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ON THE TERNARY EXTENDED PERFECT GOLAY CODE BY THE MODIFIED GC-CONSTRUCTION

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Abstract

It is shown that the ternary extended perfect Golay $[12, 6, 6]_3$-code can be constructed by the modified GC-construction. For the binary Golay $[24, 12, 8]$-code this has been shown by us earlier. In particular, the Golay $[12, 6, 6]_3$-code can be constructed using the GC-construction with the inner codes of length 3 and the outer codes of length 4.

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Key words: the ternary extended perfect Golay code, modified generalised concatenation construction.

1 Introduction. Let $E_q = \{0, 1, \ldots, q-1\}$ be the alphabet of size q. A q-ary block code C is an arbitrary nonempty subset of E_q^n. We say that C is an $(n, N, d)_q$-code, where n is the *length* of the code, $N = |C|$ is the number of codewords, i.e. the *cardinality* of C, and d is the *minimum Hamming distance*. A linear code C over the Galois field $\mathbb{F}_q$ of order q with parameters $(n, N = q^k, d)_q$ is denoted by $[n, k, d]_q$. Let $J = \{1, 2, \ldots, n\}$ be the coordinate set of E_q^n. For any vector $\mathbf{x} = (x_1, \ldots, x_n) \in E_q^n$, we denote by $\mathrm{supp}(\mathbf{x})$ its support, i.e.

$$\mathrm{supp}(\mathbf{x}) = \{i \in J : \ x_i \neq 0\}.$$

Let $\mathrm{wt}(\mathbf{x})$ be the weight of vector $\mathbf{x}$, i.e. the cardinality of its support: $\mathrm{wt}(\mathbf{x}) = |\,\mathrm{supp}(\mathbf{x})|$. The *parity* of any vector is the sum of its coordinates (in the field $\mathbb{F}_3$). In this case, the parity of any vector can be 0, 1 or 2.

Here, we restrict ourselves to the ternary codes. Thus, throughout the whole paper, we drop the size of the alphabet and use $[n, k, d]$ to denote a $[n, k, d]_3$-code over $\mathbb{F}_3$.

In the recent publication [6], we showed that the extended perfect $[24, 12, 8]_2$ Golay code can be constructed using the GC-construction (see [2] for the details on the Golay code). The goal of this work is to prove a similar result for the ternary extended perfect Golay $[12, 6, 6]_3$-code. In particular, this code is obtained using the modified GC-construction.

It is necessary to mention that the interest in cascade and generalized cascade codes is caused by the possibility of cascade decoding, which realizes the code distance by decoding short inner and outer codes (see [1, 9, 8] and the references therein). Consequently, the soft decoding is possible [4].

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2 Preliminary results. In order to make this paper more self-sufficient, following [6, 7, 5], we repeat the well known generalized concatenation (GC) construction for the ternary codes. This construction is based on a nested family of inner codes of a fixed length n_b. In this paper, we need the inner codes of length $n_b = 3$ that we are going to introduce in the next section. Here we provide several trivial statements.

Lemma 2.1. *(i) If code C of length n consists of all vectors of length n of parity 0, then it is linear.*

(ii) Let C be an arbitrary linear code and let C_1 and C_2 be its cosets, given by

$$C_1 = C + \mathbf{b}_1, \quad C_2 = C + \mathbf{b}_2.$$

If the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ are such that $\mathbf{b}_1 + \mathbf{b}_2$ is the zero vector, then, the code $C \bigcup C_1 \bigcup C_2$ is also linear.

Proof. (i) By construction, the set C is closed under the addition. Indeed, the sum of two words of parity 0, has parity 0, and the statement follows.

(ii) It is easy to verify that the code $C \cup C_1 \cup C_2$ is closed under the addition of its codewords, thus the statement follows. □

2.1 Inner codes of length 3. Consider the inner codes of length $n_b = 3$ over the field $\mathbb{F}_3$. These codes are denoted by B_i with some index i which represents the subdivision step and the tuple of i indices $(j_1, j_2, \ldots, j_i)$. Let B be the $[3, 3, 1]$-code, i.e. the set of all ternary vectors of length 3. This code can be subdivided into three $(3, 9, 2)$ subcodes: the linear $[3, 2, 2]$-code $B_1(0)$ and its cosets, the $(3, 9, 2)$ codes $B_1(1)$ and $B_1(2)$. The code $B_1(0)$ contains the trivial $[3, 1, 3]$-code $B_2(0, 0)$, which partitions the code $B_1(0)$ into the cosets of $B_2(0, 0)$, i.e.

