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A Reuniformization of 'Dynamical Systems Ansamble' in Uniform Spaces

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Let X be a complete Hausdorff uniform space. The notion 'dynamical systems ansamble' has been introduced, that is, the family of continuous operators $T_t^{\Gamma} : X \to X$, $t \in [0, \infty)$, $\Gamma \in \Omega$ (where Ω is an index set) with the following properties: 1) $T_0^{\Gamma}(x) = x$ for every $x \in X$ and $\Gamma \in \Omega$; 2) for every $\Gamma_1, \Gamma_2 \in \Omega$ and $t_1, t_2 \in [0,\infty)$ there exists an element $\Gamma \in \Omega$ such that $T_{t_1}^{\Gamma_1}(T_{t_2}^{\Gamma_2}(x)) = T_{t_1+t_2}^{\Gamma}(x)$ for every $x \in X$. Necessary and sufficient conditions for the existence of a topology-preserving reuniformization such that each operator becomes a contraction with respect to a family of contraction functions with suitable properties has been obtained.

In many applications the semigroup structure is not enough to describe some phenomena in the system theory. That is why V. I. Opoizev has introduced in [1] the notion 'dynamical systems ansamble' in a metric space. The investigation of such systems is based on the remetrization results due to P. R. Meyers [2], [3]. Our goal from the applications point of view is to extend this theory to suitable uniform spaces, with fixed point theory for contractive mappings (cf. [4], [5]). In the present paper we shall obtain a reuniformization result for 'dynamical systems ansamble' (DSA), in particular for contractive semigroup of mappings in a unifom space. This theory includes the locally convex topological vector spaces because they are completely regular and their topology is uniformizable. As far as we know, the first paper dealing with a semigroup theory in locally convex topological vector spaces and applications to partial differential equations is that of I. Miyadera [6].

1. Definitions and denotations

Let X be a complete Hausdorff uniform space with a uniformity generated by a saturated family of pseudometrics $\mathfrak{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$, where A is an index set [7]. Let G be a commutative topological semigroup, written additively. A family of continuous self-mappings of X $\{T_t : t \in G\}$ will be called a G-semigroup of operators on X, if: 1) $T_0(x) = x$ for every $x \in X$; 2) $T_{t_1}(T_{t_2}(x)) = T_{t_1+t_2}(x)$ for every $t_1, t_2 \in G$ and $x \in X$. Throughout the remainder of the paper we shall consider the case $G = R_+^1 \equiv [0,\infty)$.

Denote by 2^X, as usually the set of all subsets of X. Let T be a family of continuous operators of X into itself. Define the set-valued mapping $T_d^k : X \to 2^X$ by the following way

$$\mathsf{T}_d^k(x)=\bigcup\{T(x):T\in\bar{\mathsf{T}}^k\}.$$

where $\bar{\mathsf{T}}^k=\{T:T(x)=T_k(T_{k-1}(\dots T_1(x)\dots)),\ T_i\in\mathsf{T},\ i=1,2,\dots,k\}$. T_d^k will be called a 'dynamical systems ansamble' (DSA) with discrete time in the uniform space X.

Let us consider a family of continuous operators $\mathsf{T}=\{T_t^\Gamma:\Gamma\in\Omega,\ t\in R_+^1\}$ (where $T_t^\Gamma:X\to X$, Ω is an index set), satisfying the conditions:

1) $T_0^\Gamma(x)=x$ for every $x\in X$ and $\Gamma\in\Omega$.

2) for every $\Gamma_1,\Gamma_2\in\Omega$ there is $\Gamma\in\Omega$ so that for every $t_1,\ t_2\in R_+^1$

$$T_{t_1}^{\Gamma_1}(T_{t_2}^{\Gamma_2}(x))=T_{t_1+t_2}^{\Gamma}(x)\ \text{ for each }\ x\in X.$$

We shall call the family of set-valued mappings

$$\mathsf{T}_t(x)=\bigcup\{T_t^\Gamma(x):\Gamma\in\Omega\},\ t\in R_+^1$$

a (DSA) with a continuous time.

Let $j:A\to A$ be mapping on the index set into itself and let $j^k(\alpha)=j(j^{k-1}(\alpha))$, $j^0(\alpha)=\alpha$ $(k=1,2,\dots)$. Since j is, in general, not invertible we shall denote by $j^{-1}(\alpha_0)$ the inverse image of the element $\alpha_0\in A$, that is, $j^{-1}(\alpha_0)=\{\alpha\in A:j(\alpha)=\alpha_o\}$. Further on, we define $j^{-2}(\alpha_0)=\{\alpha\in A:j(\alpha)=\bar\alpha,$ where $\bar\alpha$ runs over all elements of $j^{-1}(\alpha_0)\}$ and inductively $j^{-n}(\alpha_0)=\{\alpha\in A:j(\alpha)=\bar\alpha,$ where $\bar\alpha$ runs over all elements of $j^{-(n-1)}(\alpha_0)\}$, $(n=1,2,\dots)$.

The uniform space X is called j-bounded if for every $x,\ y\in X$ and $\alpha\in A$, there exists a constant $Q=Q(x,\ y,\ \alpha)>0$ such that

$$\rho_{j^{-n}(\alpha)}(x,\ y)\leqq Q(x,\ y,\ \alpha)<\infty\ \ (n=0,\ 1,\ 2\ ,\dots) \tag{1}$$

and $\rho_{j(\alpha)}(x,y)\leqq\rho_\alpha(x,\ y)$. (Let us note that the last condition is satisfied in the applications cf. [5].) The inequality (1) means (for $n\geqq 0$): the constant $Q=Q(x,\ y,\ \alpha)>0$ is an upper bound of the set

$$\{\rho_\alpha(x,\ y),\ \rho_{\alpha_1}(x,\ y),\dots,\ \rho_{\alpha_n}(x,\ y),\dots\}$$

for every $\{\alpha,\ \alpha_1,\ \alpha_2,\dots,\ \alpha_n,\dots\}$, where $\alpha_1\in j^{-1}(\alpha),\ \alpha_2\in j^{-2}(\alpha),\dots,\ \alpha_n\in j^{-n}(\alpha),\dots$

Let (Φ) be a family of functions $\Phi_\alpha(u):R_+^1\to R_+^1$, $\alpha\in A$ $(R_+^1=[0,\ \infty))$ with the properties:

(C1) $\Phi_\alpha(u)$ is continuous from the right, strictly increasing and

$$0<\Phi_\alpha(u)<u\ \text{ for }\ u>0\ \text{ and }\ \Phi_{j(\alpha)}(u)\leqq\Phi_\alpha(u)\ ;$$

(C2) $\lim\limits_{n\to\infty}\Phi_\alpha(\Phi_{j^{-1}(\alpha)}(\dots\Phi_{j^{-n}(\alpha)}(u)\dots))=0.$

The last equality means that

$$\lim_{n\to\infty}\Phi_\alpha(\Phi_{\alpha_1}(\dots\Phi_{\alpha_n}(u)\dots))=0$$

for every sequence $\{\alpha, \alpha_1, \ldots, \alpha_n, \ldots\}$, where

$$\alpha_1 \in j^{-1}(\alpha),\ \alpha_2 \in j^{-2}(\alpha), \ldots,\ \alpha_n \in j^{-n}(\alpha)\ \ldots.$$

The mapping $T : X \to X$ is called: 1) Φ-contractive if $\rho_{j(\alpha)}(Tx, Ty) \leqq \Phi_\alpha(\rho_\alpha(x, y))$ for every $x, y \in X$ and $\alpha \in A$, and 2) j-regular if $\{T^n x\}_{n=0}^{\infty}$ is not a ρ_α-Cauchy sequence, then it is not $\rho_{j(\alpha)}$-Cauchy sequence.

Remark 1. Let us emphasize that the definition of j-regularity is more general than the corresponding one from [4] and [5]. That is why we shall give a new proof of the fixed point theorem for j-regular contractive mappings.

