МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2026
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2026
Proceedings of the Fifty-Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Tryavna, Bulgaria, April 5–9, 2026

A CONJECTURE ON THE MINIMUM LENGTH OF BINARY LINEAR CODES [1]

T. Maruta, T. Tochiyama, and K. Yasufuku

Abstract

We give a new conjecture on $D_{2,k}$, where $D_{q,k}$ denotes the largest d such that a linear code of dimension k with minimum weight d meeting the Griesmer bound does not exist. We also prove that $D_{2,11} = 1632$, which yields that our conjecture is valid for dimensions $k \le 11$.

2020 Mathematics Subject Classification: 94B05, 94B27, 94B65.

Key words: Griesmer bound, optimal code, divisible code.

1 Introduction. Let $\mathbb{F}_q^n$ denote the vector space of n-tuples over $\mathbb{F}_q$, the field of order q. An $[n, k, d]_q$ code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming weight $d = \min\{wt(c) \mid c \in \mathcal{C}, c \neq (0, \ldots, 0)\}$, where $wt(c)$ is the number of non-zero entries in c. A fundamental problem in coding theory is to find $n_q(k, d)$, the minimum length n for which an $[n, k, d]_q$ code exists [6]. The Griesmer bound [5, 6] gives a lower bound on the length n:

$$n \ge g_q(k, d) := \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. An $[n, k, d]_q$ code $\mathcal{C}$ is called *Griesmer* if it attains the Griesmer bound, i.e., $n = g_q(k, d)$. It is known that $n_q(k, d) = g_q(k, d)$ for $k = 1, 2$ and all d. So, we assume $k \ge 3$. For fixed q and k, it is also known that the Griesmer bound is attained for all sufficiently large d [6]. A natural question arises.

Problem. For fixed q and k, find the integer $D_{q,k}$ such that $n_q(k, d) = g_q(k, d)$ for all $d > D_{q,k}$ and $n_q(k, d) > g_q(k, d)$ for $d = D_{q,k}$. Then, determine $n_q(k, D_{q,k})$.

As for the value $D_{q,k}$, Kawabata et al. [11] posed the following conjecture.

Conjecture 1.1 ([11])**.** (a) $D_{q,k} = B_{q,k}$ *for* $q \ge k \ge 3$*;*

(b) $D_{q,k} = C_{q,k}$ *for* $3 \le q < k$*;*

[1]The work of T. Maruta is supported by JSPS KAKENHI Grant Number 24K06850
https://doi.org/10.55630/mem.2026.55.423-432

(c) $n_q(k, D_{q,k}) = g_q(k, D_{q,k}) + 1$,
where

$$B_{q,k} = (k-2)q^{k-1} - (k-1)q^{k-2},$$

$$C_{q,k} = q^{k-2-l}(((q-1)l + \alpha - 3)q^{l+1} + q^{k-\beta} + q - \alpha) \text{ for } \beta \le k \le \beta + l,$$

with integers l, α *and* β *satisfying* $l \ge 1$, $1 \le \alpha \le q-1$ *and*

$$\beta = \beta_{l,\alpha} = (q-1)\binom{l+1}{2} + (l+1)\alpha - l.$$

Note for $q = 2$ with $l \ge 2$ that $\beta = \binom{l+1}{2} + 1$ and

$$C_{2,k} = (l-2)2^{k-1} + 2^{k+\tau-l-2} + 2^{k-l-2}, \tag{2}$$

where $\tau = k - \beta$ with $0 \le \tau \le l$. As for binary linear codes, the following is known.

Theorem 1.2 ([2, 11, 16])**.** (a) $n_2(k, d) = g_2(k, d)$ *for all* d *for* $1 \le k \le 4$.

(b) $n_2(k, d) = g_2(k, d)$ *for all* $d > C_{2,k}$ *for* $k \ge 5$.

(c) $D_{2,k} = C_{2,k}$ *for* $k = 5, 6, 9, 10$.

(d) $D_{2,7} = 44 < C_{2,7} = 72$, $D_{2,8} = 104 < C_{2,8} = 152$.

(e) $D_{2,11} < C_{2,11}$.

So, Conjecture 1 excludes the case $q = 2$. Let

$$\tilde{C}_{2,k} = (l-2)2^{k-1} - (2^l - 1 - 2^{\tau+1})2^{k-l-2}. \tag{3}$$

We pose a new conjecture for binary codes as follows.

