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THE PUNCTURED DODECACODE IS UNIQUE [1]

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Abstract

The punctured dodecacode is an additive 4-ary code of length 11 and distance 5 which is uniformly packed. We show that any code with the same weight distribution is equivalent to it. This code is also shown to be nonlinear.

We also establish the nonexistence of analogs of the dodecacode and the punctured dodecacode in Doob graphs. To that end, we classify two-weight codes of weights 6 and 8 in Doob graphs and 4-ary Hamming graphs of diameter 9 and the corresponding strongly regular graphs.

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Key words: dodecacode, additive code, trace Hermitian duality, uniformly packed code, completely regular code, Doob graph, strongly regular graph.

1 Introduction. The dodecacode, introduced in [6] for the needs of quantum computing, is the most famous additive $\mathbb{F}_4$-code. It is trace Hermitian self-dual. It has length 12 and distance 6. The parameters of possible uniformly packed quaternary codes were found in [2] to belong to an infinite family of codes of length $\frac{2^{2m+1}+1}{3}$ and codimension $2m+1$. The punctured dodecacode is the case $m = 2$ of this family. As a uniformly packed code, it is completely regular [13], and its coset graph is a distance-regular graph of diameter 3 [12, 1]. In this note, we show that it is unique (not only in the Hamming scheme $H(11,4)$, but also in all schemes $D(m, 11-2m)$ with the same algebraic parameters) and not $\mathbb{F}_4$-linear.

The main part of the note consists of two sections. In Section 2, we show that the punctured dodecacode is unique as an additive 4-ary length-11 code (i.e., a code in $H(11,4) = D(0,11)$) with the given weight distribution. This part also includes some partial classification results for additive 4-ary codes (Section 2.3). In Section 3, we

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establish the nonexistence of such codes in the Doob graphs $D(m,11-m)$, $0<m\le 5$, and of analogs of the dodecacode in $D(m,12-m)$, $0<m\le 6$. This part also includes a classification of additive two-weight codes in $D(m,9-m)$, $0\le m\le 4$ (Section 3.2) and the corresponding strongly regular graphs. The results of the classification up to equivalence of the following codes, which were used for obtaining the results in Section 3, can be found in [8, Appendix]: weight-$\{6,8\}$ codes in Doob graphs $D(m,9-2m)$; additive 4-ary codes of length 6, distance at least 3 and size at least 2^6.

2 Codes in Hamming metric space.

2.1 Preliminaries. An *additive* $\mathbb{F}_4$*-code* of length n is an additive subgroup of $\mathbb{F}_4^n$. Two additive $\mathbb{F}_4$-codes are *equivalent* if one of them is obtained from the other by a permutation of coordinates and, for each coordinate independently, multiplication by a constant and/or conjugation. (Note that this equivalence is wider than the equivalence of linear codes, where the conjugation can be applied to all coordinates simultaneously.) The Hamming weight is denoted by $wt(\cdot)$. The trace denoted by Tr(), is defined by $\mathrm{Tr}(z)=z+z^2$. The two inner products that we consider here are the *Hermitian inner product*

$$\langle x,y\rangle=\sum_i x_i y_i^2,$$

and the *trace Hermitian inner product*

$$\mathrm{Tr}(\langle x,y\rangle)=\sum_i x_i y_i^2+x_i^2 y_i.$$

Thus $\mathrm{Tr}(\langle x,y\rangle)=\langle x,y\rangle+\langle y,x\rangle$. The duals of a code C with respect to these two inner products are denoted by $C^{\perp}$ and $C^{\perp_T}$, respectively. The dodecacode is an additive $\mathbb{F}_4$-code of length 12 and binary dimension 12 that is self-dual with respect to the trace Hermitian inner product. It is cyclic, and puncturing it at any coordinate yields a code of parameters $(11,2^{12},5)_4$. The dual weight distribution of the punctured dodecacode, in Magma notation, is

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[ <0,1>, <6,198>, <8,495>, <10,330> ].
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2.2 Results. The proof of the main result is based on trace Hermitian duality. We need a pair of Lemmas.

Lemma 2.1. *If an additive* $\mathbb{F}_4$*-code* C *is even, then* C *is self-orthogonal for the trace Hermitian inner product.*

Proof. (see also [6, Th. 4]) The semilinearity of the Hermitian inner product implies

$$\langle x+y,x+y\rangle=\langle x,x\rangle+\langle y,y\rangle+\langle x,y\rangle+\langle y,x\rangle.$$

Using $\langle x,x\rangle\equiv wt(x) \pmod 2$ and $\langle x,y\rangle+\langle y,x\rangle=\mathrm{Tr}(\langle x,y\rangle)$ we obtain (see also eq. (7) in [6])

$$wt(x+y)\equiv wt(x)+wt(y)+\mathrm{Tr}(\langle x,y\rangle) \pmod 2.$$

If C is even, this implies $\mathrm{Tr}(\langle x,y\rangle)=0$. □

The following is proved in [7].

Lemma 2.2. *Up to equivalence, there is a unique additive self-dual* $\mathbb{F}_4$*-code of length* 11 *and distance* 5*, where the duality is understood with respect to the trace Hermitian inner product.*

Note that equivalence includes all monomial transformations as well as conjugation of coordinates.