Two families of pseudometrics $\mathfrak{A} = \{\rho_\alpha(x, y) : \alpha \in A\}$ and $\mathfrak{A}^* = \{\rho^*_{\alpha^*}(x, y) : \alpha^* \in A^*\}$ are said to be topologically equivalent if the uniform topologies generated by them coincide. Let

$$\text{card } A = \text{card } A^*. \tag{2}$$

Let $\psi : A \to A^*$ be a bijective map. It induces map $j^* : A^* \to A^*$ by the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & A^* \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j^*} \\ A & \xrightarrow[\psi]{} & A^* \end{array}$$

that is, $j^*(\alpha^*) = \psi(j(\alpha))$, where $\alpha = \psi^{-1}(\alpha^*)$.

The families $\mathfrak{A}$ and $\mathfrak{A}^*$ (for which (2) is satisfied) are called equivalent if every Cauchy net $x_\gamma (\gamma \in \omega$; ω is directed set, cf. [8]) in ρ_α is a Cauchy net in $\rho^*_{\alpha^*}$ for every $\alpha \in A$, where $\alpha^* = \psi(\alpha)$, and vice versa — every Cauchy net x_γ in $\rho^*_{\alpha^*}$ is a Cauchy net in ρ_α for every $\alpha^* \in A^*$, where $\alpha = \psi^{-1}(\alpha^*)$.

It is not difficult to establish that the equivalence of the families $\mathfrak{A}$ and $\mathfrak{A}^*$ implies their topological equivalence. Conversely, the topological equivalence implies their equivalence when X is a complete uniform space.

$\mathfrak{A}$ and $\mathfrak{A}^*$ are said to be j-topologically equivalent if j-regularity of T implies its j^*-regularity and j-boundedness of X implies j^*-boundedness of X.

We shall use the standard denotations

$$U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta) = \{y \in X : \rho_{\alpha_1}(\xi, y) < \eta, \rho_{\alpha_2}(\xi, y) < \eta, \ldots, \rho_{\alpha_p}(\xi, y) < \eta\} \ (\eta > 0)$$

for the open neighbourhood of $\xi \in X$.

We say that the sequence $\{\mathbf{T}^n(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta))\}_{n=0}^{\infty}$ tends uniformly to ξ if for every $\varepsilon > 0$ there is n_0 such that for $n > n_0$

$$\mathbf{T}^n(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta)) \subset U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \varepsilon).$$

Now we are going to formulate the basic dynamical characteristics of (DSA) with a discrete time in the uniform space X:

(RD 1) There exists a point $\xi \in X$ (an equilibrium state) such that

$$\bigcap \{ T(\xi) : T \in \mathbf{T} \} = \{\xi\}.$$

(RD 2) (Stability) For every neighbourhood W of ξ, one can find a neighbourhood V of ξ such that $x \in V$ implies $\mathbf{T}_d^k(x) \subset W$ for every $k = 1, 2, \ldots,$

(RD 3) (Convergence) For every $x \in X$ and every finite collection $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$, $\mathbf{T}_d^k(x) \to \{\xi\}$ as $k \to \infty$, so that, if $\mathbf{T}_d^k(x) \bigcap \mathbf{cl}\, U(j(\alpha_1), j(\alpha_2), \ldots, j(\alpha_p))(\xi, \eta) = \emptyset$ then $\mathbf{T}_d^k(x) \bigcap \mathbf{cl}\, U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta) = \emptyset$.

(RD 4) (Uniform convergence) For every finite collection $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$ there is an $\eta = \eta(\alpha_1, \alpha_2, \ldots, \alpha_p) > 0$ such that $\{ \mathbf{T}_d^k(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta)) \}_{k=0}^{\infty}$ tends uniformly to ξ, so that if $x \in \mathbf{cl}\mathbf{T}_d^n(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta))$, then $x \in \mathbf{cl}\mathbf{T}^n(U(j(\alpha_1), j(\alpha_2), \ldots, j(\alpha_p))(\xi, \eta))$ for every n.

Let us note that the definition of a (DSA) with discrete time includes, for example, an iteration procedure of the type

$$x_{k+1} = T_k(x_k) \quad (k = 0, 1, 2, \ldots),$$

where the operator T_k depends on the iteration number k; (cf. [1]). This fact motivates the more general definition of (DSA) in the case of continuous time. The semigroup structure is not enough to describe such processes.

As in the discrete case we shall formulate correspondence conditions:

(RC 1) (An existence of an equilibrium state) There exists a common fixed point $\xi \in X$ for all operators $T\Gamma$ of the family $\mathbf{T}_t$, that is,

$$\bigcap \{ T\Gamma(\xi) : \Gamma \in \Omega,\ t \in R_+^1 \} = \{\xi\}.$$

(RC 2) (Stability condition) For every neighbourhood W of ξ one can find a neighbourhood V and ξ such that $x \in V$ implies $\mathbf{T}_t(x) \subset W$ for every $t > 0$.

(RC 3) (Convergence) For every $x \in X$ and every finite collection $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$, $\mathbf{T}_t(x) \to \{\xi\}$ as $t \to \infty$, so that if

$$\mathbf{T}_t(x) \bigcap \mathbf{cl}\, U(j(\alpha_1), j(\alpha_2), \ldots, j(\alpha_p))(\xi, \eta) = \emptyset$$

then

$$\mathbf{T}_t(x) \bigcap \mathbf{cl}\, U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta) = \emptyset.$$

(RC 4) (Uniform convergence) For every finite collection $\alpha_1, \alpha_2, \ldots, \alpha_p \in A$ there is a $\eta = \eta(\alpha_1, \alpha_2, \ldots, \alpha_p) > 0$ such that $\mathbf{T}_t(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi, \eta))$ tends uniformly to ξ as $t \to \infty$ and if $x \in \mathbf{cl}\mathbf{T}_t(U(\alpha_1, \alpha_2, \ldots, \alpha_p)(\xi\ \eta))$, then

$$x \in \mathbf{cl}\mathbf{T}_t(U(j(\alpha_1), j(\alpha_2), \ldots, j(\alpha_p))(\xi, \eta)).$$

A family of operators $\mathbf{T}=\{T_\Gamma : t\in R^1_+,\ \Gamma\in\Omega\}$ is said to be (Φ)-contractive if for every $x,\ y\in X,\ t>0$ and $\Gamma\in\Omega$ the following inequality holds:

$$\rho_{j(\alpha)}(T_\Gamma(x),\ T_\Gamma(y))\leqq k_\alpha^t\rho_\alpha(x,\ y).$$

Here the family (Φ) consists of functions of the type $\Phi_\alpha(u)=k_\alpha u$, $0<k_\alpha<1$. Let us note that for the reuniformization result [5] the functions $\Phi_\alpha(u)$ must be superadditive. On the other hand, the main result of the present paper (sufficiency of Theorem 2) needs $\Phi_\alpha(u)$ to be subadditive, so that finally we obtain the linear functions $\Phi_\alpha(u)=k_\alpha u$.

A family $\mathbf{T}$ of continuous operators is said to be contractifiable if for every family (Φ) there exists a family of pseudometrics $\mathfrak{U}^*=\{\rho^*_{\alpha^*}(x,\ y) : \alpha^*\in A^*\}$ and the mappings $j^* : A^*\to A^*$ such that $\mathbf{T}$ is (Φ)-contractive (with respect to $\mathfrak{U}^*$) and $\mathfrak{U}^*$ is j-topologically equivalent to $\mathfrak{U}$.

Finally, we shall formulate the following definition for a uniform continuity (UC) of the (DSA) for every $x_0\in X,\ t,\ t_0\in R^1_+$ and $\varepsilon>0$ there exist $\delta_1>0$ and $\delta_2>0$ such that $\rho_\alpha(x_0,\ x)<\delta_1,\ |t-t_0|<\delta_2$ imply $\rho_\alpha(T_{\Gamma_0}(x_0),\ T_\Gamma(x))<\varepsilon$ for every $\Gamma\in\Omega$.

2. Fixed point theorem

Theorem 1. *Every Φ-contractive j-regular mapping T of a complete uniform Hausdorff j-bounded space X into itself has a unique fixed point $\bar{x}\in X$ and* $\bar{x}=\lim_{n\to\infty} T^n x$ *for arbitrary* $x\in X$.