Conjecture 1.3. (a) $D_{2,k} = \tilde{C}_{2,k}$ *for* $k = \beta, \beta+1$, $l \ge 3$*;*

(b) $D_{2,k} = C_{2,k}$ *for* $k = \beta + l - 1, \beta + l$, $l \ge 2$.

Since $\tilde{C}_{2,7} = 44$ and $\tilde{C}_{2,8} = 104$, Conjecture 2 is valid for $5 \le k \le 10$ ($\beta = 2, 3$). In this paper, we prove that Conjecture 2 is also valid for $k = 11$:

Theorem 1.4. $D_{2,11} = \tilde{C}_{2,11}$.

2 Geometric preliminaries. In this section, we give the geometric methods to construct codes or to prove the non-existence of codes with certain parameters. We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A j-flat is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. The 0-flats, 1-flats, 2-flats, 3-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, solids, secundums* and *hyperplanes*, respectively. We denote by θ_j the number of points in a j-flat, i.e., $\theta_j = |\mathrm{PG}(j, q)| = (q^{j+1} - 1)/(q - 1)$, where $|T|$ denotes the number of elements in a set T.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix G of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted by $\mathcal{M}_\mathcal{C}$. A point P in Σ is called an *i-point* if it has multiplicity $m_\mathcal{C}(P) = i$ in

$\mathcal{M}_\mathcal{C}$ (in other words, an i-point appears exactly i times as a column of G). Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_\mathcal{C}$ and let Λ_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. We denote by $\Delta_1 + \cdots + \Delta_s$ the multiset consisting of the s sets $\Delta_1, ..., \Delta_s$ in Σ. We denote by $\Delta_2 = \Delta - \Delta_1$ when $\Delta = \Delta_1 + \Delta_2$. We write $s\Delta$ for $\Delta_1 + \cdots + \Delta_s$ when $\Delta_1 = \cdots = \Delta_s$. Then, $\mathcal{M}_\mathcal{C} = \sum_{i=1}^{\gamma_0} i\Lambda_i$. For any set S in Σ, the *multiplicity of* S (*with respect to* $\mathcal{M}_\mathcal{C}$), denoted by $m_\mathcal{C}(S)$, is defined as

$$m_\mathcal{C}(S) = \sum_{P \in S} m_\mathcal{C}(P) = \sum_{i=1}^{\gamma_0} i \cdot |S \cap \Lambda_i|.$$

Then we obtain the partition $\bigcup_{i=0}^{\gamma_0} \Lambda_i$ of Σ such that $n = m_\mathcal{C}(\Sigma)$ and that

$$n - d = \max\{m_\mathcal{C}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where $\mathcal{F}_j$ denotes the set of all j-flats in Σ. Such a partition of Σ is called an $(n, n-d)$-*arc* of Σ. Conversely an $(n, n-d)$-arc of Σ gives an $[n, k, d]_q$ code. A line ℓ with $t = m_\mathcal{C}(\ell)$ is called a *t-line*. A *t-hyperplane* and so on are defined similarly. Denote by a_i the number of i-hyperplanes in Σ. The list of a_i's is called the *spectrum* of $\mathcal{C}$. The spectrum can be calculated from the weight distribution of $\mathcal{C}$ by $a_i = A_{n-i}/(q-1)$ for $0 \le i \le n-d$, where A_w is the number of codewords of $\mathcal{C}$ with weight w. Let τ_j be the number of j-secundums in a fixed hyperplane Π of Σ. The list of τ_j's is called the *spectrum* of Π. We denote by λ_s the number of s-points in Σ. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_\mathcal{C}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

We write simply by γ_j for $\gamma_j(\Sigma)$. It holds that $\gamma_{k-2} = n - d$, $\gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, the value γ_j is uniquely determined in [13] as follows:

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \text{ for } 0 \le j \le k-1. \tag{4}$$

Lemma 2.1 ([15])**.** *Put* $\epsilon = (n-d)q - n$ *and* $t_0 = \lfloor (w+\epsilon)/q \rfloor$, *where* $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x. *Let* Π *be a* w*-hyperplane through a* t*-secundum* δ. *Then* $t \le (w+\epsilon)/q$ *and the following holds.*

(i) $a_w = 0$ *if an* $[w, k-1, d_0]_q$ *code with* $d_0 \ge w - t_0$ *does not exist.*

(ii) $\gamma_{k-3}(\Pi) = t_0$ *if an* $[w, k-1, d_1]_q$ *code with* $d_1 \ge w - t_0 + 1$ *does not exist.*

Note that a w-hyperplane Π with $\gamma_{k-3}(\Pi) = t_0$ gives a $[w, k-1, w-t_0]_q$ code if and only if $t_0 < w$. An $[n, k, d]_q$ code is called *m-divisible* if all codewords have weights divisible by an integer $m > 1$.