We are now ready for a characterization of the punctured dodecacode by its weight distribution. A characterization by the minimum distance is hopeless in view of the existence of the quadratic residue code of parameters $[11, 6, 5]_4$ (but see the next subsection for partial classification results).

Theorem 2.3. *Up to equivalence, the punctured dodecacode is a unique additive* $\mathbb{F}_4$*-code with the dual weight distribution*

```
[ <0,1>, <6,198>, <8,495>, <10,330> ].
```

Proof. Consider an additive code C of length 11 with the same weight distribution as the punctured dodecacode. Its dual, say D, contains 2^{10} codewords. It is even since its weights are 6, 8, 10. By Lemma 2.1, D is trace Hermitian self-orthogonal. Consider a weight-5 codeword x in $C = D^{\perp_T}$. Such a word exists by the MacWilliams identity. The code $N = D \cup (x + D)$ is additive and self-orthogonal of size 2^{11}, hence self-dual. (Note that $\mathrm{Tr}(\langle x, x\rangle) = 0$). It has distance 5. By Lemma 2.2, it is unique, up to equivalence. By construction, D is the even part of N, as an even subcode of size half the size of N. The result follows. □

We could prove the following result by inspection, but a conceptual proof is better. The following was shown to the last author by Jon-Lark Kim.

Proposition 2.4. *The punctured dodecacode is not* $\mathbb{F}_4$*-linear.*

Proof. The code D of the preceding proof, being even and $\mathbb{F}_4$-linear, would be Hermitian self-orthogonal by [11]. But there is no Hermitian self-orthogonal code of length 11 and distance 6 over $\mathbb{F}_4$ by [5]. □

2.3 Partial classification results. In order to show the uniqueness of the punctured dodecacode, we have also obtained some partial classification results.

Shortening an additive code over $\mathbb{F}_4$ with parameters $C = (n, 2^{k_2}, d)_4$ reduces the dimension at most by two, i. e., results in a code $C' = (n-1, 2^{k_2'}, \geq d)_4$ where $k_2' \in \{k_2 - 2, k_2 - 1, k_2\}$. In order to find all inequivalent additive $(11, 2^{12}, 5)_4$ codes over $\mathbb{F}_4$, we start with the $(5, 1, 5)_4$ code generated by the all-one vector. Then we try to increase the dimension by one, while the minimum distance is at least 5. Having found all $(n, 2^k, d \geq 5)_4$ codes with covering radius ≤ 4, i. e., codes for which the dimension cannot be increased, we add a zero coordinate to the codes and continue. We exclude codes whose dimension is too small to reach $(11, 2^{12}, 5)_4$.

Table 1 shows the number of inequivalent additive $(n, 2^k, 5)_4$ codes that we have found. We only found two $(11, 2^{12}, 5)_4$ codes, corresponding to the punctured dodecacode and the $\mathbb{F}_4$-linear quadratic residue code.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
5	1	1	1	0										
6	1	2	5	1	0	0								
7	1	3	15	32	43	1	0							
8	1	4	34	322	5114	26299	1579	0	0					
9								≥89116	2298	0	0			
10										≥20935	37	0	0	
11												≥24	2	0

Table 1: The number $N(n,k,5)$ of equivalence classes of additive $(n,2^k,\geq 5)_4$ codes. According to our approach, we classify $(n,2^k,\geq 5)_4$ codes by lengthening $N(n-1,k-2,5)+N(n-1,k-1,5)+N(n-1,k,5)$ codes of length $n-1$. The notation $\geq N_0$ means that we lengthened only $N(n-1,k-1,5)+N(n-1,k,5)$ codes of size 2^{k-1} and 2^k and N_0 is the number of codes having at least one coordinate with at most two symbols.

There is also a unique additive $(7,2^5,5)_4$ code, with generator matrix

$$\begin{pmatrix} 1 & 0 & \omega & \omega^2 & 0 & \omega^2 & 1 \\ \omega & 0 & 0 & 1 & \omega^2 & \omega & 1 \\ 0 & 1 & \omega & 1 & \omega^2 & \omega^2 & \omega \\ 0 & \omega & \omega & \omega^2 & 1 & 0 & \omega^2 \\ 0 & 0 & 1 & \omega^2 & \omega & 1 & \omega \end{pmatrix},$$

where $\omega^2 = \omega + 1$. It is cyclic and has the weight distribution

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[ <0,1>, <5,21>, <6,7>, <7,3> ].
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Note that the best $\mathbb{F}_4$-linear codes have parameters $[7,2,5]_4$ and $[7,3,4]_4$. Hence, this code has twice as many codewords as the largest $\mathbb{F}_4$-linear code of length 7 and distance 5.

