(i+1)$-th bit $1 \to 0$: $E = -2^i, i = 0, 1, \ldots, b-1$.

E2 The $(i+1)$-th bit $0 \to 1$: $E = 2^i, i = 0, 1, \ldots, b-1$.

All other error configurations in a single b-byte correspond to combinations of E1 and E2, thus E is a sum of powers (positive or negative) of 2.

In [9] we propose a construction of a class of integer codes over $\mathbb{Z}_{2^b+1}$ that can correct up to two bit errors per a b-byte. Such error configurations have been observed in experiments with many binary channels, e.g., with wireless optical channels. The construction is based on cyclotomic cosets of 2 modulo $q = 2^b + 1$.

Theorem 4.1 ([9])**.** *Any integer code over $\mathbb{Z}_q$, $q = 2^b + 1$, having parity-check matrix $\mathbf{H} = (h_1, h_2, \ldots, h_n)$, where h_j is a leader of cyclotomic coset with length $2b$ can correct a single $\pm 2^i$-error, $i = 0, 1, \ldots, b-1$.*

Theorem 4.2 ([9])**.** *Let $\mathcal{E}$ be a set of cyclotomic cosets containing C_1. If there exist $h_2, \ldots, h_n \in \mathbb{Z}_A$ such that $\mathcal{E}, h_2\mathcal{E}, \ldots, h_n\mathcal{E}$ are disjoint sets of different cosets then the code with parity-check matrix $\mathbf{H} = (1, h_2, \ldots, h_n)$ can correct error values that belong to $\mathcal{E}$ and, of course, bit error configurations corresponding to them.*

The considered set of error values is a proper subset of

$$\mathcal{E} = \bigcup C_{2^k \pm 1}, \ k = 1, \ldots, \lfloor b/2 \rfloor.$$

Example 4.3 ([9])**.** *Let* $\mathbf{b = 10}$, $\mathbf{n = 3}$. *In this case a code that corrects up two bits errors in a single b-byte per codeword has parity-check matrix* $\mathbf{H} = (1\ 19\ 27)$*:*

$$\mathcal{E} \subset C_1 \cup C_3 \cup C_5 \cup C_7 \cup C_9 \cup C_{15} \cup C_{17} \cup C_{31}, \quad C_{33} = C_{31},$$

$$19\mathcal{E} \subset C_{19} \cup C_{57} \cup C_{35} \cup C_{39} \cup C_{171} \cup C_{105} \cup C_{43} \cup C_{109},$$
$$27\mathcal{E} \subset C_{27} \cup C_{59} \cup C_{55} \cup C_{51} \cup C_{53} \cup C_{155} \cup C_{107} \cup C_{47}.$$

Length $n = 3$ is the maximal possible value for such a code.

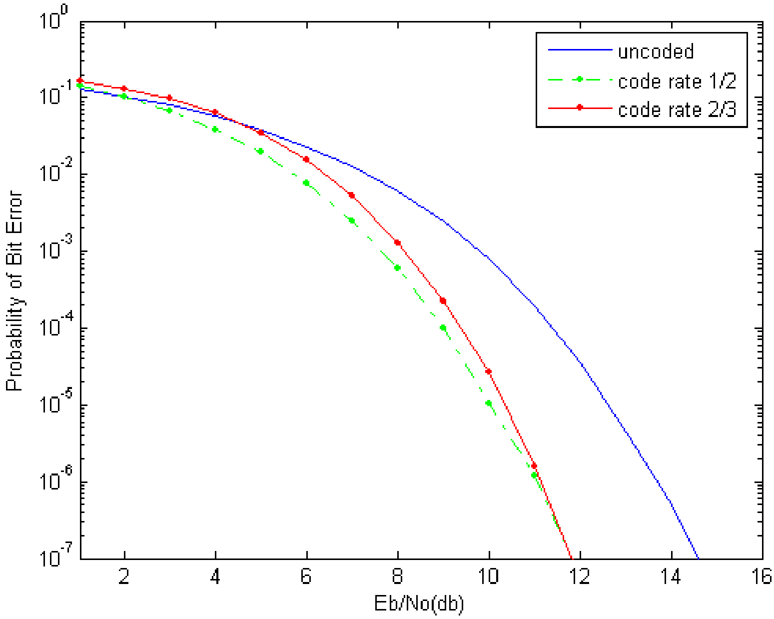

Figure 4: BER for the code given in Example 4.1 and the code with $\mathbf{H} = (1\ 23)$ over $\mathbb{Z}_{512}$, see [9].

Figure 4 presents in graphical form probability of bit error (BER) versus signal/noise ratio per bit in decibels for a noisy wireless optical channel. To simulate the channel we have used the MATLAB file given in [4]. The blue solid curve gives BER in the case when no error-correcting code is used. The red (solid line with dots) curve corresponds to applying the code given in Example 4.1. The green curve corresponds to the code defined by $\mathbf{H} = (1\ 23)$ over $\mathbb{Z}_{512}$. The code from Example 4.1 demonstrates slightly worse performance which is the price of the higher code rate.

5 Conclusion. The successful applications to several devices for communications and storage which are presented in this paper demonstrate that integer codes with their simple encoding and decoding are at least commensurable with other codes with similar parameters. Moreover, in many situations they are preferable.

REFERENCES.