Lemma 2.2 ([9, 10, 15])**.** *Let* $\mathcal{C}$ *be an* m*-divisible* $[n, k, d]_q$ *code with* $q = p^h$, p *prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \ldots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \ldots, \alpha_1, \alpha_0),$$

where $m = p^e$ for some $1 \le e < h(k-2)$ satisfying $\lambda_0 > 0$ and $n(q-1) > dq$. Then there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with

$$t = q^{k-2}/m,\ n^* = ntq - \frac{d}{m}\theta_{k-1},\ d^* = ((n-d)q - n)t \tag{5}$$

whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \dots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \dots, \lambda_1, \lambda_0).$$

$\mathcal{C}^*$ is called a *projective dual* of $\mathcal{C}$, see also [3] and [7]. Note that a generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as j-points in the dual space Σ^* of Σ for $0 \le j \le w-1$ [15]. From the principle of projective geometry, $\mathcal{C}$ and $(\mathcal{C}^*)^*$ are equivalent, and $(n^*)^* = n$, $(d^*)^* = d$. Hence, the following holds.

Lemma 2.3. *Assume there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ satisfying (5) with $q = p^h$, p prime, $m = p^e$, $1 \le e < h(k-2)$, $\lambda_0(\mathcal{C}^*) > 0$. Then, so does an m-divisible $[n, k, d]_q$ code.*

Lemma 2.4 ([10])**.** *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with multiset $\mathcal{M}_\mathcal{C}$ over $\Sigma = \mathrm{PG}(k-1, q)$. Assume $d > q^t$ and that $\mathcal{M}_\mathcal{C}$ contains a t-flat Δ with $1 \le t \le k-1$. Let $\mathcal{C}'$ be an $[n-\theta_t, k, d']_q$ code with multiset $\mathcal{M}_{\mathcal{C}'} = \mathcal{M}_\mathcal{C} - \Delta$. Then, $d' \ge d - q^t$.*

The method to construct new codes from a given $[n, k, d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1, q)$ is called *geometric puncturing*, see [14]. As a consequence for $q = 2$, we get the following lemma, which is employed in Section 4.

Lemma 2.5. *Assume that a Griesmer $[n, k, d]_2$ code with multiset $\mathcal{M}_\mathcal{C}$ over $\Sigma = \mathrm{PG}(k-1, 2)$ exists and that $d = s \cdot 2^{k-1} - (d_{k-2}2^{k-2} + d_{k-3}2^{k-3} + \cdots + d_{t+1}2^{t+1} + 2^t)$ with integer $t \ge 2$ and $d_i \in \mathbb{F}_2$ for $t+1 \le i \le k-2$. If $\mathcal{M}_\mathcal{C}$ contains a multiset $F_1 + F_2 + \cdots + F_{t-1}$, where F_j is a j-flat for $1 \le j \le t-1$, then, $n_2(k, d') = g_2(k, d')$ for $d - 2^t + 1 \le d' \le d$.*

For the divisibility of Griesmer codes, we need the following results.

Lemma 2.6 ([17])**.** *Let $\mathcal{C}$ be a Griesmer $[n, k, d]_p$ code with a prime p. If p^e divides d, then $\mathcal{C}$ is p^e-divisible.*

Lemma 2.7 ([11])**.** *If there exists no p^e-divisible $[n, k, d]_q$ code with $q = p^h$, p prime, $\frac{dq}{q-1} < n < \theta_{k-1}$ and $1 \le e < h(k-2)$, then neither does there exist a p^e-divisible $[n, k+1, d]_q$ code.*

Lemma 2.8 ([11])**.** *Let $\mathcal{C}$ be an m-divisible $[n, k, d]_q$ code. If $\mathcal{M}_\mathcal{C}$ has an s-point P with $s \ge 1$, then the projection of $\mathcal{M}_\mathcal{C} - sP$ from P onto a hyperplane Π with $P \notin \Pi$ gives an m-divisible $[n-s, k-1, d']_q$ code with $d' \ge d$.*

