МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2026
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2026
Proceedings of the Fifty-Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Tryavna, Bulgaria, April 5–9, 2026

ON MAXIMUM DISTANCE SEPARABLE AND COMPLETELY REGULAR CODES [1]

J. Borges, J. Rifà, and V. A. Zinoviev

Abstract

We investigate when a maximum distance separable (MDS) code over $\mathbb{F}_q$ is also completely regular (CR). For lengths $n = q+1$ and $n = q+2$ we provide a complete classification of the MDS codes that are CR or at least uniformly packed in the wide sense (UPWS). For the more restricted case $n \le q$ with $q \le 5$ we obtain a full classification (up to equivalence) of all nontrivial MDS codes: there are none for $q = 2$; only the ternary Hamming code for $q = 3$; four nontrivial families for $q = 4$; and exactly six linear MDS codes for $q = 5$ (three of which are CR and one admits a self-dual version). Additionally, we close two gaps left open in a previous classification of self-dual CR codes with covering radius $\rho \le 3$: we precisely determine over which finite fields the MDS self-dual completely regular codes with parameters $[2,1,2]_q$ and $[4,2,3]_q$ exist.

2020 Mathematics Subject Classification: 94B60, 94B25.

Key words: maximum distance separable codes, completely regular codes.

1 Introduction. Let $\mathbb{F}_q$ denote the finite field of order q, where q is a prime power, and let $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Let C be a linear $[n,k,d]_q$ *code* with $0 < k < n$.

For any $\mathbf{v} \in \mathbb{F}_q^n$ its *distance* to the code C is $d(\mathbf{v}, C) = \min\{d(\mathbf{v}, \mathbf{x}) \mid \mathbf{x} \in C\}$. The *external distance* of C (see [7]) is the number s of nonzero weights in $C^\perp$. The *packing radius* of C is $e = \lfloor (d-1)/2 \rfloor$ and the *covering radius* is $\rho = \max_{\mathbf{v} \in \mathbb{F}_q^n}\{d(\mathbf{v}, C)\}$. Clearly $e \le \rho$, with equality if and only if C is perfect. It is well known that any nontrivial perfect code (with more than two codewords) satisfies $e \le 3$ [19, 20].

Definition 1.1 ([7])**.** *A code C is* completely regular *(CR) if for any $\mathbf{v} \in \mathbb{F}_q^n$ and any $t \in \{0, \ldots, n\}$, the value $B_{\mathbf{v},t} = |\{\mathbf{x} \in C \mid d(\mathbf{v}, \mathbf{x}) = t\}|$ depends only on t and $d(\mathbf{v}, C)$.*

[1]This work has been partially supported by the Spanish Ministerio de Ciencia e Innovación under Grants PID2022-137924NB-I00 (AEI/FEDER UE), RED2022-134306-T, and also by the Catalan AGAUR Grant 2021SGR-00643. The research of the third author of the paper was carried out at the Institute for Information Transmission Problems of the Russian Academy of Sciences within the program of fundamental research on the topic "Mathematical Foundations of the Theory of Error-Correcting Codes" and was also partly supported by the Higher School of Modern Mathematics MIPT, Moscow College of Physics and Technology, 1 Klimentovskiy per., Moscow, Russia.

https://doi.org/10.55630/mem.2026.55.350-356

For a more detailed description and basic properties of CR codes over finite fields, see [5, 6], or [15, Chap. 2], for example.

A more general class of codes is the class of uniformly packed codes in the wide sense [2].

Definition 1.2 ([2])**.** *A code $C \subseteq \mathbb{F}_q^n$ with covering radius ρ is* uniformly packed in the wide sense *(UPWS) if there exist rational numbers $\beta_0, \ldots, \beta_\rho$ such that*

$$\sum_{i=0}^{\rho} \beta_i B_{\mathbf{x},i} = 1, \tag{1}$$

for any $\mathbf{x} \in \mathbb{F}_q^n$. The numbers $\beta_0, \ldots, \beta_\rho$ are called the packing coefficients*.*

If C is a $[n,k,d]_q$ code, the Singleton bound [16] establishes that $d \leq n-k+1$. If C attains equality, then C is a *maximum distance separable* (MDS) code.