3 Codes in Doob metric space.

3.1 Preliminaries. A set C of vertices in a connected regular graph is called a *completely regular code* with covering radius ρ and *intersection array* $I = \{b_0, \ldots, b_{\rho-1}; c_1, \ldots, c_\rho\}$ (for short, CR-ρ, I-CR, or just CR code) if each vertex in $C^{(i)}$ has exactly b_i neighbors in $C^{(i+1)}$ and exactly c_i neighbors in $C^{(i-1)}$, $b_\rho = c_0 = 0$, where $C^{(i)}$ is the set of vertices at distance i from C and ρ is the largest i such that $C^{(i)}$ is nonempty. A connected graph is called *distance-regular* with intersection array I if every singleton is an I-CR code. Distance-regular graphs of diameter 2 are called *strongly regular*.

The Doob graph $D(m,n)$ is the Cartesian product of m copies of the Shrikhande graph on 16 vertices and n copies of the complete graph K_4 of order 4. Usually, for Doob graphs it is implied that $m > 0$, while $D(0,n)$ is the 4-ary Hamming graph $H(n,4)$. It is known that $D(m,n)$ is a distance-regular graph with intersection array $\{3d, 3d-3, \ldots, 3; 1, 2, \ldots, d\}$, where $d = 2m+n$ is its diameter; in particular, all $D(m,n)$ have the same intersection array as $H(2m+n,4)$. If we associate the vertices of $D(m,n)$ with the elements of the module $M = \mathbb{R}_{16}^m \times \mathbb{F}_4^n$ over the Galois ring $\mathbb{R}_{16} = \mathrm{GR}(4^2)$, then we can define linear codes as submodules of M and additive codes as subgroups of the additive group of M. In particular, treating $\mathbb{R}_{16}$ as a module over $\mathbb{Z}_4$, we see that such additive codes are submodules of $(\mathbb{Z}_4^2)^m \times (\mathbb{Z}_2^2)^n$. In [9], the concept of additive codes in $D(m,n)$ was further generalized by allowing some of the last n coordinates to have the structure $\mathbb{Z}_4$ instead of $\mathbb{Z}_2^2$. The formal definition below is based on that extended concept of additive codes.

We first recall that the *Cayley graph* on an Abelian group G with a connecting set $S \subset G \setminus \{\mathrm{Id}\}$, $S = S^{-1}$, is the graph with the vertex set G where two vertices $a, b \in G$ are adjacent if and only if $b - a \in S$. Let the *Doob graph* $D(m, n' + n'')$ be defined as the Cayley graph on the additive group of the module $V = (\mathbb{Z}_4^2)^m \times (\mathbb{Z}_2^2)^{n'} \times \mathbb{Z}_4^{n''}$ with connecting set $S = S^* \cup S' \cup S''$, where S^* (S', S'', respectively) consists of vectors of V with one of the first m symbols (the next n' symbols, the last n'' symbols, respectively) lying in $\{01, 10, 11, 03, 30, 33\}$ ($\{01, 10, 11\}$, $\{1, 2, 3\}$, respectively) and all other symbols are zero, 00 or 0 (we naturally consider vectors of V as words of length $m + n' + n''$ where the first m symbols are from $\mathbb{Z}_4^2$, the next n' symbols are from $\mathbb{Z}_2^2$, and the last n'' symbols are from $\mathbb{Z}_4$). The (Doob) weight $\mathrm{wt}(x)$ of a vector x in V is the sum of the weights of all symbols of x, where

$$\text{for } x^* \in \mathbb{Z}_4^2,\ \mathrm{wt}(x^*) = \begin{cases} 0 & \text{if } x^* = 00, \\ 1 & \text{if } x^* \in \{01, 03, 10, 30, 11, 33\}, \\ 2 & \text{otherwise;} \end{cases}$$

$$\text{for } x' \in \mathbb{Z}_2^2,\ \mathrm{wt}(x') = \begin{cases} 0 & \text{if } x' = 00, \\ 1 & \text{if } x' \in \{01, 10, 11\}; \end{cases}$$

$$\text{for } x'' \in \mathbb{Z}_4,\ \mathrm{wt}(x'') = \begin{cases} 0 & \text{if } x'' = 0, \\ 1 & \text{if } x'' \in \{1, 2, 3\}. \end{cases}$$

The (Doob) distance between $x, y \in V$ is defined as $\mathrm{wt}(y - x)$ and coincides with the natural shortest-path distance in the graph $D(m, n' + n'')$.

As in the case of codes in the Hamming space $H(n, 4)$, the weight distributions of a code and its dual are connected by the MacWilliams transform (the formulas are the same as in $H(2m + n' + n''))$. However, the duality should be defined properly, with respect to a special inner product $\langle \cdot, \cdot \rangle_-$ (see [3, Theorem 6.6]): for $x^*, y^* \in (\mathbb{Z}_4^2)^m$, $x', y' \in \mathbb{Z}_4^{n'}$, $x'', y'' \in (\mathbb{Z}_2^2)^{n''}$,

$$\big\langle (x^*, x', x''), (y^*, y', y'') \big\rangle_- = \sum_{i=1}^{m} (x^*_{2i-1} y^*_{2i-1} - x^*_{2i} y^*_{2i}) + 2 \cdot \sum_{i=1}^{2n'} x'_i y'_i + \sum_{i=1}^{n''} x''_i y''_i. \quad (1)$$

In particular, similarly to the codes in $H(n, q)$, a code with distance at least 3 is CR-2 (completely regular with covering radius 2) if and only if its dual has two nonzero weights.