3 Dimension-optimal binary divisible codes. Let $q = p^h$, p prime. Assume that an m-divisible $[n, k, d]_q$ code $\mathcal{C}$ exists for some integer $m > 1$. We say that $\mathcal{C}$ is *dimension-optimal* if an m-divisible $[n, k', d]_q$ code does not exist for any $k' > k$. We mainly consider d-divisible dimension-optimal $[n, k, d]_2$ codes for $d = 2^t$ with a positive integer t. Denote

Table 1: $k_2(n,4)$

n	4	6	7	8	10	11	12	13	14	15	16	17	18	19	20
$k_2(n,4)$	1	2	3	4	4	4	5	5	6	7	8	8	8	8	9

Table 2: $k_2(n,8)$

n	8	12	14	15	16	20	22	23	24	26	27	28
$k_2(n,8)$	1	2	3	4	5	4	4	5	6	5	6	7
n	29	30	31	32	34	35	36	37	38	39	40	41
$k_2(n,8)$	7	8	9	10	8	8	9	8	9	10	11	9

by $k_q(n,d)$ the maximum dimension k such that a d-divisible $[n,k,d]_q$ code exists. Körner and Kurz [12] determined $k_2(n,4)$ for $n \le 20$, $k_2(n,8)$ for $n \le 41$ and $k_2(n,16)$ for $n \le 79$, see Tables 1-3.

For $i = 1,2$, let $\mathcal{C}_i$ be an $[n_i,k_i,d_i]_q$ code with generator matrix G_i. Then, their *direct sum* $\mathcal{C}_1 \oplus \mathcal{C}_2$ is the $[n_1+n_2, k_1+k_2, \min\{d_1,d_2\}]_q$ code with generator matrix $G_1 \oplus G_2 = \begin{bmatrix} G_1 & O \\ O & G_2 \end{bmatrix}$, see [8]. For the multisets $\mathcal{M}_{\mathcal{C}_1}$ and $\mathcal{M}_{\mathcal{C}_2}$ obtained from G_1 and G_2, respectively, we denote by $\mathcal{M}_{\mathcal{C}_1} \oplus \mathcal{M}_{\mathcal{C}_2}$ the multiset $\mathcal{M}_{\mathcal{C}_1 \oplus \mathcal{C}_2}$ of the direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$. The following is obvious.

Lemma 3.1. *Let $\mathcal{C}_i$ be an m-divisible $[n_i,k_i,d_i]_q$ code for $i = 1,2$. Then, their direct sum $\mathcal{C}_1 \oplus \mathcal{C}_2$ is also m-divisible.*

A code $\mathcal{C}$ is called an *s-fold simplex code* if $\mathcal{M}_{\mathcal{C}} = s\mathrm{PG}(k-1,q)$ for some positive integer s, whose parameters are $[s\theta_{k-1}, k, sq^{k-1}]_q$. $\mathcal{C}$ is sq^{k-1}-divisible since every non-zero codeword has weight sq^{k-1}. Taking $s = 2^{t+1-k}$ with $q = 2$, one can get the part (a) of the next lemma.

Let $q = 2$. We denote a j-flat in $\mathrm{PG}(k-1,2)$ by Π_j. So, Π_0 and Π_{k-1} stand for a point and the whole space, respectively. We denote by $\mathcal{E}_3$ an elliptic quadric in $\mathrm{PG}(3,2)$, which gives a 2-divisible $[5,4,2]_2$ code with spectrum $(a_1,a_3) = (5,10)$.

Lemma 3.2. *Let $\mathcal{C}$ be an $[n,k,d]_2$ code and let t be a positive integer.*

(a) *$\mathcal{C}$ is 2^t-divisible $[2^{t+1-k}\theta_{k-1}, k, 2^t]_2$ code if $\mathcal{M}_{\mathcal{C}} = 2^{t+1-k}\Pi_{k-1}$ for $t \ge k$.*

Table 3: $k_2(n,16)$

n	16	24	28	30	31	32	40	44	46	47	48	52	54	55	56	58
$k_2(n,16)$	1	2	3	4	5	6	4	4	5	6	7	5	6	7	8	7
n	59	60	61	62	63	64	68	70	71	72	74	75	76	77	78	79
$k_2(n,16)$	8	9	9	10	11	12	8	8	9	10	8	9	10	10	11	12

(b) $\mathcal{C}$ *is* 2^{k-2}*-divisible* $[2^{k-1}, k, 2^{k-1} - 2^{k-2}]_2$ *code if* $\mathcal{M}_{\mathcal{C}} = \Pi_{k-1} - \Pi_{k-2}$.