The Griesmer bound [10, 18] is $n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$, $(k \geq 1)$. If C attains equality, then C is a *Griesmer* code.

In this paper we study the possible parameters of MDS codes (which are also Griesmer codes) over $\mathbb{F}_q$. For length $n = q+1$ and $n = q+2$ we give a complete classification of the MDS codes that are completely regular or at least uniformly packed in the wide sense. For $q \leq 5$, we obtain a full classification (up to equivalence) of all nontrivial MDS codes. For $q = 2$, as is well known, there are only trivial cases. For $q = 3$, there is only one nontrivial case, which is a self-dual Hamming code. For $q = 4$, the nontrivial cases are a Hamming code, its dual code, a self-dual code, and the so-called hexacode. The case $q = 5$ is the most interesting. We determine that there are 6 nontrivial such codes over $\mathbb{F}_5$, up to equivalence. Three of them are CR codes. In addition, we point out in which cases such codes are self-dual. Moreover, we specify exactly in which finite fields the MDS codes with parameters $[2,1,2]_q$ and $[4,2,3]_q$ are also self-dual. Such codes are completely regular as stated in [6].

The main conjecture on MDS codes is the following

Conjecture 1.3. *If C is a nontrivial* MDS $[n,k,d]_q$ *code ($2 \leq k \leq n-2$), then $n \leq q+1$ unless q is even and $k = 3$ or $k = q-1$, in which case $n \leq q+2$.*

This conjecture was first considered by Segre [13] and is proven for a large number of cases. For our purposes, the following case, proven in [1] is useful.

Proposition 1.4. *If C is an* MDS $[n,k,d]_p$ *code with p prime and $2 \leq k \leq n-2$, then $n \leq p+1$.*

2 Preliminary results.

The next result can be derived from Remark 5.13 in [12].

Proposition 2.1. *Let q be an odd prime power. Then -1 is a square in $\mathbb{F}_q$ if and only if $q \equiv 1 \pmod 4$.*

Recall that the quadratic character χ of $\mathbb{F}_q$ is defined as $\chi(a) = 1$ if a is a square, and $\chi(a) = -1$ if a is not ($a \in \mathbb{F}_q^*$). The next result is Lemma 6.24 in [12].

Proposition 2.2. *Let q be an odd prime power and $t, a_1, a_2 \in \mathbb{F}_q^*$. The equation*

$$a_1x_1^2 + a_2x_2^2 = t \tag{2}$$

has exactly $q - \chi(-a_1a_2)$ solutions, where χ is the quadratic character of $\mathbb{F}_q$.

In [6] it is stated that any $[4, 2, 3]_q$ code is CR and taking a generator matrix of the form

$$G = \begin{pmatrix} 1 & 0 & \alpha & \beta \\ 0 & \xi & \beta & -\alpha \end{pmatrix}, \tag{3}$$

where $\alpha, \beta \in \mathbb{F}_q^*$, such code is also self-dual if $1 + \alpha^2 + \beta^2 = 0$ and $\xi^2 = 1$. But in [6] it is not specified in which finite fields these conditions can be satisfied. By using Proposition 2.2, we can precise such finite fields.

Corollary 2.3. *A self-dual $[4, 2, 3]_q$ code exists if and only if $q \notin \{2, 5\}$.*

For completely regular codes, we will use the following properties.

Lemma 2.4. *Let C be a linear code with minimum distance d, covering radius ρ, packing radius e and external distance s.*

(i) $\rho \leq s$ *[7].*

(ii) If $d \geq 2s - 1$*, then* C *is* CR *[7].*

(iii) $\rho = s$ *if and only if* C *is* UPWS *[3].*

(iv) If C *is* CR*, then* $\rho = s$ *[17].*

(v) If C *is* UPWS *and* $\rho = e + 1$*, then* C *is* CR *[9, 14].*

Note that, by (iii) and (iv), a CR code is always a UPWS code.