$$B_1(0) = B_2(0,0) \bigcup B_2(0,1) \bigcup B_2(0,2) = \bigcup_{t=0}^{2} B_2(0,t).$$

We write out these codes explicitly

$$B_2(0,0) = \begin{bmatrix} (000) \\ (111) \\ (222) \end{bmatrix}, B_2(0,1) = \begin{bmatrix} (012) \\ (120) \\ (201) \end{bmatrix}, B_2(0,2) = \begin{bmatrix} (021) \\ (102) \\ (210) \end{bmatrix}.$$

Thus, we have defined a nested chain of codes of length 3:
the trivial $[3, 3, 1]_3$-code $B_0 = B$,
the trivial $[3, 2, 2]_3$-code $B_1(0)$,
the trivial $[3, 1, 3]_3$-code $B_2(0, 0)$.
These codes induce the following alphabets sizes of the outer codes:

$$q_1 = q_2 = q_3 = 3.$$

Recall that the sum of a matrix $B = [\mathbf{b}_i]$ (here $\mathbf{b}_i$ is the i-th row) and vector $\mathbf{x}$ is defined as $B + \mathbf{x} = [\mathbf{b}_i + \mathbf{x}]$, where the vector $\mathbf{x}$ is added to every row.

Thus, we have subdivided the set of all vectors of length 3 into $[3,1,3]$-codes $B_2(i,j)$. Every such code B can be presented as: $B = B_2(0,0) + \mathbf{b}$ for some vector $\mathbf{b}$. This presentation defines the following additive operation of two different codes $B = B_2(0,0) + \mathbf{b}$ and $B' = B_2(0,0) + \mathbf{b}'$. The sum $B'' = B + B'$ of two codes is defined as $B'' = B_2(0,0) + \mathbf{b}''$, where $\mathbf{b}'' = \mathbf{b} + \mathbf{b}'$.

Enumerate all ternary vectors of length 3. Assume that the s-th row $\mathbf{b} \in \mathbb{F}_3^3$ of the above matrices $B_2(i,j)$ is the s-th codeword of the corresponding code. In particular, if the word $\mathbf{b}$ belongs to the code $B_2(i,j)$ then, it is assigned index (i,j,s), and we write $\mathbf{b} = \mathbf{b}(i,j,s)$. So, any vector $\mathbf{b}(i,j,s) \in \mathbb{F}_3^3$ is the s-th codeword of the code $B_2(i,j)$.

The introduced enumeration of the ternary vectors of length 3 is linear in all three indices under the addition of codes as well as vectors. Thus it is useful to define a map μ from the set of vectors $\mathbf{x} = (x_1, x_2, x_3)$ of length 3 over $\mathbb{F}_3$ into the ring $\mathbb{Z}_{3^3}$ of integers

$$\mu(x_1, x_2, x_3) = \sum_{t=1}^{3} x_t 3^{t-1}.$$

The following statement is obvious:

Lemma 2.2. *Let $B_2(i,j)$ and $B_2(i',j')$ be two arbitrary codes of length 3, and let $\mathbf{b}(i,j,s) \in B_2(i,j)$ and $\mathbf{b}(i',j',s') \in B_2(i',j')$ be two arbitrary codewords of these codes. The sum of these codes*

$$B'' = B_2(i,j) + B_2(i',j')$$

and its codewords are such that

$$\mathbf{b} = \mathbf{b}(i,j,s) + \mathbf{b}(i',j',s').$$

Then

(i) The code B'' has index (i'', j'') equal to the coordinate-wise sum (in the field $\mathbb{F}_3$) of the corresponding indices, i.e.

$$i'' = i + i', \;\; j'' = j + j'.$$

(ii) The code B'' is equal to

$$B = B_2(0,0) + \mathbf{b}'',$$

where the vector $\mathbf{b}''$ is the sum of two vectors $\mathbf{b}'' = \mathbf{b} + \mathbf{b}'$.

(iii) The vector $\mathbf{b}''$ is of the form $\mathbf{b}''(i'', j'', s'')$, whose indices i'', j'' and s'' are the sums (in the field $\mathbb{F}_3$) of indices of the corresponding terms, i.e.

$$i'' = i + i', \;\; j'' = j + j', \;\; s'' = s + s'.$$

(iv) The codeword $\mathbf{b}(i'', j'', s'')$, which is the sum of two codewords $\mathbf{b}(i,j,s)$ and $\mathbf{b}(i',j',s')$ of the codes $B_2(i,j)$ and $B_2(i',j')$ respectively, can be presented as $\mathbf{b}(i'', j'', s'')$ of the code $B_2(i'', j'')$, which is the sum of the codes $B_2(i,j)$ and $B_2(i',j')$ with these codewords.