3.2 Two-weight codes of size 64. Our first aim is to classify additive codes of size 64 in $D(m, n' + n'')$, $2m + n' + n'' = 9$, with nonzero weights 6 and 8 and dual distance at least 3. The last condition means that either $m = n'' = 0$ or the code has at least one element z of order 4, which can only be possible if $2m + n'' \geq 6$ (otherwise, $0 < \mathrm{wt}(2z) < 6$). In [8, Appendix A], we list all, up to equivalence, additive weight-$\{6, 8\}$ (two-weight with weights 6 and 8 and one-weight with weight 6 or 8) codes in Doob graphs $D(m, n' + n'')$ of diameter $2m + (n' + n'') = 9$ such that $2m + n'' \geq 6$ and in

$$
B_1 : \left(\begin{array}{cccc|c|}
21 & 10 & 10 & 10 & 10 \\
31 & 01 & 01 & 01 & 01 \\ \hline
20 & 02 & 20 & 00 & 00 \\
02 & 22 & 02 & 00 & 00
\end{array}\right), \quad
B_2 : \left(\begin{array}{cccc|c|}
21 & 10 & 10 & 10 & 10 \\
12 & 11 & 01 & 01 & 01 \\ \hline
20 & 02 & 20 & 00 & 00 \\
22 & 20 & 02 & 00 & 00
\end{array}\right),
$$

$$
B_3 : \left(\begin{array}{cccc|c|}
21 & 21 & 10 & 00 & 10 \\
20 & 10 & 01 & 10 & 01 \\
12 & 01 & 01 & 01 & 01 \\ \hline
\end{array}\right), \quad
B_4 : \left(\begin{array}{cccc|c|}
21 & 21 & 10 & 00 & 10 \\
20 & 10 & 01 & 10 & 01 \\
13 & 01 & 01 & 01 & 11 \\ \hline
\end{array}\right),
$$

$$
B_5 : \left(\begin{array}{cccc|c|}
21 & 21 & 10 & 00 & 10 \\
20 & 10 & 01 & 10 & 01 \\
32 & 03 & 01 & 01 & 10 \\ \hline
\end{array}\right), \quad
B_6 : \left(\begin{array}{cccc|c|}
21 & 21 & 10 & 00 & 10 \\
31 & 20 & 01 & 10 & 00 \\
21 & 33 & 01 & 01 & 01 \\ \hline
\end{array}\right).
$$

Table 2: Generator matrices of additive weight-$\{6,8\}$ codes of type $\mathbb{Z}_4^2\mathbb{Z}_2^2$ (B_1, B_2) and type $\mathbb{Z}_4^3$ (B_3, B_4, B_5, B_6) in $D(4, 1+0)$. Here and further, the two vertical lines in the matrix separate the three groups of coordinates (one or two of which can be empty), while the horizontal line separates the rows of order 4 and the rows of order-2.

$D(0, 9+0) = H(9,4)$. The approach of the classification is rather straightforward. We start with the collection consisting of one trivial code having only the all-zero codeword. For each code C from the collection, we try to continue it by adding a new element c to the generator set in all possible ways and checking if the resulting code $\mathrm{span}(C \cup \{c\})$ has the required weights and is not equivalent to any code from our collection. If it is a new weight-$\{6,8\}$ code, we add it to the collection. When all codes from the collection are checked for augmentation, the process stops.

In Tables 2, 3, 4, as a recapitulation, we list only weight-$\{6,8\}$ codes of size 64. The following proposition summarizes the computational results.

Proposition 3.1. *In $D(m, n'+n'')$, $2m+n'+n''=9$, there are exactly 22 equivalence classes of weight-$\{6,8\}$ codes of size 64. Namely, $B_1, \ldots, B_6$ in $D(4, 1+0)$ (Table 2), $C_1, \ldots, C_8$ in $D(3, 0+3)$ (Table 3), $D_1, \ldots, D_8$ in $D(0, 9+0) = H(9,4)$ (Table 4). The weight distribution of all these codes is $\{0^1, 6^{36}, 8^{27}\}$; the dual codes (with respect to the inner product (1)) are completely regular with intersection array $\{27, 16; 1, 12\}$; the coset graphs of the dual codes are strongly regular with parameters $(64, 27, 10, 12)$; these strongly regular graphs belong to only 7 equivalence classes, corresponding to the groups $\{B_1, C_2, D_2, D_6\}$, $\{B_2, B_3, C_1, C_3, C_8, D_1, D_5\}$, $\{B_4\}$, $\{B_5, C_4, C_6\}$, $\{B_6, C_5, C_7\}$, $\{D_3, D_4, D_7\}$, $\{D_8\}$ of codes.*