The only self-dual perfect code is the ternary Hamming $[4, 2, 3]_3$ code (see, for example, [6, Lemma 13]). Since a Hamming code has parameters

$$[n = (q^m - 1)/(q - 1), n - m, 3]_q,$$

it is an MDS code if and only if $m = 2$. In such case, the parameters are $[q + 1, q - 1, 3]_q$ and the dual simplex code has parameters $[q + 1, 2, q]_q$. Since a simplex code has only one nonzero weight, Hamming codes have external distance $s = 1$. Hence a Hamming code is CR by Lemma 2.4 (ii).

The following equivalence is easy to see, but not usually mentioned.

Proposition 2.5. *Let C be an $[n, k, d]_q$ code with $k \geq 2$. Then C is an* MDS *code if and only if C is a Griesmer code and $d \leq q$.*

All the following results on MDS codes can be found in [11, Sect. 7.4].

Proposition 2.6. *Let C be an* MDS *$[n, k, d]_q$.*

(i) $C^\perp$ *is* MDS.

(ii) The number of minimum weight codewords in C (i.e. codewords of weight d) is $A_d = \binom{n}{d}(q-1)$.

(iii) If $2 \leq k \leq n-2$, then $k \leq \min\{n-2, q-1\}$ and $n \leq 2(q-1)$.

In [8], the number of weights of MDS codes is determined.

Proposition 2.7 ([8])**.** *Let C be an* MDS *$[n,k,d]_q$ code and let s' be the number of nonzero weights of C.*

(i) If C is an $[n, n-1, 2]_2$ code with $n > 2$, then $s' = \lfloor n/2 \rfloor \neq k$.

(ii) If C is a $[q+1, 2, q]_q$ code, then $s' = 1$.

(iii) If C is a $[2^m+2, 3, 2^m]_{2^m}$ code, then $s' = 2$.

In all other cases $s' = k$, assuming that Conjecture 1.3 is true.

3 **Trivial** MDS **codes.**

Trivial MDS $[n,k,d]_q$ codes are those with $k = 1$ or $k \geq n-1$.

For $k = 1$, we have the MDS repetition codes C_R with parameters $[n, 1, n]_q$. Such codes and their duals, the $[n, n-1, 2]_q$ codes, are CR (by Lemma 2.4 (ii), since the codes C_R have just one nonzero weight). For $q = 2$, C_R codes have covering radius $\rho = \lfloor n/2 \rfloor$ (see [4, Thm. 1]) and identical external distance $s = \lfloor n/2 \rfloor$ by Proposition 2.7. Hence, binary C_R codes are UPWS by Lemma 2.4 (iii). Moreover, since the packing radius is $e = \lfloor (n-1)/2 \rfloor$, we have that $e = \rho$ if n is odd, and $e+1 = \rho$ if n is even. Therefore, C_R is a perfect code if n is odd, and a quasi-perfect UPWS code if n is even. In any case, C_R is a CR code by Lemma 2.4. For $q > 2$ and $n > 2$, we have $\rho = n - \lceil n/q \rceil$ (see [4, Thm. 1]) and the external distance is $s = n-1$ (Proposition 2.7). Hence, C_R is UPWS if and only if $q \geq n$.

Note that if a C_R code is self-dual then $n = 2$. In such case, the codes $[2,1,2]_q$ are CR. Take a generator matrix $G = (1 \;\; \alpha)$ for a $[2,1,2]_q$ C_R code. Then C_R is self-dual if and only if $\alpha^2 = -1$. In fact, these codes or direct sums of them are the only self-dual CR codes with $d \leq 2$ (see [6, Thm. 36(i)]). However, in [6] the finite fields where such codes exist are not specified. As a direct consequence of Proposition 2.1 we can establish the following result.