Proof: Define the sequence $a_n^\alpha=\rho_\alpha(T^n x,\ T^{n+1}x)$ with an arbitrary $x\in X$. Then

$$a_n^\alpha=\rho_\alpha(T^n x,\ T^{n+1}x)\leqq\Phi_{j^{-1}(\alpha)}(\rho^{-1}_{j(\alpha)}(T^{n-1}x,\ T^n x))\leqq\dots$$

$$\Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\dots\Phi_{j^{-n}(\alpha)}(\rho_{j^{-n}(\alpha)}(x,\ Tx))\dots))\leqq$$

$$\Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\dots\Phi_{j^{-n}(\alpha)}(Q(\alpha,\ x,\ Tx))\dots))$$

that implies $\lim_{n\to\infty} a_n^\alpha=0$.

Let us suppose, by contradiction, that $\{T^n x\}_{n=0}^\infty$ is not Cauchy sequence. But T is j-regular and then we can find $\varepsilon_1>0$ and finite collection of pseudometrics $\rho_{\alpha_i}(i=1,\ 2,\dots,\ l)$ such that for every ν there are $m(\nu)>\nu$ and $p(\nu)>0$ for which $\rho_{\alpha_i}(T^{m(\nu)+p(\nu)}x,\ T^{m(\nu)}x)\geqq\varepsilon_1$. Then by j-regularity of T $\rho_{j(\alpha_i)}(T^{m'(\nu)+p'(\nu)}x,\ T^{m'(\nu)}x)\geqq\delta_1$ for some $\delta_1>0$ and $m'(\nu),\ p'(\nu)$ $(\nu=1,\ 2,\dots)$. Consequently $\rho_{\alpha_i}(T^{m(\nu)+p(\nu)}x,\ T^{m(\nu)}x)\geqq\varepsilon_0$ and

$$\rho_{j(\alpha_i)}(T^{m'(\nu)+p'(\nu)}x,\ T^{m'(\nu)}x)\geqq\varepsilon_0,\text{where } \varepsilon_0=\min\{\varepsilon_1,\ \delta_1\}.$$

The following cases are possible:

1) the intersections $\{m(\nu)\}_{\nu=1}^{\infty} \bigcap \{m'(\nu)\}_{\nu=1}^{\infty}$ and $\{p(\nu)\}_{\nu=1}^{\infty} \bigcap \{p'(\nu)\}_{\nu=1}^{\infty}$ are infinite. Then we can denote the elements of the intersections by $\{\bar{m}(\nu)\}_{\nu=1}^{\infty}$ and $\{\bar{p}(\nu)\}_{\nu=1}^{\infty}$. Let $\bar{\mathrm{p}}$ be the smallest positive integer for which

$$\rho_{\alpha_i}(T^{\bar{m}+\bar{p}}x,\ T^{\bar{m}}x) \geqq \varepsilon_0 \text{ and therefore } \rho_{\alpha_i}(T^{\bar{m}+\bar{p}-1}x,\ T^{\bar{m}}x) < \varepsilon_0.$$

We put $h_{\nu}^{\alpha i} = \rho_{\alpha_i}(T^{\bar{m}}x,\ T^{\bar{m}+\bar{p}}x)$. Then

$$\varepsilon_0 \leqq h_{\nu}^{\alpha i} \leqq \rho_{\alpha_i}(T^{\bar{m}}x,\ T^{\bar{m}+\bar{p}-1}x) + \rho_{\alpha_i}(T^{\bar{m}+\bar{p}-1}x,\ T^{\bar{m}+\bar{p}}x) \leqq \varepsilon_0 + a_{\bar{m}+\bar{p}-1}^{\alpha_i}.$$

From the last inequality by $\nu \to \infty$ we obtain $\lim\limits_{\nu\to\infty} h_{\nu}^{\alpha i} = \varepsilon_0$. On the other hand,

$$\varepsilon_0 \leqq \rho_{j(\alpha_i)}(T^{\bar{m}+\bar{p}}x,\ T^{\bar{m}}x) \leqq \rho_{j(\alpha_i)}(T^{\bar{m}+\bar{p}}x,\ T^{\bar{m}+\bar{p}+1}x) + \rho_{j(\alpha_i)}(T^{\bar{m}+\bar{p}+1}x,\ T^{\bar{m}+1}x)$$

$$+ \rho_{j(\alpha_i)}(T^{\bar{m}+1}x,\ T^{\bar{m}}x) \leqq a_{\bar{m}+\bar{p}}^{j(\alpha_i)} + \Phi_{\alpha_i}(\rho_{(\alpha_i)}(T^{\bar{m}+\bar{p}}x,\ T^{\bar{m}}x)) + a_{\bar{m}}^{j(\alpha_i)}$$

which by $\nu \to \infty$ yields $\varepsilon_0 \leqq \Phi_{\alpha_i}(\varepsilon_0)$ and therefore $\{T^n x\}_{n=0}^{\infty}$ is a Cauchy sequence.

2) the intersections $\{m(\nu)\}_{\nu=1}^{\infty} \bigcap \{m'(\nu)\}_{\nu=1}^{\infty}$ and $\{p(\nu)\}_{\nu=1}^{\infty} \bigcap \{p'(\nu)\}_{\nu=1}^{\infty}$ are finite. Then we can reject the common members of the sequences and the obtained sequences we denote by $\{m(\nu)\}_{\nu=1}^{\infty}$, $\{m'(\nu)\}_{\nu=1}^{\infty}$, $\{p(\nu)\}_{\nu=1}^{\infty}$, $\{p'(\nu)\}_{\nu=1}^{\infty}$ so that $\{m(\nu)\}_{\nu=1}^{\infty} \bigcap \{m'(\nu)\}_{\nu=1}^{\infty} = \emptyset$ and $\{p(\nu)\}_{\nu=1}^{\infty} \bigcap \{p'(\nu)\}_{\nu=1}^{\infty} = \emptyset$. Consequently $\rho_{\alpha_i}(T^{m'(\nu)+p'(\nu)}x,\ T^{m'(\nu)}x) < \varepsilon_0$. Then

$$\varepsilon_0 \leqq \rho_{j(\alpha_i)}(T^{m'(\nu)+p'(\nu)}x,\ T^{m'(\nu)}x) \leqq \rho_{j(\alpha_i)}(T^{m'(\nu)+p'(\nu)+1}x,\ T^{m'(\nu)+p'(\nu)}x)$$

$$+ \rho_{j(\alpha_i)}(T^{m'(\nu)+p'(\nu)+1}x,\ T^{m'(\nu)+1}x) + \rho_{j(\alpha_i)}(T^{m'(\nu)+1}x,\ T^{m'(\nu)}x)$$

$$\leqq a_{m'+p'}^{j(\alpha_i)} + \Phi_{\alpha_i}(\rho_{\alpha_i}(T^{m'(\nu)+p'(\nu)}x,\ T^{m'(\nu)}x)) + a_{m'}^{j(\alpha_i)} \leqq a_{m'+p'}^{j(\alpha_i)} + \Phi_{\alpha_i}(\varepsilon_0) + a_{m'}^{j(\alpha_i)}.$$

After $\nu \to \infty$ we obtain $\varepsilon_0 \leqq \Phi_{\alpha_i}(\varepsilon_0)$.

In view of the completeness of X there is $\bar{x} \in X$ such that $\bar{x} = \lim\limits_{n\to\infty} T^n x$. The continuity of T implies $T\bar{x} = T(\lim\limits_{n\to\infty} T^n x) = \lim\limits_{n\to\infty} T^{n+1}x = \bar{x}$.

The uniqueness of the fixed point of T follows from the estimates ($\bar{x} = T\bar{x}$ and $\bar{\bar{x}} = T\bar{\bar{x}}$)

$$\rho_{\alpha}(\bar{x},\ \bar{\bar{x}}) = \rho_{\alpha}(T\bar{x},\ T\bar{\bar{x}}) \leqq \Phi_{j^{-1}(\alpha)}(\rho_{j^{-1}(\alpha)}(\bar{x},\ \bar{\bar{x}})) \leqq \ldots$$

$$\Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\ldots \Phi_{j^{-n}(\alpha)}(Q(\alpha,\ \bar{x},\ \bar{\bar{x}}))\ldots))$$

which completes the proof.

Remark 2. Each iteration of T has a unique fixed point. Indeed, let s be positive integer. Then $T^s\bar{x}=T^s(\lim_{n\to\infty} T^n x)=\lim T^{n+s}x=\bar{x}$ and $\rho_\alpha(\bar{x},\ \bar{x})=\rho_\alpha(T^s\bar{x},\ T^s\bar{x})\leqq\Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\ldots\Phi_{j^{-s}(\alpha)}(\ldots\Phi_{j^{-s-\eta(\alpha)}}(Q(\alpha,\ \bar{x},\ \bar{x}))\ldots)\ldots))$.