Remark 3.2. *By construction, the graphs from codes of group type $\mathbb{Z}_4^i\mathbb{Z}_2^j$ are strongly regular Cayley graphs over $\mathbb{Z}_4^i\mathbb{Z}_2^j$ (in our case, (i,j) is $(0,6)$, $(2,2)$, or $(3,0)$); however, we cannot say that we have found all such graphs. The reason is that being representable as a coset graph of a code in a Doob graph or a Hamming graph $H(n,4)$ implies certain relations between vectors from the connecting set of a Cayley graph. More strongly regular Cayley graphs with the same parameters can hypothetically be constructed as coset graphs of binary linear or $\mathbb{Z}_2\mathbb{Z}_4$-linear two-weight codes.*

$$
C_1 : \left(\begin{array}{ccc||ccc}
21 & 10 & 10 & 1 & 1 & 0 \\
12 & 01 & 01 & 1 & 0 & 1 \\ \hline
22 & 02 & 20 & 0 & 0 & 0 \\
02 & 22 & 02 & 0 & 0 & 0
\end{array}\right), \quad
C_2 : \left(\begin{array}{ccc||ccc}
21 & 10 & 10 & 1 & 1 & 0 \\
10 & 21 & 01 & 1 & 0 & 1 \\ \hline
22 & 02 & 20 & 0 & 0 & 0 \\
02 & 22 & 02 & 0 & 0 & 0
\end{array}\right),
$$

$$
C_3 : \left(\begin{array}{ccc||ccc}
21 & 21 & 21 & 0 & 0 & 0 \\
31 & 31 & 00 & 1 & 1 & 0 \\
02 & 10 & 10 & 1 & 0 & 1
\end{array}\right), \quad
C_4 : \left(\begin{array}{ccc||ccc}
21 & 21 & 21 & 0 & 0 & 0 \\
31 & 31 & 00 & 1 & 1 & 0 \\
22 & 30 & 10 & 1 & 0 & 1
\end{array}\right),
$$

$$
C_5 : \left(\begin{array}{ccc||ccc}
21 & 21 & 21 & 0 & 0 & 0 \\
30 & 20 & 10 & 1 & 1 & 0 \\
33 & 11 & 20 & 1 & 0 & 1
\end{array}\right), \quad
C_6 : \left(\begin{array}{ccc||ccc}
21 & 21 & 10 & 2 & 0 & 0 \\
20 & 10 & 01 & 1 & 1 & 0 \\
12 & 01 & 01 & 1 & 0 & 1
\end{array}\right),
$$

$$
C_7 : \left(\begin{array}{ccc||ccc}
21 & 21 & 10 & 2 & 0 & 0 \\
20 & 10 & 01 & 1 & 1 & 0 \\
33 & 20 & 11 & 1 & 0 & 1
\end{array}\right), \quad
C_8 : \left(\begin{array}{ccc||ccc}
21 & 21 & 10 & 2 & 0 & 0 \\
21 & 30 & 01 & 1 & 1 & 0 \\
30 & 02 & 01 & 1 & 0 & 1
\end{array}\right).
$$

Table 3: Generator matrices of additive weight-$\{6,8\}$ codes of type $\mathbb{Z}_4^2\mathbb{Z}_2^2$ (C_1, C_2) and type $\mathbb{Z}_4^3$ (C_3, C_4, C_5, C_6, C_7, C_8) in $D(3,0+3)$.

$$
D_1 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
11\,11\;\;01\,01\,00\,00\,01\,01\,00 \\
11\,01\;\;01\,00\,11\,00\,10\,00\,10 \\
10\,11\;\;11\,00\,10\,00\,01\,00\,01
\end{pmatrix}, \quad
D_2 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
11\,11\;\;01\,01\,00\,00\,01\,01\,00 \\
11\,01\;\;01\,00\,11\,00\,10\,00\,10 \\
01\,00\;\;01\,10\,10\,00\,01\,00\,01
\end{pmatrix},
$$

$$
D_3 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
01\,10\;\;01\,00\,10\,00\,01\,01\,00 \\
01\,11\;\;11\,00\,01\,00\,10\,00\,10 \\
10\,10\;\;00\,01\,01\,00\,01\,00\,01
\end{pmatrix}, \quad
D_4 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
01\,10\;\;01\,00\,10\,00\,01\,01\,00 \\
01\,11\;\;11\,00\,01\,00\,10\,00\,10 \\
11\,00\;\;01\,11\,01\,00\,01\,00\,01
\end{pmatrix},
$$

$$
D_5 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
11\,01\;\;01\,00\,11\,00\,01\,01\,00 \\
11\,11\;\;01\,01\,00\,00\,10\,00\,10 \\
10\,11\;\;11\,00\,10\,00\,01\,00\,01
\end{pmatrix}, \quad
D_6 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,01\,01\,00\,00\,00 \\
01\,01\;\;10\,10\,00\,00\,10\,10\,00 \\
11\,01\;\;01\,00\,11\,00\,01\,01\,00 \\
11\,11\;\;01\,01\,00\,00\,10\,00\,10 \\
01\,00\;\;01\,10\,10\,00\,01\,00\,01
\end{pmatrix},
$$

$$
D_7 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,10\,00\,10\,00\,00 \\
11\,11\;\;01\,10\,01\,00\,00\,10\,00 \\
11\,01\;\;01\,00\,00\,01\,01\,01\,00 \\
10\,01\;\;11\,01\,00\,01\,00\,00\,10 \\
11\,00\;\;01\,11\,11\,00\,01\,00\,01
\end{pmatrix}, \quad
D_8 : \begin{pmatrix}
10\,10\;\;10\,10\,10\,10\,00\,00\,00 \\
01\,01\;\;01\,01\,10\,00\,10\,00\,00 \\
11\,11\;\;01\,10\,01\,00\,00\,10\,00 \\
11\,01\;\;01\,00\,00\,01\,01\,01\,00 \\
10\,11\;\;00\,11\,01\,01\,00\,00\,10 \\
00\,11\;\;11\,01\,01\,00\,01\,00\,01
\end{pmatrix}.
$$