Theorem 3.1. *Let C be a self-dual* CR *$[n,k,d]_q$ code. If $d \leq 2$, then C is necessarily an* MDS *$[2,1,2]_q$ code, or a direct sum of j codes with these parameters (and C is not* MDS *for $j > 1$). Such codes exist if and only if q is even or $q \equiv 1 \pmod 4$.*

4 **Nontrivial** MDS **codes.**

Nontrivial MDS $[n,k,d]_q$ codes are those with $2 \leq k \leq n-2$. Assuming that the MDS *conjecture* (Conjecture 1.3) is true, we have that $n \leq q+1$ except in some exceptional cases where $n = q+2$. We start with these extremal cases.

According to Conjecture 1.3, a nontrivial MDS code of length $n = q+2$ is a $[2^m + 2, 3, 2^m]_{2^m}$ code or a $[2^m+2, 2^m-1, 4]_{2^m}$ code, where $m > 1$.

Theorem 4.1. *Let C be an* MDS *$[q+2,k,d]_q$ code, where $q = 2^m$ and $m > 1$.*

(i) If C is an MDS *$[2^m+2, 3, 2^m]_{2^m}$ code with odd m, then C is not* UPWS *(and so, not* CR*).*

(ii) If C is an MDS $[2^m+2, 2^m-1, 4]_{2^m}$ *code, then C is* CR*.*

For $n = q+1$ we have MDS codes with parameters $[q+1, k, q-k+2]_q$. We start with the particular case $k = 2$.

Proposition 4.2. *Let C be an* MDS $[q+1, 2, q]_q$ *code.*

(i) If q is odd, then C is UPWS *and* CR *if and only if $q = 3$.*

(ii) If q is even, then C is UPWS*. For $q = 2^2$, C is* CR*. For $q = 2^3$, C is not* CR*.*

Now, we have the following strong result for any $k \geq 2$.

Theorem 4.3. *If C is a nontrivial* MDS $[q+1, k, q-k+2]_q$ *code, then C is not* UPWS *(and so not* CR*) with the following exceptions:*

(i) $k = q-1$, in which case C is a Hamming perfect and CR *code.*

(ii) $k = 2$ with q even or $q = 3$, in which case C is UPWS*. For $q = 3$, C is also* CR*.*

(iii) $k = q-2$ with q even, in which case C is UPWS*. Moreover, for any $m > 1$, there exists a* CR $[q+1, q-2, 4]_q$ *code, where $q = 2^m$.*

For $n \leq q$, we begin by establishing the following result.

Theorem 4.4. *If C is an* MDS $[n, k, d]_q$ *code with $n \leq q$, then C is* UPWS*.*

In general, it seems hard to determine in which cases an MDS code C is also CR. However, for small fields we may consider some particular cases.

It is well known that all binary MDS codes are trivial. For $q = 3$, we have $n \leq 4$, by Proposition 2.6 (iii). Hence, we have the only nontrivial case of $[4, 2, 3]_3$ codes which are the self-dual CR Hamming ternary codes. If $q = 4$ then, by Proposition 2.6 (iii), we have $n \leq 6$. So, we have the possible parameters $[6, 4, 3]_4$, $[6, 3, 4]_4$, $[6, 2, 5]_4$, $[5, 3, 3]_4$, $[5, 2, 4]_4$ and $[4, 2, 3]_4$. The case of a $[6, 3, 4]_4$ code corresponds to the well-known two-weight *hexacode* (see, e.g., [11, p. 383]). Such code is CR by Lemma 2.4 (ii) since $s = 2$. In fact, it is a particular case of Theorem 4.1 (ii). The other two cases with $n = 6$ can be discarded after the following result.