[1] R. AHLSWEDE, H. AYDINIAN, and L. KHACHATRIAN. Undirectional error control codes and related combinatorial problems. In *Proc. of the Eighth International Workshop on Algebraic and Combinatorial Coding Theory (ACCT'2002)*, pages 6–9, 2002.

[2] T. ALEXANDROVA, H. KOSTADINOV, and N. MANEV. General construction of integer codes correcting single error in t^2-TQAM. *Computers and Electrical Engineering*, 127:110572, 2025.
[3] Y. CASSUTO, M. SCHWARTZ, V. BOHOSSIAN, and J. BURCK. Codes for asymmetric limited-magnitude errors with application to multi-level flash memories. *IEEE Trans. Inf. Theory*, 56:1582–1595, 2010.
[4] Z. GHASSEMLOOY, W. POPOOLA, and S. RAJBHANDARI. *Optical Wireless Communications, System and Channel Modelling with MATLAB*. CRC Press. ISBN 13: 978-1-4398-5235-4.
[5] A.J. HAN VINCK and H. MORITA. Codes over the ring of integers modulo m. *IEICE Trans. on Fundamentals*, E81-A(10):2013–2018, 1998.
[6] T. KLOVE and B. BOSE. Systematic, single limited magnitude error correcting codes for flash memories. *IEEE Trans. Inf. Theory*, 57:4477–4487, 2011.
[7] T. KLOVE, J. LOU, I. NAYDENOVA, and S. YARI. Some codes correcting asymmetric errors of limited magnitude. *IEEE Trans. Inf. Theory*, 57:7459–7472, 2011.
[8] H. KOSTADINOV and N. MANEV. Integer codes correcting asymmetric errors in nand flash memory. *Mathematics*, 9:1269, 2021.
[9] H. KOSTADINOV and N. MANEV. A general construction of integer codes correcting specific errors in binary communication channels. *Mathematics*, 11:2521, 2023.
[10] H. KOSTADINOV and N. MANEV. New integer coded hexagonal QAM schemes and their performance in AWGN channel. In *Studies in Computational Intelligence*. Volume 1076, pages 100–110. Springer, 2023.
[11] H. KOSTADINOV, H. MORITA, N. IIJIMA, A.J. HAN VINCK, and N. MANEV. Soft decoding of integer codes and their application to coded modulation. *IEICE Trans. on Fundamentals*, E39-A(7):1363–1370, 2010.
[12] H. KOSTADINOV, H. MORITA, and N. MANEV. Integer codes correcting single errors of specific types $(\pm e_1, \pm e_2, \ldots, \pm e_s)$. *IEICE Trans. on Fundamentals*, E86-A(7):1843–1849, 2003.
[13] H. KOSTADINOV, H. MORITA, and N. MANEV. Derivation on bit error probability of coded QAM using integer codes. *IEICE Trans. on Fundamentals*, E87-A(12):3397–3403, 2004.
[14] M.J. MIHALJEVIĆ, A. RADONJIĆ, L. WANG, and S. XU. Security enhanced symmetric key encryption employing an integer code for the erasure channel. *Symmetry*, 14:1709, 2022.
[15] H. MORITA. Nearest-neighbor error correcting codes on a hexagonal signal constellation. In *ISIT 2015*, pages 2480–2484, 2015.
[16] N. K. POKHREL and P. K. DAS. Low-density and high-density asymmetric CT-burst correcting integer codes. *Advances in Mathematics of Communications*, 18(3):842–856, 2024.
[17] A. RADONJIC. (Perfect) integer codes correcting single errors. *IEEE Commun. Lett.*, 22(1):17–20, 2018.
[18] A. RADONJIC. Integer codes correcting double errors and triple-adjacent errors within a byte. *IEEE Transactions on Very Large Scale Integration (VLSI) Systems*, 28(8):1901–1908, 2020.
[19] A. RADONJIC and V. VUJICIC. Integer codes correcting burst errors within a byte. *IEEE Trans Comput*, 62(2):411–415, 2013.
[20] A. RADONJIC and V. VUJICIC. Integer codes correcting single errors and burst asymmetric errors within a byte. *Inf. Process. Lett.*, 121:45–50, 2017.
[21] A. RADONJIC and V. VUJICIC. Integer codes correcting sparse byte errors. *Cryptography and Communications*, 11:1069–1077, 2019.
[22] R. VARSHAMOV. A class of codes for asymmetric channels and a problem from the additive theory of numbers. *IEEE Trans. on Information Theory*, 19(1):92–95, 1973.
[23] R. VARSHAMOV and G. TENENGOLZ. A code for correcting a single asymmetric error. *Automatika i Telematika*, 26(2):288–292, 1965. (in Russian).

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КРАТЪК ОБЗОР НА ЦЕЛОЧИСЛЕНИТЕ КОДОВЕ И ТЕХНИТЕ ПРИЛОЖЕНИЯ

Т. Александрова, Х. Костадинов и Н. Л. Манев

Абстракт

Статията представлява кратък преглед на известните приложения на кодове над пръстени от цели числа по модул дадено естествено число за контрол на грешките в цифровите комуникации и съхранението на данни. Представените приложения показват, че целочислените кодове с тяхното просто кодиране и декодиране са поне съизмерими с други кодове с подобни параметри и в много ситуации са за предпочитане за използване в практиката.

Ключови думи: кодове над пръстени от цели числа, квадратурно-амплитудна модулация, флаш памети, оптични канали.