Table 4: Generator matrices of additive weight-$\{6,8\}$ codes in $D(0,9+0) = H(9,4)$.

$$
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,12\,01\,01\,01 & 01 \\ \hline 20\,22\,20\,00\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,13\,01\,01\,01 & 11 \\ \hline 20\,20\,20\,00\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,13\,01\,01\,01 & 11 \\ \hline 20\,22\,00\,02\,00 & 00 \end{array}\right),
$$

$$
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,13\,01\,01\,01 & 11 \\ \hline 20\,02\,20\,02\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,32\,03\,01\,01 & 10 \\ \hline 20\,22\,20\,00\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,20\,10\,01\,10 & 01 \\ 00\,32\,03\,01\,01 & 10 \\ \hline 20\,02\,20\,02\,00 & 00 \end{array}\right),
$$

$$
\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10 \\ 00\,31\,20\,01\,10 & 00 \\ 00\,21\,33\,01\,01 & 01 \\ \hline 20\,22\,20\,00\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,10\,10\,10 & 10 \\ 00\,31\,01\,01\,01 & 01 \\ \hline 00\,20\,02\,20\,00 & 00 \\ 00\,02\,22\,02\,00 & 00 \\ 20\,02\,20\,00\,00 & 00 \end{array}\right),\quad
\left(\begin{array}{c|c} 00\,21\,10\,10\,10 & 10 \\ 00\,12\,11\,01\,01 & 01 \\ \hline 00\,20\,02\,20\,00 & 00 \\ 00\,22\,20\,02\,00 & 00 \\ 20\,02\,20\,00\,00 & 00 \end{array}\right).
$$

Table 5: Lengthenings of type $\mathbb{Z}_4^3\mathbb{Z}_2^1$ and type $\mathbb{Z}_4^2\mathbb{Z}_2^3$.

3.3 Lengthening to codes in Doob graphs of diameter 11 and 12. We aim to classify all additive codes of size 1024 with nonzero weights 6, 8, 10 in Doob graphs $D(m, n'+n'')$, $m > 0$, of diameter $2m + n' + n'' = 11$. If such a code C_{11} exists, then shortening it in one of the first m positions or two of the last $n' + n''$ positions leads to a weight-$\{6, 8\}$ code C_9 of size 64 in $D(m-1, n'+n'')$, $D(m, (n'-2)+n'')$ (if $n' \geq 2$), $D(m, (n'-1)+(n''-1))$ (if $n', n'' \geq 1$), or $D(m, n'+(n''-2))$ (if $n'' \geq 2$). As such codes C_9 exist only in $D(4, 1+0)$ and $D(3, 0+3)$, we conclude that the only option for C_{11} is $D(m, n'+n'') = D(5, 1+0)$. Now, our approach is quite straightforward. Starting with one of the six weight-$\{6, 8\}$ codes found in $D(4, 1+0)$, we extend the code to $D(5, 1+0)$ by adding 00 to the beginning of each codeword. Next, we add a row to the generator matrix in all possible ways and check that the resulting code has a larger cardinality (128 or 256 in the first iteration) and desired nonzero weights 6, 8, 10. After filtering only inequivalent representatives from the found collection of codes, we repeat the same procedure and repeat iterating until no new codes are found. Summarizing the results, we have found only codes of size 128 and 256, with weight distributions $\{0^1, 6^{42}, 8^{55}, 10^{30}\}$ and $\{0^1, 6^{54}, 8^{111}, 10^{90}\}$, respectively; as a corollary, we have Proposition 3.3 below. The check matrices of inequivalent representatives are given in Tables 5 and 6 (note that we do not claim that there are no other additive weight-$\{6, 8, 10\}$ codes of size at least 128 in $D(5, 1)$; our search was restricted only to those codes that have a shortening of size 64 in $D(4, 1)$).

Proposition 3.3. *In Doob graphs of diameter* 11*, there are no additive codes of size* 1024 *with weights* 6*,* 8*,* 10*.*

Corollary 3.4. *In Doob graphs $D(m, 11-2m)$, $m > 0$, there is no additive completely regular code C with intersection array* $\{33, 30, 15; 1, 2, 15\}$.