(c) $\mathcal{C}$ *is* 2^t*-divisible* $[5 \cdot 2^{t-1}, 4, 2^t]_2$ *code if* $\mathcal{M}_{\mathcal{C}} = 2^{t-1}\mathcal{E}_3$.

Körner and Kurz [12] proved the following with the aid of a computer.

Lemma 3.3. *Let* $\mathcal{C}$ *be an* $[n, k, d]_2$ *code. Then,* d*-divisible* $[n, k_2(n, d), d]_2$ *codes are unique up to equivalence for*

(a) $n = 4, 6\text{-}8, 10, 17$ *when* $d = 4$;

(b) $n = 8, 12, 14\text{-}16, 20, 22, 23, 24, 27\text{-}29, 31, 32, 34\text{-}37, 39, 40$ *when* $d = 8$;

(c) $n = 16, 24, 30\text{-}32, 40, 46\text{-}48, 54\text{-}56, 58\text{-}61, 63, 64, 68, 70\text{-}72, 74\text{-}77, 79$ *when* $d = 16$.

Table 4: 8-divisible $[n, k_2(n, 8), 8]_2$ codes

n	$k_2(n,8)$	construction
22	4	$8\Pi_0 \oplus 2\Pi_2$
23	5	$8\Pi_0 \oplus \Pi_3$
24	6	$8\Pi_0 \oplus (\Pi_4 - \Pi_3)$
27	6	$4\Pi_1 \oplus \Pi_3$
28	7	$4\Pi_1 \oplus (\Pi_4 - \Pi_3)$
29	7	$2\Pi_2 \oplus \Pi_3$
31	9	$\Pi_3 \oplus (\Pi_4 - \Pi_3)$
32	10	$(\Pi_4 - \Pi_3) \oplus (\Pi_4 - \Pi_3)$
35	8	$4\mathcal{E}_3 \oplus \Pi_3$

Most of the dimension-optimal binary divisible codes in Lemma 3.3 can be constructed by Lemmas 3.1 and 3.2, see Table 4 and Table 5. In the tables, "construction" gives the multiset $\mathcal{M}_{\mathcal{C}}$ for the unique d-divisible $[n, k_2(n, d), d]_2$ code $\mathcal{C}$.

Table 5: 16-divisible $[n, k_2(n, 16), 16]_2$ codes

n	$k_2(n,16)$	construction
72	10	$8\mathcal{E}_3 \oplus (\Pi_5 - \Pi_4)$
74	8	$16\Pi_0 \oplus 4\Pi_2 \oplus 2\Pi_3$
75	9	$16\Pi_0 \oplus 4\Pi_2 \oplus \Pi_4$
76	10	$16\Pi_0 \oplus 4\Pi_2 \oplus (\Pi_5 - \Pi_4)$
77	10	$16\Pi_0 \oplus 2\Pi_3 \oplus \Pi_4$
79	12	$16\Pi_0 \oplus \Pi_4 \oplus (\Pi_5 - \Pi_4)$

In this paper, the point P in $\mathrm{PG}(r, 2)$ with coordinate vector $(p_0, p_1, \ldots, p_r)$ is denoted by $(p_0, p_1, \ldots, p_r)$ or simply $p_0 p_1 \ldots p_r$, and the hyperplane defined by the equation $a_0 x_0 + a_1 x_1 + \cdots + a_r x_r = 0$ is denoted by $[a_0 a_1 \ldots a_r]$. For two distinct points $P(p_0, p_1, \ldots, p_r)$ and $Q(q_0, q_1, \ldots, q_r)$ in $\mathrm{PG}(r, 2)$, we denote the point $(p_0 + q_0, p_1 + q_1, \ldots, p_r + q_r)$ by $P + Q$.

Let $\boldsymbol{e}_i = 0 \cdots 010 \cdots 0$ be the point of $\mathrm{PG}(r, 2)$ the only i-th entry of which is 1. Note that $\mathcal{E}_3 = \{1000, 0100, 0010, 0001, 1111\}$ is an elliptic quadric in $\mathrm{PG}(3, 2)$.