3. Main results

Lemma 1. *Condition* (*RD* 4) *implies condition* (*RD* 2).

Proof: Let us suppose that the assertion is not true. Then there exists a neighbourhood $W=W(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta_0)$ such that for every sequence of neighbourhoods $V_k=V_k(\alpha_1,\ \alpha_2,\ldots,\ \alpha_p)(\xi,\ \eta_k)$, $(k=1,\ 2,\ldots)$ for which $V_k\to\{\xi\}$ as $k\to\infty$ one can find elements $x_k\in V_k$, indices $n_k>0$ and $y_k\in\mathbf{T}^{n_k}(x_k)$ such that $y_k\bar{\in}W(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta_0)$. We may assume that

$$x_k\in V_k\subset U(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta),$$

where $U(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta)$ is from condition (RD 4).

Then either: 1) $n_k\to\infty$ as $k\to\infty$ and $y_k\bar{\in}W$ contradicts to (RD 4), or 2) the sequence $\{n_k\}_{k=1}^{\infty}$ is bounded. Then $\{n_k\}_{k=1}^{\infty}$ contains only finite number different elements. Let us choose a subsequence $\{n_{k_s}\}_{s=1}^{\infty}$ such that $n_{k_1}=n_{k_2}=\ldots=n_{k_s}=\ldots$ and let us take n_{k_s} operators from $\mathbf{T}^{n_{k_s}}$ such that

$$y_{k_s}=T_1(T_2(\ldots T_{n_{k_s}}(x_{k_s})\ldots)).$$

Since $\lim_{s\to\infty} x_{k_s}=\xi$ and $T_1,\ T_2,\ldots,\ T_{n_{k_s}}$ are continuous, then $\lim_{s\to\infty} y_{k_s}=\xi$. Consequently $y_{k_s}\bar{\in}W(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta_0)$ contradicts to the continuity of the operators $T_1,\ T_2,\ldots,\ T_{n_{k_s}}$ which completes the proof.

Lemma 2. *Condition* (*RC* 4) *implies* (*RC* 2).

Proof: As in the proof of Lemma 1 there exists a neighbourhood W of ξ such that for every sequence of neighbourhoods $V_k=V_k(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta_k)$ for which $V_k\to\{\xi\}$ one can find $x_k\in V_k$, numbers $t_k\in R_+^1$ and $y_k\in\mathbf{T}_{t_k}(x_k)$ such that $y_k\bar{\in}W$. We may assume that

$$x_k\in V_k\subset U(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta)=U,$$

where U is from (RC 4). Then either 1) $\{t_k\}$ is unbounded. Then there is $t_{k_s}\to\infty$ and $y_{k_s}\bar{\in}W$ contradicts to (RC 4), or 2) $\{t_k\}$ is bounded. In this case we choose a subsequence $\{t_{k_s}\}_{s=1}^{\infty}$ which tends to some $t_0\in R_+^1$. Obviously $\lim_{s\to\infty} x_{k_s}=\xi$. Besides, there is $\Gamma_0\in\Omega$ such that $y_{k_s}=T_{t_{k_s}}^{\Gamma_0}(x_{k_s})$. Therefore for every $\alpha_i\in\{\alpha_1,\ldots,\ \alpha_p\}$ we have

$$\rho_{\alpha_i}(y_{k_s},\ \xi)=\rho_{\alpha i}(T_{t_{k_s}}^{\Gamma_0}(x_{k_s}),\ \xi)=\rho_{\alpha_i}(T_{t_{k_s}}^{\Gamma_0}(x_{k_s}),\ T_{t_{k_s}}^{\Gamma_0}(\xi))\leqq\rho_{\alpha_i}(T_{t_{k_s}}^{\Gamma_0}(x_{k_s}),\ T_{t_0}^{\Gamma_0}(\xi))$$

$$+\rho_{\alpha_i}(T_{t_0}^{\Gamma_0}(\xi),\ T_{t_{k_s}}^{\Gamma_0}(\xi)).$$

The right-hand side of the last inequality tends to zero as $s\to\infty$ because (DSA) is uniformly continuous (UC). Consequently $\lim_{s\to\infty}\rho_{\alpha_i}(y_{k_s}, \xi)=0$ which contradicts to $y_{k_s}\bar{\in} W$.

Lemma 3. *If for every finite collection* $\alpha_1,\ldots,\alpha_p\in A$ *there exists a compact neighbourhood* $K=K(\alpha_1,\ldots,\alpha_p)(\xi,\eta)$ *of* ξ *with non-empty int* K, *then conditions* (*RD* 2) *and* (*RD* 3) *imply* (*RD* 4).

Proof: We shall show that $U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)=\operatorname{int}K(\alpha_1,\ldots,\alpha_p)(\xi,\eta)$ satisfies condition (RD 4). Indeed, let $W=W(\alpha_1,\ldots,\alpha_p)(\xi,\varepsilon)$ be an arbitrary neighbourhood of ξ. From (RD 3) we find a smallest positive integer $N(x)(\alpha_1,\ldots,\alpha_p)$ such that $\mathbf{T}^k(x)\subset W$ for every $k\geqq N(x)(\alpha_1,\ldots,\alpha_p)$. To prove the Lemma we must establish that $\sup\{N(x)(\alpha_1,\ldots,\alpha_p):x\in K\}$ is finite for every finite collection $\alpha_1,\ldots,\alpha_p\in A$. Suppose, by contradiction, that the supremum does not exist for some finite collection $\bar{\alpha}_1,\ldots,\bar{\alpha}_p\in A$. Therefore we can find a sequence $\{x_k\}_{k=1}^{\infty}$, $x_k\in K$ such that $\lim_{k\to\infty}N(x_k)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)=\infty$. But K is compact and we choose a subsequence $\{x_{k_s}\}_{s=1}^{\infty}$ which tends to some $x^*\in K$ and also $\lim_{s\to\infty}N(x_{k_s})(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)=\infty$. In view of (RD 2) one can assert: to the neighbourhood $W(\alpha_1,\ldots,\alpha_p)(\xi,\varepsilon)$ corresponds a neighbourhood $V(\alpha_1,\ldots,\alpha_p)(\xi,\delta)$ such that from $x\in V$ follows $\mathbf{T}^k(x)\subset W$ for every $k\geqq 0$. But (RD 3) guarantees an existence of a smallest number $\tilde{N}=\tilde{N}(x)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)$ such that $\mathbf{T}^{\tilde{N}}(x)\subset V$ for $x\in V$. Obviously $N(x)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)\leqq\tilde{N}(x)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)$ $(N=0)$ and therefore $\lim_{s\to\infty}N(x_{k_s})(\alpha_1,\ldots,\alpha_p)=\infty$ implies $\lim_{s\to\infty}\tilde{N}(x_{k_s})(\alpha_1,\ldots,\alpha_p)=\infty$ for every finite collection $\alpha_1,\ldots,\alpha_p\in A$ and moreover for $\alpha_i=\bar{\alpha}_i(i=1,2,\ldots,3)$. But $\tilde{N}(x^*)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)$ is finite and for every x sufficiently close to x^* the equality $\tilde{N}(x)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)=\tilde{N}(x^*)(\bar{\alpha}_1,\ldots,\bar{\alpha}_p)$ holds. The obtained contradiction proves the Lemma.

Lemma 4. *If for every finite collection* $\alpha_1,\ldots,\alpha_p\in A$ *there exists a compact neighbourhood* $K(\alpha_1,\ldots,\alpha_p)(\xi,\eta)$ *of* ξ *with non-empty int* K, *then conditions* (*RC* 2), (*RC* 3) *imply* (*RC* 4).

The proof is analogous to the one of Lemma 3.

Remark 3. In what follows we do not use condition (RC 2) (resp. (RD 2)). It is easy to see that (RC 2) is an abstract analog of the classical notion — stability of the equilibrium state; (RC 2) and (RC 3) — of an asymptotic stability; (RC 4) — of a uniform asymptotic stability.

Now we are going to formulate the main reuniformization result.