Proof. Since the given intersection array is the same as that of the punctured dodecacode D^- in $H(11, 4)$ and since the graphs $D(m, 11-2m)$ and $H(11, 4)$ have the same

$$\left(\begin{array}{c|c} 00\,21\,10\,10\,10 & 10\\ 00\,31\,01\,01\,01 & 01\\ 21\,11\,11\,10\,00 & 01\\ \hline 00\,20\,02\,20\,00 & 00\\ 00\,02\,22\,02\,00 & 00 \end{array}\right), \left(\begin{array}{c|c} 00\,21\,10\,10\,10 & 10\\ 00\,12\,11\,01\,01 & 01\\ 21\,11\,11\,10\,00 & 01\\ \hline 00\,20\,02\,20\,00 & 00\\ 00\,22\,20\,02\,00 & 00 \end{array}\right), \left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10\\ 00\,20\,10\,01\,10 & 01\\ 00\,12\,01\,01\,01 & 01\\ \hline 20\,22\,20\,00\,00 & 00\\ 02\,20\,00\,02\,00 & 00 \end{array}\right)$$

$$\left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10\\ 00\,20\,10\,01\,10 & 01\\ 00\,13\,01\,01\,01 & 11\\ \hline 20\,20\,20\,00\,00 & 00\\ 02\,22\,00\,02\,00 & 00 \end{array}\right), \left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10\\ 00\,20\,10\,01\,10 & 01\\ 00\,32\,03\,01\,01 & 10\\ \hline 20\,22\,20\,00\,00 & 00\\ 02\,20\,00\,02\,00 & 00 \end{array}\right), \left(\begin{array}{c|c} 00\,21\,21\,10\,00 & 10\\ 00\,31\,20\,01\,10 & 00\\ 00\,21\,33\,01\,01 & 01\\ \hline 20\,22\,20\,00\,00 & 00\\ 02\,22\,02\,02\,00 & 00 \end{array}\right),$$

$$\left(\begin{array}{ccccc|c} 00 & 21 & 10 & 10 & 10 & 10\\ 00 & 31 & 01 & 01 & 01 & 01\\ \hline 00 & 20 & 02 & 20 & 00 & 00\\ 00 & 02 & 22 & 02 & 00 & 00\\ 20 & 02 & 20 & 00 & 00 & 00\\ 02 & 22 & 02 & 00 & 00 & 00 \end{array}\right), \left(\begin{array}{ccccc|c} 00 & 21 & 10 & 10 & 10 & 10\\ 00 & 12 & 11 & 01 & 01 & 01\\ \hline 00 & 20 & 02 & 20 & 00 & 00\\ 00 & 22 & 20 & 02 & 00 & 00\\ 20 & 02 & 20 & 00 & 00 & 00\\ 02 & 22 & 02 & 00 & 00 & 00 \end{array}\right).$$

Table 6: Lengthenings of type $\mathbb{Z}_4^3\mathbb{Z}_2^2$ and type $\mathbb{Z}_4^2\mathbb{Z}_2^4$.

parameters as distance-regular graphs, the weight distributions of C and D^- coincide, see, e.g., [10, Proposition 1.21]. The weight distributions of the dual codes $C^\perp$ (with respect to the inner product (1)) and $D^{-\perp}$ also coincide by the MacWilliams transform [3, Theorem 6.6]. □

Proposition 3.5. *In Doob graphs of diameter* 12*, there are no additive codes of size* 4096 *with weights* 6*,* 8*,* 10*,* 12*.*

Proof. For the graph $D(m, 12-2m)$, $0 < m < 6$, the claim follows from Proposition 3.3 (indeed, if we have a code with required parameters, then shortening it in the last coordinate leads to a code of size 1024 in $D(m, 11-2m)$, which does not exist).

For the case $m = 6$, we consider an additive code of type $\mathbb{Z}_4^\delta\mathbb{Z}_2^\gamma$ in $D(6,0)$, with weights 6, 8, 10, 12. The type-$\mathbb{Z}_2^{\gamma+\delta}$ subcode has Doob distance at least 6, and it is equivalent to an additive $(6, 2^{\gamma+\delta}, 3)_4$ code, a 4-ary code of length 6. In its turn, by concatenation (mapping the 4 quaternary symbols to the binary triples 000, 011, 101, 110), each of such codes can be mapped to a binary linear $(18, 2^k, \{6, 8, 10, 12\})$ code, $k = \gamma + \delta$. Such codes are classified with the software [4]: there are 2859, 258, and 3 equivalence classes of such codes for $k = 6, 7, 8$, respectively. By inverse concatenation, we get 646 equivalence classes of additive $(6, 2^6, 3)_4$ codes (including one distance-4 code, known as the hexacode), see [8, Appendix B.1], 14 equivalence classes of additive $(6, 2^7, 3)_4$ codes, see [8, Appendix B.2], and no additive $(6, 2^8, 3)_4$ codes (only 626 of the 2859 binary $(18, 2^6, \{6, 8, 10, 12\})$ codes can be represented as quaternary by concatenation; on the other hand, some codes can be represented in more than one way; that is why we say that the number of equivalence classes of quaternary codes is larger than 626). So, now we know all additive distance-6 codes of type $\mathbb{Z}_2^k$, $k \geq 6$ in $D(6,0)$. To confirm the results,

$n\backslash k$	0	1	2	3	4	5	6	7	8
3	1	1	1	0					
4	1	2	5	3	1	0			
5	1	3	14	32	40	9	1	0	
6	1	4	30	181	885	1660	646	14	0

Table 7: The number $N(n,k,3)$ of equivalence classes of additive $(n,2^k,\geq 3)_4$ codes.

we have made a search similar to that in Section 2.3, see Table 7.