Lemma 3.4 ([12]). *There are exactly 11* 16*-divisible* $[78, 10, 16]_2$ *codes up to equivalence, all of whose multisets contain a 1-point.*

Lemma 3.5 ([12]). *There are exactly 23* 16*-divisible* $[79, 10, 16]_2$ *codes up to equivalence, all of whose multisets contain a 1-point.*

4 Proof of Theorem 1.4.

Let d be a positive integer and let $\mathcal{C}$ be an $[n, k, d]_q$ code with effective length n. Since $[g_q(k, sq^{k-1}), k, sq^{k-1}]_q$ codes (s-fold simplex codes) exist for any positive integer s, we assume that d is not divisible by q^{k-1}. Then, d can be uniquely expressed with $s = \lceil d/q^{k-1} \rceil$ as

$$d = sq^{k-1} - \sum_{j=1}^{r} q^{u_j - 1}, \tag{6}$$

where r and u_j's are integers satisfying

$$k - 1 \geq u_1 \geq u_2 \geq \cdots \geq u_r \geq 1 \text{ and } u_j > u_{j+q-1} \text{ for } 1 \leq j \leq r - q + 1. \tag{7}$$

The latter condition of (7) means that at most $q-1$ of $u_1, \ldots, u_r$ can take any given value. If $r \leq s$, it is easy to construct Griesmer codes from the multiset of an s-fold simplex code by deleting $F_1 + \cdots F_r$,where F_j denotes a $(u_j - 1)$-flat. In general, a Griesmer code which is obtained from an s-fold simplex code by deleting a union of disjoint flats (at most $q - 1$ of any given dimension) is called a *code of Belov type*.

Lemma 4.1. *There exists a* $[g_q(k, d), k, d]_q$ *code if* $r \leq s$.

Assume $r \geq s + 1$ and let $u = \sum_{i=1}^{s+1} u_i$. The following theorem was proved by Belov et al. [1] for binary linear codes and by Hill [6] and Dodunekov [4] for codes over $\mathbb{F}_q$.

Theorem 4.2 ([6]). *When* $r \geq s + 1$, *there exists a* $[g_q(k, d), k, d]_q$ *code of Belov type if and only if* $u \leq sk$.

Assume l, k, β, τ are integers with $l \geq 3$, $\beta = \binom{l+1}{2} + 1$, $\beta \leq k \leq \beta + l$, $\tau = k - \beta$. Recall that $C_{2,k} = (l-2)2^{k-1} + 2^{k+\tau-l-2} + 2^{k-l-2}$ from (2). A Griesmer $[g_2(k, d), k, d]_2$ code for $d = C_{2,k}$ is not of Belov type by Theorem 4.2 since $u = \sum_{i=1}^{s+1} u_i$ from (6) for $d = C_{2,k}$ satisfies $u = (l - 1)k + 1$ with $s = l - 1$. Actually, $C_{2,k}$ is the largest value of d such that a Griesmer $[g_2(k, d), k, d]_2$ code of Belov type does not exist for given $k \geq 5$ [11].

Lemma 4.3. *Let* $\mathcal{C}$ *be a Griesmer* $[g_2(k, d), k, d]_2$ *code for* $d = C_{2,k}$ *and let* $\mathcal{C}^*$ *be a projective dual. Then,*

(a) $g_2(k, C_{2,k}) = (l - 2)\theta_{k-1} + \theta_{k+\tau-l-2} + \theta_{k-l-2} + l + 1.$

(b) $\mathcal{C}^*$ *is a* 2^{l-1}*-divisible* $[(l - 1)2^l + 1, k, 2^{l-1}]_2$ *code for* $\tau = 0$.

(c) $\mathcal{C}^*$ *is a* 2^l*-divisible* $[(l - 1)2^{l+1} + 2^\tau + 1, k, 2^l]_2$ *code for* $\tau \geq 1$.

Proof. The part (a) is obtained from (1). Since $\mathcal{C}$ is 2^{k-l-1}-divisible (resp. 2^{k-l-2}-divisible) for $\tau = 0$ (resp. $\tau \geq 1$) by Lemma 2.6, one can get (b) and (c) applying Lemma 2.2. □

Lemma 4.4. *$n_2(k,d) = g_2(k,d)$ for $\tilde{C}_{2,k} < d \le (l-2)2^{k-1}$.*

Proof. Let $d = \tilde{C}_{2,k} + 1 = (l-2)2^{k-1} - (2^l - 1 - 2^{\tau+1})2^{k-l-2} + 1$ from (3). Since $(2^l - 1 - 2^{\tau+1})2^{k-l-2} - 1 = \theta_{k-3} - 2^{k-l+\tau-1} - 2^{k-l-2}$, we can calculate $u = \sum_{i=1}^{s+1} u_i$ from (6) for $d = \tilde{C}_{2,k} + 1$ as $u = (l-2)k$. Hence, there exists a Griesmer code of Belov type. One can prove similarly for other values of d with $\tilde{C}_{2,k} < d \le (l-2)2^{k-1}$. □