Let $(X, \mathfrak{U})$ be a complete Hausdorff j-bounded space. The family $\mathbf{T}=\{T_t^{\Gamma}:\Gamma\in\Omega,\ t\in R_+^1\}$, $T_t^{\Gamma}:X\to X$ consists of continuous and j-regular operators.

Theorem 2. *The family* $\mathbf{T}$ *is contractifiable if and only if conditions* (*RC* 1), (*RC* 3) *and* (*RC* 4) *are satisfied.*

The closed ball of radius $r>0$ with a center at x in pseudometrics $\rho_{\alpha_1}, \rho_{\alpha_2}, \ldots, \rho_{\alpha_p}$ we shall denote, as usually $B(\alpha_1, \ldots, \alpha_p)(x, r)=\{y\in X : \rho_{\alpha_1}(x, y)\leqq r, \ldots, \rho_{\alpha_p}(x, y)\leqq r\}$.

The following Lemmas precede the proof of Theorem 1.

Lemma 5. *For every $x\in X$, $t_0>0$, $\eta>0$, $\Gamma\in\Omega$ and for every finite collection $\alpha_1, \ldots, \alpha_p\in A$, there exists $\delta>0$ such that*

$$T_t^{\Gamma}(B(\alpha_1, \ldots, \alpha_p)(x, \delta))\subseteqq B(\alpha_1, \ldots, \alpha_p)(T_t^{\Gamma}(x), \eta)$$

for all $t\in[0, t_0]$.

Proof: Suppose, by contradiction, that there exist a sequence $\delta_n\to 0$ $(n\to\infty)$, $t(n)\in[0, t_0]$, $\Gamma_0\in\Omega$, a finite collection $\alpha_1^0, \ldots, \alpha_p^0\in A$ and $x_n\in B(\alpha_1^0, \ldots, \alpha_p^0)(x, \delta_n)$ such that

$$\rho_{\alpha_i^0}(T_{t(n)}^{\Gamma_0}(x), T_{t(n)}^{\Gamma_0}(x_n))>\eta \quad (i=1, 2, \ldots, p).$$

Let us choose a subsequence $t(n_k)$ such that $t(n_k)\to\bar{t}\in[0, t_0]$ as $k\to\infty$. Then $(i=1, 2, \ldots, p)$

$$\rho_{\alpha_i^0}(T_{t(n_k)}^{\Gamma}(x), T_{t(n_k)}^{\Gamma}(x_{n_k}))\leqq\rho_{\alpha_i^0}(T_{t(n_k)}^{\Gamma}(x), T_t^{\Gamma}(x))+\rho_{\alpha_i^0}(T_t^{\Gamma}(x), T_t^{\Gamma}(x_{n_k}))$$
$$+\rho_{\alpha_i^0}(T_t^{\Gamma}(x_{n_k}), T_{t(n_k)}^{\Gamma}(x_{n_k})).$$

Since $t(n_k)\to\bar{t}$ as $k\to\infty$, then condition (UC) implies that the first and third summands on the right tend to zero. The second summand tends to zero in view of the continuity of $T_{\bar{t}}^{\Gamma}$ and $\delta_n\to 0$ as $n\to\infty$. The obtained contradiction proves the Lemma.

Lemma 6. *If for some $t_0>0$ $\mathbf{T}_{t_0}$ is contractifiable (that is, for every $\Gamma\in\Omega$ $T_{t_0}^{\Gamma}$ is contradictifiable), then for every finite collection $\alpha_1, \ldots, \alpha_p\in A$ there is $\eta_0=\eta_0(\alpha_1, \ldots, \alpha_p)$ such that the neighbourhood $U=U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0)$ satisfies the conditions*

$$\mathbf{T}_t(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0))\subseteqq U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0) \tag{3}$$

for all $t\in R_+^1$ and $\mathbf{T}_t(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0))$ tends uniformly to ξ

$$\mathbf{T}_t(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0))\to\{\xi\} \tag{4}$$

as $t\to\infty$.

Proof: Since $T_{t_0}^{\Gamma}$ is contractifiable, we may conclude that for arbitrary finite collection $\alpha_1, \ldots, \alpha_p\in A$ there is a neighbourhood $U_{t_0}=U_{t_0}(\alpha_1, \ldots, \alpha_p)(\xi, \bar{\eta})$ of ξ such that $\{T_{t_0}^{\Gamma, n}(U_{t_0}(\alpha_1, \ldots, \alpha_p)(\xi, \bar{\eta}))\}_{n=0}^{\infty}$ tends uniformly to ξ for every $\Gamma\in\Omega$, where the denotation $T_{t_0}^{\Gamma, n}$ stands for n-th iterate of $T_{t_0}^{\Gamma}$. The existence of $U_{t_0}(\alpha_1, \ldots, \alpha_p)(\xi, \bar{\eta})$ follows from Theorem 2 [5].

As in [5], it can further be assumed that $T^{\Gamma}_{t_0}(U_{t_0}) \subseteqq U_{t_0}$. For each $\bar{t} \in [0, t_0]$ we define the open set

$$U^*_{\bar{t}} = U^*_{\bar{t}}(\alpha_1, \ldots, \alpha_p) = \{x \in X : T^{\Gamma}_{\bar{t}}(x) \in U_{t_0}\}$$

and put

$$U = U(\alpha_1, \ldots, \alpha_p) = \bigcap \{U^*_{\bar{t}} : \bar{t} \in [0, t_0]\}.$$

In order to prove (3), for any given $t \geqq 0$, choose $\bar{t} \in [0, t_0]$ and write $t + \bar{t} = nt_0 + \sigma$ ($\sigma \in [0, t_0]$, n is integer). Obviously

$$U(\alpha_1, \ldots, \alpha_p) \subset U^*_{\sigma}(\alpha_1, \ldots, \alpha_p)$$

and then

$$T^{\Gamma}_{t+\bar{t}}(U(\alpha_1, \ldots, \alpha_p)) = T^{\Gamma}_{nt_0+\sigma}(U(\alpha_1, \ldots, \alpha_p)) = T^{\Gamma,n}_{t_0}(T^{\Gamma}_{\sigma}(U(\alpha_1, \ldots, \alpha_p)))$$

$$\subseteqq T^{\Gamma,n}_{t_0}(U_{t_0}(\alpha_1, \ldots, \alpha_p)) \subseteqq U_{t_0}(\alpha_1, \ldots, \alpha_p)$$

so that $T^{\Gamma}_{t}(U(\alpha_1, \ldots, \alpha_p)) \subseteqq U^*_{t}(\alpha_1, \ldots, \alpha_p)$ and hence (3) holds.

To prove (4) we shall apply Lemma 5. Consider any $\eta > 0$ $\alpha_1, \ldots, \alpha_p \in A$, $\Gamma \in \Omega$ and $x \equiv \xi$, $t \equiv t_0$. By Lemma 5 there exists a $\sigma > 0$ such that

$$T^{\Gamma}_{t}(B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)) \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \eta)$$

for all $\bar{t} \in [0, t_0]$. Then we take N so large that

$$T^{\Gamma,n}_{t_0}(U_{t_0}(\alpha_1, \ldots, \alpha_p)(\xi, \bar{\eta}) \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)$$

for all $n \geqq N$.