Our construction consists of the following steps. First, using the standard *GC*-construction we construct the linear $[12,5,6]_3$-code C_0. Then, we find two cosets C_1 and C_2 of this code at distance 6 from the code C_0, and also $d(C_1,C_2)=6$, so that the union of three codes C_0, C_1 and C_2 is the code with the minimum Hamming distance 6. Thus, using the classical GC-construction construct the $[12,5,6]$-code C_0. Here, we use only two outer codes instead of (according to the nested chain of the inner codes) the possible three. Indeed, there is no code A_1 with distance 6, since the length is 4.

Outer codes A_i for the construction of the code C_0: The ternary perfect $[4,2,3]$-Hamming code A_2, the ternary $[4,3,2]$-code A_3 with parity check.

We will provide these codes explicitly. The outer $[4,2,3]_3$-Hamming code A_2 is the union of three subcodes $A_{2,0}$, $A_{2,1}$ and $A_{2,2}$ listed below

$$A_{2,0}=\begin{bmatrix} (0000) \\ (0111) \\ (0222) \end{bmatrix},\ A_{2,1}=\begin{bmatrix} (1012) \\ (1201) \\ (1120) \end{bmatrix},\ A_{2,2}=\begin{bmatrix} (2021) \\ (2102) \\ (2210) \end{bmatrix}.$$

We also list below the outer code A_3 which is the union of three subcodes $A_{3,0}$, $A_{3,1}$ and $A_{3,2}$:

$$A_{3,0}=\begin{bmatrix} (0000) \\ (0012) \\ (0021) \\ (0102) \\ (0111) \\ (0120) \\ (0201) \\ (0210) \\ (0222) \end{bmatrix},\ A_{3,1}=\begin{bmatrix} (1002) \\ (1011) \\ (1020) \\ (1102) \\ (1111) \\ (1120) \\ (1201) \\ (1210) \\ (1222) \end{bmatrix},\ A_{3,2}=\begin{bmatrix} (2001) \\ (2010) \\ (2022) \\ (2100) \\ (2112) \\ (2121) \\ (2202) \\ (2211) \\ (2220) \end{bmatrix}.$$

Now we construct the linear $[12,5,6]$-code C_0 using the standard GC-construction with the given inner and outer codes. Lemma 2.1 implies that this code is linear. We are interested in the distance between the code C_0 and its cosets C_1 and C_2.

3 Ternary extended perfect Golay code. We want to find two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ so that the union of three codes

$$C=C_0\bigcup C_1\bigcup C_2,$$

where $C_1=C_0+\mathbf{b}_1$ and $C_2=C_0+\mathbf{b}_2$ is the ternary Golay code $\mathcal{G}_{12}$.

Lemma 3.1. *Consider the following two vectors* $\mathbf{b}_1$ *and* $\mathbf{b}_2$

$$\mathbf{b}_1=(001,001,010,211),\ \ \mathbf{b}_2=(002,002,020,122),$$

which satisfy $\mathbf{b}_1+\mathbf{b}_2=\mathbf{0}$. *Then*

(i) The minimal weight of the codewords of C_1 and C_2 is equal to 6.

(ii) The $[12,6,6]$-code C, obtained as the union of codes C_0, C_1 and C_2 is linear and has the minimal distance 6.

Proof. By Lemma 2.1 and direct verification. □

The following is the main theorem.

Theorem 3.2. *The code C, defined above as the union of codes C_0, C_1, and C_2, has parameters* $[12, 6, 6]$. *Then it is the ternary extended perfect Golay* $[12, 6, 6]$*-code* $\mathcal{G}_{12}$.

Proof. Lemma 2.1 and Lemma 3.1 imply that the code C is the ternary linear $[12, 6, 6]$-code. It follows from the classical result of Vera Pless [3] that the constructed $[12, 6, 6]$-code is the ternary extended perfect Golay code. □

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ЗА ТРОИЧНИЯ РАЗШИРЕН СЪВЪРШЕН КОД НА ГОЛЕЙ ЧРЕЗ МОДИФИЦИРАНАТА GC-КОНСТРУКЦИЯ

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Абстракт

Показано е, че троичният разширен съвършен код на Голей $[12, 6, 6]_3$ може да бъде конструиран чрез модифицираната GC-конструкция, като за бинарния код на Голей $[24, 12, 8]$ това беше доказано от нас по-рано. В частност, кодът на Голей $[12, 6, 6]_3$ може да бъде конструиран с GC-конструкция с вътрешни кодове с дължина 3 и външни кодове с дължина 4.

Ключови думи: троичен разширен съвършен код на Голей, модифицирана обобщена конкатенационна конструкция.