In the case $k=6$, we have $\gamma=0$ and $\delta=6$, and for a putative additive code C of type $\mathbb{Z}_4^6$ in $D(6,0)$ its type-$\mathbb{Z}_4^2$ subcode is exactly $2C$. So, we can start with a generator matrix of $2C$ (which comes from a binary generator matrix of an additive $(6,2^6,3)_4$ code by replacing 1s by 2s) and try to lift it row-by-row. From the 646 additive $(6,2^6,3)_4$ codes we found, there are only two codes such that any row of the generated matrix (but not all six rows) can be lifted, with (additive) generator matrices

$$\begin{pmatrix} 01\ 00\ 10\ 00\ 00\ 10 \\ 00\ 00\ 00\ 10\ 10\ 10 \\ 00\ 00\ 01\ 01\ 00\ 01 \\ 00\ 01\ 00\ 01\ 01\ 00 \\ 10\ 00\ 11\ 01\ 01\ 10 \\ 00\ 10\ 10\ 00\ 10\ 10 \end{pmatrix}, \qquad \begin{pmatrix} 10\ 01\ 01\ 01\ 01\ 11 \\ 00\ 10\ 01\ 01\ 00\ 01 \\ 00\ 01\ 10\ 00\ 01\ 01 \\ 00\ 00\ 01\ 10\ 01\ 10 \\ 00\ 01\ 00\ 01\ 10\ 10 \\ 01\ 00\ 00\ 01\ 01\ 01 \end{pmatrix}$$

(the second code has distance 4 and is equivalent to the well-known hexacode). By lifting row-by-row (and keeping only inequivalent codes at each step), we find that for each of the two matrices, it is not possible to lift the first 3 rows to produce a code of type $\mathbb{Z}_4^3\mathbb{Z}_2^3$ in $D(6,0)$. It follows that a distance-6 even-weight code of type $\mathbb{Z}_4^6$ does not exist in $D(6,0)$.

For the type $\mathbb{Z}_4^5\mathbb{Z}_2^2$, it is slightly more complicated. The reason is that $2C$ is of type $\mathbb{Z}_2^5$, which is not the whole type-$\mathbb{Z}_2^7$ subcode of C. So, if we have a putative type-$\mathbb{Z}_2^7$ subcode (from a 4-ary additive $(6,2^7,3)_4$ code) where the rows of some concrete generator matrix cannot be lifted, this does not lead to a direct contradiction. The strategy is to try to lift each nonzero codeword of the putative type-$\mathbb{Z}_2^7$ subcode. For each of the 14 candidates C for such a subcode, obtained from the 14 additive $[6,3.5,3]_4$ codes by replacing 1s by 2s, we try to lift each nonzero codeword. A codeword b is called *liftable* if there is another word c such that $b-2c$ and the additive code obtained as the additive closure of $C\cup\{c\}$ has only (Doob) weights 0, 6, 8, 10, 12 (not necessarily all of them). Clearly, if there is a type-$\mathbb{Z}_4^5\mathbb{Z}_2^2$ code with required weights, then its type-$\mathbb{Z}_2^7$ subcode must have at least $31=2^5-1$ liftable codewords. However, among the 14 candidates for such a subcode, 4 codes have only 7 liftable codewords and 10 codes have only 1 liftable codeword. This means that a required code does not exist. □

4 Conclusion and open problems. We have shown that the punctured dodecacode is characterized by its weight distribution and that it is not $\mathbb{F}_4$-linear. It is the case

$m = 2$ of an infinite family of putative uniformly packed codes introduced in [2]. The case $m = 3$ of that family would lead us to consider a dual code D of binary dimension $2(2m+1) = 2 \times 7 = 14$ and length 43. Its weight distribution would be, in Magma notation:

```
[ <0, 1>, <28, 3612>, <32, 8127>, <36, 4644> ].
```

While this code would still be even and trace Hermitian self-orthogonal, the classification of self-orthogonal codes either for the Hermitian or trace Hermitian codes is far beyond the tables in [5] and [7]. A computer search for an analog of the dodecacode with length 44 and a cyclic automorphism group has failed to produce an example.

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СКЪСЕНИЯТ ДОДЕКАКОД Е ЕДИНСТВЕН

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Абстракт

Скъсеният додекакод е адитивен 4-ичен код с дължина 11 и разстояние 5, който е равномерно опакован. Показваме, че всеки код със същото разпределение на теглата е еквивалентен на него. Освен това е установено, че този код е нелинеен. Също така доказваме несъществуването на аналози на додекакода и на скъсения додекакод в графите на Doob. За тази цел класифицираме двутеглови кодове с тегла 6 и 8 в графи на Doob и в 4-ични графи на Хеминг с диаметър 9, както и съответните силно регулярни графи.

Ключови думи: додекакод, адитивен код, следова ермитова двойственост, равномерно опакован код, напълно регулярен код, граф на Doob, силно регулярен граф.