Then for $t \geqq Nt_0$ we have $t = nt_0 + \sigma$, $n \geqq N$, $\sigma \in [0, t_0]$. Therefore

$$T^{\Gamma}_{t}(U(\alpha_1, \ldots, \alpha_p)) \subseteqq T^{\Gamma}_{t}(U^*_0) = T^{\Gamma}_{t}(U_{t_0}) = T^{\Gamma}_{\sigma}(T^{\Gamma,n}_{t_0}(U_{t_0}))$$

$$\subseteqq T^{\Gamma}_{t}(B(\alpha_1, \ldots, \alpha_p)(\xi, \delta)) \subseteqq B(\alpha_1, \ldots, \alpha_p)(\xi, \eta).$$

This establishes (4)

It remains to show that $U(\alpha_1, \ldots, \alpha_p)$ is an open neighbourhood of ξ. Let $x \in U(\alpha_1, \ldots, \alpha_p)$ be arbitraty element. One can assert that there exists a $\delta > 0$ such that

$$T^{\Gamma}_{t}(B(\alpha_1, \ldots, \alpha_p)(x, \delta)) \subset U_{t_0}(\alpha_1, \ldots, \alpha_p)(\xi, \bar{\eta})$$

for all $t \in [0, t_0]$, $\Gamma \in \Omega$, $\alpha_1, \ldots, \alpha_p \in A$. This implies

$$B(\alpha_1, \ldots, \alpha_p)(x, \delta) \subset U(\alpha_1, \ldots, \alpha_p).$$

Consequently U is an open set. Indeed, if no such $\delta>0$ existed, then there exists a sequence $\delta_n\to 0$, $t(n)\in[0, t_0]$, $(\alpha_1,\ldots,\alpha_p)\in A$, $\Gamma\in\Omega$ and $x_n\in B(\alpha_1,\ldots,\alpha_p)(x, \delta_n)$ such that $T^{\Gamma}_{t(n)}(x_n)\in X\setminus U_{t_0}$. Choosing a subsequence $t(n_k)$, we may assume $t(n_k)\to t\in[0, t_0]$ as $k\to\infty$. Then the inequality

$$\rho_{\alpha_i}(T^{\Gamma}_{t(n_k)}(x_{n_k}),\ T^{\Gamma}_{t}(x))\leqq\rho_{\alpha_i}(T^{\Gamma}_{t(n_k)}(x_{n_k}),\ T^{\Gamma}_{t}(x_{n_k}))$$

$$+\rho_{\alpha_i}(T^{\Gamma}_{t}(x_{n_k}),\ T^{\Gamma}_{t}(x))\ \ (i=1,\ 2,\ldots,\ p)$$

implies

$$\lim_{k\to\infty} T^{\Gamma}_{t(n_k)}(x_{n_k})=T^{\Gamma}_{t}(x).$$

This is a contradiction to the fact that each element $T^{\Gamma}_{t(n_k)}(x_{n_k})$ lies in the closed set $X\setminus U_{t_0}$. On the other hand, $x\in U$ implies $T^{\Gamma}_{t}(x)\in U_{t_0}$.

Thus Lemma 6 is proved.

4. Proof of theorem 2. Sufficiency

The proof will be made in three steps.

1. We shall construct a family of pseudometrics $\bar{\mathfrak{A}}=\{\bar\rho_{\bar\alpha}(x,\ y):\bar\alpha\in\bar A\}$, j-topologically equivalent to $\mathfrak{A}=\{\rho_\alpha(x, y):\alpha\in A\}$, with respect to which each T^{Γ}_{t} is nonexpanding, that is

$$\bar\rho_{\bar j(\bar\alpha)}(T^{\Gamma}_{t}(x),\ T^{\Gamma}_{t}(y))\leqq\bar\rho_{\bar\alpha}(x,\ y)$$

for every $x,\ y\in X$. Here we choose $\bar A=A$, $\bar j(\bar\alpha)=j(\alpha)$, $\psi:A\to\bar A$, $\psi\equiv Id$.

Let us put

$$\bar\rho_{\alpha}(x,\ y)=\sup\{\rho_{j^k(\alpha)}(T^{\Gamma}_{t}(x),\ T^{\Gamma}_{t}(y)):t\in R^1_+,\ \Gamma\in\Omega,\ k=0,\ 1,\ 2,\ldots\}.$$

In what follows we shall show that $\bar\rho_\alpha(x,\ y)$ is finite for every $x,\ y\in X$ and $\alpha\in A$. Indeed, we observe that the set $C^{\Gamma}_{x}=\{T^{\Gamma}_{\sigma}(x):\sigma\in[0,\tau]\}$ is compact because it is a continuous image of the compact $[0,\tau]$, $\tau>0$. On the other hand, conditions (RC 1), (RC 3) and (RC 4) imply that every mapping T^{Γ}_{t} satisfies: E1), E2), E3) and (H) from the paper [5]. In view of Theorem 2 from [5], the last three conditions are necessary and sufficient to be T^{Γ}_{τ} contractifiable. Let $\mathfrak{N}_\tau=\{d_\lambda(x,\ y):\lambda\in\Lambda\}$ be a family of pseudometrics on $X\times X$ such that T^{Γ}_{τ} is a $(\Phi)_\tau$-contractive (with respect to the family $(\Phi)_\tau$), where $j_\tau:\Lambda\to\Lambda$. Then $M^{\Gamma}_{x,\lambda}=\max\{d_\lambda(z,\xi):z\in C^{\Gamma}_{x}\}$. Writing arbitrary $t>0$ as $t=n\tau+\sigma$, $\sigma\in[0,\tau)$, $n\geqq 0$, we have

$$d_\lambda(T^{\Gamma}_{t}(x),\ \xi)=d_\lambda(T^{\Gamma}_{t}(x),\ T^{\Gamma,n}_{\tau}(\xi))=d_\lambda(T^{\Gamma,n}_{\tau}(T^{\Gamma}_{\tau}(x)),\ T^{\Gamma,n}_{\tau}(\xi))\leqq\ldots$$

$$\leqq\Phi_\lambda(\Phi_{j_\tau^{-1}(\lambda)}(\ldots\Phi_{j_\tau^{-n}(\lambda)}(d_{j_\tau^{-n}(\lambda)}(T^{\Gamma}_{\sigma}(x),\ \xi))\ldots))$$

$$\leqq \Phi_\lambda(\Phi_{j_\tau^{-1}(\lambda)}(\ldots \Phi_{j_\tau^{-n}(\lambda)}(M^\Gamma_{x,j_\tau^{-n}(\lambda)})\ldots))$$

$$\leqq \Phi_\lambda(\Phi_{j_\tau^{-1}(\lambda)}(\ldots \Phi_{j_\tau^{-n}(\lambda)}(Q(\lambda,\ T^\Gamma_\sigma(x),\ \xi))\ldots)).$$

The last inequality follows from the fact that X is j_τ-bounded (cf. [5]) and hence $M^\Gamma_{x,j_\tau^{-n}(\lambda)} \leqq Q$ $(n=0,1,2,\ldots)$. Since $n\to\infty$ as $t\to\infty$, it follows that $\lim\limits_{t\to\infty} d_\lambda(T^\Gamma_t(x), \xi)=0$ for $\lambda\in\Lambda$. For arbitrary $x,\ y\in X$ we have

$$\rho_\alpha(T^\Gamma_t(x),\ T^\Gamma_t(y)) \leqq \rho_\alpha(T^\Gamma_t(x),\ \xi)+\rho_\alpha(T^\Gamma_t(y),\ \xi).$$

The right-hand side tends to zero as $t\to\infty$ for every $\alpha\in A$. Thus there exists a $t_0>0$ such that

$$\rho_\alpha(T^\Gamma_t(x),\ T^\Gamma_t(y))<1$$

for all $t>t_0$. Since $\{(T^\Gamma_t(x),\ T^\Gamma_t(y)) : t\in[0,\ t_0]\}$ is a compact subset of $X\times X$, then $\rho_\alpha(T^\Gamma_t(x),\ T^\Gamma_t(y))$ is finite for every fixed $\alpha\in A$. The j-boundedness of X assures $\bar\rho_\alpha(x,\ y)<\infty$.

Obviously $\rho_\alpha(x,y)\geqq 0$ and $\rho_\alpha(x,y)=\rho_\alpha(y,x)$ imply $\bar\rho_\alpha(x,y)\geqq 0$ and $\bar\rho_\alpha(x,y)=\bar\rho_\alpha(y,\ x)$. We shall verify the triangle inequality for $\bar\rho_\alpha(x,\ y)$. Indeed, for any $t\in R^1_+$ and $\Gamma\in\Omega$, we have

$$\rho_{j^k_{(\alpha)}}(T^\Gamma_t(x),\ T^\Gamma_t(y)) \leqq \rho_{j^k_{(\alpha)}}(T^\Gamma_t(x),\ T^\Gamma_t(z))+\rho_{j^k_{(\alpha)}}(T^\Gamma_t(z),\ T^\Gamma_t(y))$$

$$\leqq \bar\rho_\alpha(x,z)+\bar\rho_\alpha(z,\ y)\ \ (k=0,\ 1,\ 2,\ldots)$$

which implies the desired result.