Lemma 4.5. *There do not exist codes with parameters* $[6, 4, 3]_4$ *or* $[6, 2, 5]_4$*.*

For the remaining cases, note that a $[5, 3, 3]_4$ code is a Hamming code and a $[5, 2, 4]_4$ code is the corresponding simplex dual. Finally, a $[4, 2, 3]_4$ code exists and it is CR. In fact, any $[4, 2, 3]_q$ code is CR by Lemma 2.4 since $s \leq 2$ (see also Proposition 23 in [6]). By Corollary 2.3 a $[4, 2, 3]_4$ code is self-dual taking a generator matrix

$$G = \begin{pmatrix} 1 & 0 & \alpha & \alpha^2 \\ 0 & 1 & \alpha^2 & \alpha \end{pmatrix},$$

where α is a primitive element in $\mathbb{F}_4$.

The next smallest interesting finite field where to study MDS codes is $\mathbb{F}_5$. Now, we determine the possible parameters for nontrivial MDS codes over $\mathbb{F}_5$. Since Conjecture 1.3 is proven for prime fields, and applying Proposition 2.6, we only have to consider the cases: $(n, k) \in \{(6, 4), (6, 3), (6, 2), (5, 3), (5, 2), (4, 2)\}$.

Theorem 4.6. *If C is a nontrivial* MDS $[n,k,d]_5$ *code, then C is one of the following codes, up to equivalence:*

(i) A CR $[6,4,3]_5$ *code or its dual* $[6,2,5]_5$ *which is not* UPWS*.*

(ii) A $[6,3,4]_5$ *code, which is not* UPWS*. A self-dual version of such code exists.*

(iii) A CR $[5,3,3]_5$ *code or its dual* $[5,2,4]_5$ *which is* UPWS*.*

(iv) A $[4,2,3]_5$ *code which is* CR *and not self-dual.*

5 Summary. In summary we have found the following infinite families of MDS codes which are CR codes:

(i) The trivial $[n,n-1,2]_q$ codes for any n and q, and the $[n,1,n]_2$ codes.

(ii) Any $[2^m+2,2^m-1,4]_{2^m}$ code for $m>0$.

(iii) The Hamming $[q+1,q-1,3]_q$ codes for any q.

(iv) The $[2^m+1,2^m-2,4]_{2^m}$ codes equivalent to doubly-extended Reed-Solomon codes, for any $m>1$.

In addition, for $n \leq q$, we have the sporadic codes (not included above) with parameters: $[4,2,3]_4$, the punctured code of a $[5,2,4]_4$ CR code (item (iv) above, for $m=2$); $[5,3,3]_5$ (see Theorem 4.6 (iii)); and $[4,2,3]_5$ (see Theorem 4.6 (iv)).

REFERENCES.

[1] S. BALL. On sets of vectors of a finite vector space in which every subset of basis size is a basis. *J. Eur. Math. Soc.*, 14:733–748, 2012.

[2] L. A. BASSALYGO, G. V. ZAITSEV, and V. A. ZINOVIEV. Uniformly packed codes. *Problems Inform. Transmiss.*, 10(1):9–14, 1974.

[3] L. A. BASSALYGO and V. A. ZINOVIEV. A note on uniformly packed codes. *Problems Inform. Transmiss.*, 13(3):22–25, 1977.

[4] M. C. BHANDARI and C. DURAIRAJAN. On the covering radius of simplex codes. *Journal of Discrete Mathematical Sciences and Cryptography*, 6:59–69, 2003.

[5] J. BORGES, J. RIFÀ, and V. A. ZINOVIEV. On completely regular codes. *Problems Inform. Transmiss.*, 55(1):1–45, 2019.

[6] J. BORGES and V. A. ZINOVIEV. On self-dual completely regular codes with covering radius $\rho \leq 3$. *Finite Fields and their Applications*, 105:102617, 2025.

[7] P. DELSARTE. An Algebraic Approach to the Association Schemes in Coding Theory. Technical report 10, Philips Res. Rep. Suppl., 1973.

[8] M. F. EZERMAN, M. GRASSL, and P. SOLÉ. The weights in MDS codes. *IEEE Trans. Inform. Theory*, IT-57:392–396, 2011.