Lemma 4.5. *$n_2(11,d) = g_2(11,d)$ for $2^{11} = 2048 < d \le C_{2,11} = 2112$.*

Proof. Note that $l = 4$ for $k = 11$. Let $\mathcal{C}$ be an 8-divisible $[49,11,8]_2$ code with multiset $\mathcal{M}_{\mathcal{C}} = 4\mathcal{E}_3 \oplus 2\Pi_2 \oplus \Pi_3$. Then, the spectrum of $\mathcal{C}$ is $(a_{17}, a_{25}, a_{33}, a_{41}) = (525, 1160, 330, 32)$, and the projective dual $\mathcal{C}^*$ of $\mathcal{C}$ is a $[g_2(11,2112), 11, C_{2,11} = 2112]_2$ code. Let Λ_0^* be the set of 0-points for $\mathcal{C}^*$, which corresponds to the set of 41-hyperplanes for $\mathcal{C}$. We shall find j-flats F_j, $j = 1,2,\ldots,5$ which are disjoint from Λ_0^*. We take $\mathcal{E}_3$, Π_2 and Π_3 in $\Pi_{10} = \mathrm{PG}(10,2)$ as

$$\mathcal{E}_3 = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, \sum_{i=1}^{4} \boldsymbol{e}_i\}, \quad \Pi_2 = \langle \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 \rangle, \quad \Pi_3 = \langle \boldsymbol{e}_8, \boldsymbol{e}_9, \boldsymbol{e}_{10}, \boldsymbol{e}_{11} \rangle.$$

Let Δ be the solid containing $\mathcal{E}_3$ and let H be a hyperplane of Π_{10}. Then, $m_{\mathcal{C}}(H) = 41$ if and only if either H contains Δ and one of Π_2, Π_3, or H contains $\Pi_2 \cup \Pi_3$ and $H \cap \Delta$ is a 12-plane. So, we get the 32-set $\Lambda_0^* = \mathcal{T}_3 \cup \Pi_2^* \cup \Pi_3^*$ with

$$\begin{aligned}
\mathcal{T}_3 &= \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\} \cup \{\boldsymbol{e}_i + \boldsymbol{e}_j \mid 1 \le i < j \le 4\}, \\
\Pi_2^* &= \langle \boldsymbol{e}_5, \boldsymbol{e}_6, \boldsymbol{e}_7 \rangle, \quad \Pi_3^* = \langle \boldsymbol{e}_8, \boldsymbol{e}_9, \boldsymbol{e}_{10}, \boldsymbol{e}_{11} \rangle.
\end{aligned}$$

We take $F_1, F_2, \ldots, F_5$ as

$$\begin{aligned}
F_1 &= \langle \boldsymbol{e}_i + \boldsymbol{e}_{i+8} \mid i = 1,2 \rangle, \; F_2 = \langle \boldsymbol{e}_i + \boldsymbol{e}_{i+7} \mid i = 1,2,3 \rangle, \\
F_3 &= \langle \boldsymbol{e}_i + \boldsymbol{e}_{i+4} \mid i = 1,2,3,4 \rangle, \; F_4 = \langle \boldsymbol{e}_i + \boldsymbol{e}_{i+6} \mid i = 1,2,\ldots,5 \rangle, \\
F_5 &= \langle \boldsymbol{e}_i + \boldsymbol{e}_{i+5} \mid i = 1,2,\ldots,6 \rangle.
\end{aligned}$$

Then, it can be checked that $F_i \cap F_j = \emptyset$ for $1 \le i < j \le 5$ except for $(i,j) = (4,5)$ and that $\Lambda_0^* \cap (F_1 \cup F_2 \cup \cdots \cup F_5) = \emptyset$. Since $F_4 \cap F_5 = \{Q := 11111011111\}$ and Q is a 3-point for $\mathcal{C}^*$, $\mathcal{M}_{\mathcal{C}^*}$ contains the multiset $F_1 + F_2 + \cdots + F_5$. Hence, our assertion follows from Lemma 2.5 since $2112 = 3 \cdot 2^{10} - (2^9 + 2^8 + 2^7 + 2^6)$. □

The following lemma can be confirmed by the exhaustive computer search.