The following chain of inequalities shows that every $T\Gamma$ is nonexpansive with respect to the family $\mathfrak{A}=\{\bar\rho_\alpha(x,\ y) : \alpha\in A\}$:

$$\bar\rho_{j(\alpha)}(T^\Gamma_t(x),\ T^\Gamma_t(y))=\sup\{\rho_{j^k_{(\alpha)}}(T^{\Gamma_1}_s(T^\Gamma_t(x)),$$

$$T^{\Gamma_1}_s(T^\Gamma_t(y))) : s\in R^1_+,\ \Gamma_1\in\Omega,\ k=1,\ 2,\ldots\}$$

$$\leqq \sup\{\rho_{j^k_{(\alpha)}}(T^{\Gamma_2}_{s+t}(x),\ T^{\Gamma_2}_{s+t}(y)) : s\in R^1_+,\ \Gamma_2\in\Omega,\ k=1,\ 2,\ldots\}$$

$$\leqq \sup\{\rho_{j^k_{(\alpha)}}(T^{\Gamma_2}_s(x),\ T^{\Gamma_2}_s(y)) : s\in R^1_+,\ \Gamma_2\in\Omega,\ k=0,\ 1,\ 2,\ldots\}=\bar\rho_\alpha(x,y).$$

It only remains to show that the families $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ are j-topologically equivalent. Indeed, let $\{x_\gamma\}_{\gamma\in S}$ be a net, where S is a directed set. Since $\rho_\alpha(x,\ y)\leqq\bar\rho_\alpha(x,\ y)$ we have

$$\lim_{\gamma\in S}\bar\rho_\alpha(x,\ x_\gamma)=0 \text{ implies } \lim_{\gamma\in S}\rho_\alpha(x,\ x_\gamma)=0$$

so it should be proved the converse implication.

Let $x\in X$, $\eta>0$ and $\alpha_1,\ldots,\alpha_p\in A$ be arbitrary chosen and let $U=U(\alpha_1,\ldots,\alpha_p)(\xi,\eta_0)$ be a neighbourhood of ξ chosen as in Lemma 6. Without loss of generality we may assume $\eta>0$ small enough that $B(\alpha_1,\ldots\alpha_p)(\xi,\eta)\subseteqq U$. From $\lim_{t\to\infty} T_t^\Gamma(x)=\xi$ follows that there is a $t_0>0$ such that $\rho_{\alpha_i}(\xi, T_t^\Gamma(x))<\frac{\eta}{2}$ for all $t\geqq t_0$ and $\Gamma\in\Omega$. By Lemma 5 there is a $\delta'>0$ such that

$$T_s^\Gamma(B(\alpha_1,\ldots,\alpha_p)(x,\delta'))\subseteqq B(\alpha_1,\ldots,\alpha_p)(T_s^\Gamma(x),\frac{\eta}{2}) \tag{5}$$

for all $s\in[0,t_0]$. On the other hand, since $T_t^\Gamma(U)\to\{\xi\}$ as $t\to\infty$, there is a $t_1>0$ such that

$$T_t^\Gamma(U)\subseteqq B(\alpha_1,\ldots,\alpha_p)(\xi,\frac{\eta}{2})$$

for all $t>t_1$ and by Lemma 5 there exists a $\delta''>0$ such that

$$T_s^\Gamma(B(\alpha_1,\ldots,\alpha_p)(x,\delta''))\subseteqq B(\alpha_1,\ldots,\alpha_p)(T_s^\Gamma(x),\frac{\eta}{2}) \tag{6}$$

for all $s\in[0,t_0+t_1]$. Let $\delta=\min\{\delta',\delta''\}$. We shall show that

$$T_t^\Gamma(B(\alpha_1,\ldots,\alpha_p)(x,\delta))\subseteqq B(\alpha_1,\ldots,\alpha_p)(T_t^\Gamma(x),\eta) \tag{7}$$

for all $t\in R_+^1$.

From (6) it follows that the inclusion (7) holds for $t\in[0,t_0+t_1]$. Suppose then that $t>t_0+t_1$. If we choose in (5) $s=t_0$ and having in mind

$$\rho_{\alpha_i}(\xi,T_{t_0}^\Gamma(x))<\frac{\eta}{2}(i=1,2,\ldots,p)\text{ and }B(\alpha_1,\ldots,\alpha_p)(\xi,\eta)\subseteqq U$$

we have

$$T_{t_0}^\Gamma(B(\alpha_1,\ldots,\alpha_p)(x,\delta))\subseteqq B(\alpha_1,\ldots,\alpha_p)(\xi,\eta)\subseteqq U(\alpha_1,\ldots,\alpha_p)(\xi,\eta_0).$$

Since $t-t_0>t_1$ the inclusion

$$T_t^\Gamma(U(\alpha_1,\ldots,\alpha_p)(\xi,\eta_0))\subseteqq B(\alpha_1,\ldots,\alpha_p)(\xi,\frac{\eta}{2})$$

yields

$$T_t^\Gamma(B(\alpha_1,\ldots,\alpha_p)(x,\delta))=T_{t-t_0}^{\Gamma_1}(T_{t_0}^{\Gamma_2}(B(\alpha_1,\ldots,\alpha_p)(x,\delta)))$$

$$\subseteq T_t^{\Gamma^{-1}} t_0(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta_0)) \subseteq B(\alpha_1, \ldots, \alpha_p)(\xi, \frac{\eta}{2}),$$

which implies (7). The inclusion (7) together with the j-boundedness of X gives: if $\rho_{\alpha_i}(x, x_\gamma) < \delta$ then $\rho_{\alpha_i}(T_t^\Gamma(x_\gamma), T_t^\Gamma(x)) < \eta$ which proves the implication $\lim_{\gamma \in S} \rho_\alpha(x_\gamma, x) = 0 \Rightarrow \lim_{\gamma \in S} \bar{\rho}_\alpha(x_\gamma, x) = 0$. Therefore the family of pseudometrics $\mathfrak{A} = \{\bar{\rho}_\alpha(x, y) : \alpha \in A\}$ is topologically equivalent to $\mathfrak{A}$ and in view of the completeness of X is equivalent to $\mathfrak{A}$.

Now we are going to show that $\bar{\mathfrak{A}}$ is j-topologically equivalent to $\mathfrak{A}$. Indeed,

$$\bar{\rho} j^{-n}{}_{(\alpha)}(x, y) = \sup\{\rho j^{-n+k}{}_{(\alpha)}(T_t^\Gamma(x), T_t^\Gamma(y)) : t \in R_+^1, \Gamma \in \Omega, k = 0, 1, 2 \ldots\} \leqq \bar{Q}(\alpha, x, y)$$

for some constant $\bar{Q}(\alpha, x, y) > 0$ because the sequence

$$\{\rho j^{-n}{}_{(\alpha)}(T_t^\Gamma(x), T_t^\Gamma(y)), \ldots, \rho j^{-1}{}_{(\alpha)}(T_t^\Gamma(x), T_t^\Gamma(y))\}$$

is bounded and the $\sup\{\rho j^k{}_{(\alpha)}(T_t^\Gamma(x), T_t^\Gamma(y)) : t \in R_+^1, \Gamma \in \Omega, k = 0, 1, \ldots\}$ is finite. Therefore X is j-bounded.