[9] J. M. GOETHALS and H. C. A. VAN TILBORG. Uniformly packed codes. *Philips Res.*, 30:9–36, 1975.

[10] J. H. GRIESMER. A bound for error-correcting codes. *IBM J. Res. Develop.*, 4:532–542, 1960.

[11] W. C. HUFFMAN and V. PLESS. *Fundamentals of error-correcting codes.* Cambridge Univ. Press, New York, 2003.

[12] RUDOLF LIDL and HARALD NIEDERREITER. *Finite fields*, number 20. Cambridge university press, 1997.

[13] B. SEGRE. Curve razionali normali ek-archi negli spazi finiti. *Ann. Mat. Pur. Appl.*, 39(1):357–379, 1955.

[14] N. V. SEMAKOV, V. A. ZINOVIEV, and G. V. ZAITZEV. Uniformly packed codes. *Problems Inform. Transmiss.*, 7:38–50, 1971.

[15] M. SHI and P. SOLÉ, editors. *Completely Regular Codes in Distance Regular Graphs.* CRC Press, Boca Raton FL, 2025.
[16] R. C. SINGLETON. Maximum distance q-ary codes. *IEEE Trans. Inform. Theory*, IT-10:116–118, 1964.
[17] P. SOLÉ. Completely regular codes and completely transitive codes. *Discrete Maths.*, 81:193–201, 1990.
[18] G. SOLOMON and J. J. STIFFLER. Algebraically punctured cyclic codes. *Inform. and Control*, 8:170–179, 1965.
[19] A. TIETÄVÄINEN. On the nonexistence of perfect codes over finite fields. *SIAM J. Appl. Math.*, 24(1):88–96, 1973.
[20] V. A. ZINOVIEV and V. K. LEONTIEV. On non-existence of perfect codes over Galois fields. *Probl. Upravlen. Teor. Inform.*, 2(2):123–132, 1973. English translation in Probl. Control Inf. Theory, vol. 2, no. 2, pp. 16–24, 1973.

Josep Rifà, Joaquim Borges
Department of Information and Communications Engineering
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona), Spain
e-mail: `josep.rifa@uab.cat, joaquim.borges@uab.cat`

Victor Alexandrovich Zinoviev
A.A. Kharkevich Institute for Information Transmission Problems
Russian Academy of Sciences
Bol'shoi Karetnyi 19, Moscow, Russia
Russia and Higher School of Modern
Mathematics of the Moscow College of Physics and Technology in Dolgoprudny,
1 Klimentovskiy per., Moscow, Russia
e-mail: `vazinov@iitp.ru`

ЗА МАКСИМАЛНО РАЗДАЛЕЧИМИ И НАПЪЛНО РЕГУЛЯРНИ КОДОВЕ

J. Borges, J. Rifà, and V. A. Zinoviev

Абстракт

Изследваме кога един максимално раздалечим (MDS) код над $\mathbb{F}_q$ е също и напълно регулярен (CR). За дължини $n = q + 1$ и $n = q + 2$ даваме пълна класификация на MDS кодовете, които са CR или поне равномерно опаковани в широк смисъл (UPWS). За по-ограничения случай $n \leq q$ при $q \leq 5$ получаваме пълна класификация (с точност до еквивалентност) на всички нетривиални MDS кодове: няма такива за $q = 2$; само троичният код на Хеминг за $q = 3$; четири нетривиални семейства за $q = 4$; и точно шест линейни MDS кода за $q = 5$ (три от които са CR, а един допуска самодуална версия). Освен това запълваме две празнини, оставени в предишна класификация на самодуални CR кодове с радиус на покритие $\rho \leq 3$: точно определяме над кои крайни полета съществуват MDS самодуални напълно регулярни кодове с параметри $[2, 1, 2]_q$ и $[4, 2, 3]_q$.

Ключови думи: максимално раздалечим код, напълно регулярен код.