Lemma 4.6. *There exists no 16-divisible $[n,10,32]_2$ code for $n = 72, 76, 77, 78$.*

Lemma 4.7. *A $[g_2(11,d), 11, d]_2$ code does not exist for $d = \tilde{C}_{2,11}$.*

Proof. Assume $\mathcal{C}$ is a putative $[g_2(11,d), 11, d]$ code with $d = \tilde{C}_{2,11}$, which is a 32-divisible $[3265, 11, 1632]_2$ code by Lemma 2.6. Then, its projective dual $\mathcal{C}^*$ is a 16-divisible $[83, 11, 16]_2$ code. If there exists a t-point Q with $t > 16$ in $\mathcal{M}_{\mathcal{C}^*}$, then the multiset $\mathcal{M}_{\mathcal{C}^*} - 16Q$ over $\Pi_{10} = \mathrm{PG}(10,2)$ gives a 16-divisible $[67, 11, d']_2$ code $\mathcal{C}'$. But such a code does not exist for $d' \ge 48$ by the Griesmer bound. For $d' = 16, 32$, the projective dual of $\mathcal{C}'$ does not exist by the Griesmer bound. Hence, $\mathcal{C}^*$ has at most 16-points, and the spectrum of $\mathcal{C}$ satisfies $a_i = 0$ for any $i \notin \{1633 - 32j \mid 0 \le j \le 16\}$. Suppose $a_{1121} > 0$. Then, by Lemma 2.1, a 1121-hyperplane gives a $[1121, 10, 560]_2$ code, which does not exist by the Griesmer bound. Suppose $a_{1153} > 0$. Then, by Lemma 2.1, a 1153-hyperplane gives a Griesmer $[1153, 10, 576]_2$ code whose projective dual is a 4-divisible $[17, 10, 4]_2$ code, which does not exist since $k_2(17, 4) = 8$. In this way, one can prove that $a_{1633-32j} = 0$ for $j = 1, 2, 8, 9, 10, 12, 13, 14, 15, 16$. So, the possible t-points in $\mathcal{M}_{\mathcal{C}^*}$ satisfy

$$t \in \{3, 4, 5, 6, 7, 11\}. \tag{8}$$

Suppose $\mathcal{M}_{\mathcal{C}^*}$ has a 11-point P. By Lemmas 2.8, 4.6 and the Griesmer bound, the projection of $\mathcal{M}_{\mathcal{C}^*} - 11P$ from P onto a hyperplane Π with $P \notin \Pi$ gives a 16-divisible $[72, 10, 16]_2$ code. On the other hand, 16-divisible $[72, 10, 16]_2$ codes are unique up to equivalence by Lemma 3.3 and the multiset of such a code consists of five 8-points and 32 1-points from Table 5, which contradicts (8). Hence, $\mathcal{M}_{\mathcal{C}^*}$ has no 11-point. Using Lemmas 3.3, 3.4, 3.5 and Table 5, one can vanish possible t-points of $\mathcal{M}_{\mathcal{C}^*}$ for $t = 4, 5, 6, 7$, similarly. Then, $\mathcal{M}_{\mathcal{C}^*}$ consists only of 3-points, which is impossible. This completes the proof. □

Now, Theorem 1.4 follows from Lemmas 4.4, 4.5 and 4.7.

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Tatsuya Maruta, Taiki Tochiyama and Keita Yasufuku
Department of Mathematics
Osaka Metropolitan University
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka 558-8585 Japan
e-mail: maruta@omu.ac.jp, tt1816@icloud.com, keita0125sve@icloud.com

ХИПОТЕЗА ЗА МИНИМАЛНАТА ДЪЛЖИНА НА БИНАРНИ ЛИНЕЙНИ КОДОВЕ

T. Maruta, T. Tochiyama и K. Yasufuku

Абстракт

Даваме нова хипотеза за $D_{2,k}$, където $D_{q,k}$ означава най-голямото d, за което не съществува линеен код с размерност k и минимално тегло d, удовлетворяващ границата на Griesmer. Също така доказваме, че $D_{2,11} = 1632$, откъдето следва, че нашата хипотеза е вярна за размерности $k \leq 11$.

Ключови думи: граница на Грийсмър, оптимален код, делим код.