In order to complete the first step it suffices to show that T_t^Γ is j-regular for every $x \in X$. Suppose that $\{T_t^{\Gamma, n}(x)\}_{n=0}^\infty$ is not $\bar{\rho}_\alpha$-Cauchy sequence; we must prove that it is not $\bar{\rho}_{j(\alpha)}$-Cauchy sequence. Let us fix $\varepsilon > 0$, $t \in R_+^1$ and $\alpha \in A$ such that

$$\bar{\rho}_\alpha(T_t^{\Gamma, m}(x), T_t^{\Gamma, m+s}(x)) \geqq \varepsilon.$$

From the definition of $\bar{\rho}_\alpha$ we find $k_0 > 0$, $\Gamma^0 \in \Omega$, $t_0 \in R_+^1$ such that

$$\rho j^{k_0}{}_{(\alpha)}(T_{t_0}^{\Gamma_0, m}(x), T_{t_0}^{\Gamma_0, m+s}(x)) \geqq \varepsilon.$$

But $T_{t_0}^{\Gamma_0}$ is j-regular and then there is $\delta > 0$ such that

$$\bar{\rho} j^{k_0+1}{}_{(\alpha)}(T_{t_0}^{\Gamma_0, m}(x), T_{t_0}^{\Gamma_0, m+s}(x)) \geqq \delta$$

which implies

$$\bar{\rho}_{j(\alpha)}(T_t^{\Gamma, m}(x), T_t^{\Gamma, m+s}(x)) \geqq \delta.$$

2. Second step. Let us note that the properties (3) and (4) implies

$$T_t^\Gamma(\mathbf{cl} U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)) \subseteq \mathbf{cl} U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)$$

and

$$\{\mathbf{cl} T_t(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta))\} \to \{\xi\} \text{ as } t \to \infty.$$

Define the function

$$\nu(\alpha_1,\ldots,\alpha_p)(x)=\begin{cases}-\inf\{t\in R^1_+ : \mathbf{T}_t(x)\subset \mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta),\\ \qquad x\in X\setminus\mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta),\\ \sup\{t\in R^1_+ : x\in\mathbf{cl}\mathbf{T}_t(U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)),\\ \qquad x\in\mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)\setminus\{\xi\},\\ \infty,\ x=\xi \text{ or } x\neq\xi \text{ but}\\ \qquad \rho_{\alpha_{i_0}}(x,\xi)=0 \text{ for some } i_0\in\{1,2,\ldots,p\}.\end{cases}$$

If $\rho_{\alpha i}(x,\xi)>0$ $(i=1,2,\ldots,p)$, then the number $\nu=\nu(\alpha_1,\ldots\alpha_p)(x)$ is finite.

The supremum (and infimum) in the above definition is actually attained. Indeed, the uniform continuity of (DSA) implies

$$\lim \rho_{\alpha_i}(T_t^{\Gamma}(x),\ T^{\Gamma}_{\nu}(\alpha_1,\ldots,\alpha_p)(x))=0 \quad (i=1,2,\ldots,p)$$

$$t\to\nu(\alpha_1,\ldots,\alpha_p)(x)$$

$$t<\nu(\alpha_1,\ldots,\alpha_p)(x)$$

for every $\Gamma\in\Omega$.

When $x\in\mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)$ we have $0\leqq\nu(\alpha_1,\ldots,\alpha_p)(x)\leqq\infty$.

When $x\in X\setminus\mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)$, then $-\infty<\nu(\alpha_1,\ldots,\alpha_p)(x)<0$. Besides, $\nu(\alpha_1,\ldots,\alpha_p)(x)\neq 0$ because η is strictly positive.

It is easy to establish that $(x\neq\xi)$

1) when $\nu(j(\alpha_1),\ldots,j(\alpha_p))(x)<0$ the condition (cf. (RC 3)) if

$$\mathbf{T}_t(x)\bigcap\mathbf{cl}U(j(\alpha_1),\ldots,j(\alpha_p))(\xi,\eta)=\emptyset\Rightarrow$$

$$\mathbf{T}_t(x)\bigcap\mathbf{cl}U(\alpha_1,\ldots,\alpha_p)(\xi,\eta)=\emptyset$$

implies $\nu(j(\alpha_1),\ldots,j(\alpha_p))(x)\geqq\nu(\alpha_1,\ldots,\alpha_p)(x)$, and

2) when $\nu(j(\alpha_1),\ldots,j(\alpha_p))(x)>0$ the condition (cf. (RC 4)) if

$$x\in\mathbf{cl}\mathbf{T}_t(U(\alpha_1,\ldots,\alpha_p)(\xi,\eta))\Rightarrow$$

$$x\in\mathbf{cl}\mathbf{T}_t(U(j(\alpha_1),\ldots,j(\alpha_p)))(\xi,\eta)$$

(8) implies $\nu(j(\alpha_1),\ldots,j(\alpha_p))(x)\geqq\nu(\alpha_1,\ldots,\alpha_p)(x)$.

When $x=\xi$ the inequality is $\infty\geqq\infty$. The case $\nu(j(\alpha_1),\ldots,j(\alpha_p))(x)<\infty$ and $\nu(\alpha_1,\ldots,\alpha_p)(x)=\infty$ is impossible since the assumption $\rho_{j(\alpha)}(x,y)\leqq\rho_{\alpha}(x,y)$ gives

the implication: if $\rho_\alpha(x, y)=0 \Rightarrow \rho_{j(\alpha)}(x, y)=0$. Let us note that this assumption is satisfied in the applications (cf. [5]).

In what follows, we shall show that

$$\nu(\alpha_1, \ldots, \alpha_p)(T_t^\Gamma(x)) \geqq \nu(\alpha_1, \ldots, \alpha_p)(x)+t.$$

Let $0 \leqq \nu(\alpha_1, \ldots, \alpha_p)(x) \leqq \infty$, that is,

$$x \in \mathbf{cl}\mathbf{T}_{\nu(\alpha_1, \ldots, \alpha_p)(x)}(x)(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)).$$

Then

$$T_t^\Gamma(x) \in \mathbf{T}_t(\mathbf{cl}\mathbf{T}_{\nu(\alpha_1, \ldots, \alpha_p)(x)}(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)))$$

and thus

$$T_t^\Gamma(x) \in \mathbf{cl}\mathbf{T}_t(\mathbf{T}_{\nu(\alpha_1, \ldots, \alpha_p)(x)}(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)))$$

$$= \mathbf{cl}\mathbf{T}_{t+\nu(\alpha_1, \ldots, \alpha_p)(x)}(U(\alpha_1, \ldots, \alpha_p)(\xi, \eta))$$

from which the desired inequality follows.

When $-\infty < \nu(\alpha_1, \ldots, \alpha_p)(x) < 0$, we have

$$\nu(\alpha_1, \ldots, \alpha_p)(T_t^{\Gamma_1}(x)) = \nu(\alpha_1, \ldots, \alpha_p)(T_t^{\Gamma_2}(x))$$

for every $\Gamma_1, \Gamma_2 \in \Omega$, so that we can write

$$\nu(\alpha_1, \ldots, \alpha_p)(\mathbf{T}_t(x)) = \nu(\alpha_1, \ldots, \alpha_p)(T_t^\Gamma(x)), \quad \Gamma \in \Omega.$$

Then

$$\nu(\alpha_1, \ldots, \alpha_p)(T_t^\Gamma(x)) = \nu(\alpha_1, \ldots, \alpha_p)(\mathbf{T}_t(x)) = -\inf\{\tau : \mathbf{T}_\tau(\mathbf{T}_t(x))$$

$$\subset \mathbf{cl}\ U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)\} = -\inf\{\tau : \mathbf{T}_{\tau+t}(x) \subset \mathbf{cl}\ U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)\}$$

$$= -\inf\{\bar{\tau}-t : \mathbf{T}_{\bar{\tau}}(x) \subset \mathbf{cl}\ U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)\}$$

$$\geqq -\inf\{\bar{\tau} : \mathbf{T}_{\bar{\tau}}(x) \subset \mathbf{cl}\ U(\alpha_1, \ldots, \alpha_p)(\xi, \eta)\}+t.$$

Therefore

$$\nu(\alpha_1, \ldots, \alpha_p)(T_t^\Gamma(x)) \geqq \nu(\alpha_1, \ldots, \alpha_p)(x)+t.$$

Putting $\nu(\alpha_1, \ldots, \alpha_p)(x, y) = \min\{\nu(\alpha_1, \ldots, \alpha_p)(x), \nu(\alpha_1, \ldots, \alpha_p)(y)\}$ we can easily obtain the inequality

$$\nu(\alpha_1, \ldots, \alpha_p)(T_t^\Gamma(x), T_t^\Gamma(y)) \geqq \nu(\alpha_1, \ldots, \alpha_p)(x, y)+t.$$

Combining the last inequality with (8), we obtain

$$\nu(j(\alpha_1),\ldots,\ j(\alpha_p))(T_t^{\Gamma}(x),\ T_t^{\Gamma}(y)) \geqq \nu(\alpha_1,\ldots,\ \alpha_p)(x,\ y)+t.$$

Further on, we pass to the family $\mathfrak{U}^* = \{\bar{\rho}_{\alpha^*}(x, y) : {}_{\alpha^*} \in A^*\}$. It is well-known that the set A^* consisting of all finite subsets of A has the same cardinality as A (cf. [9], p. 26). Let $\Psi : A \to A^*$ be some bijection $\psi(\alpha) = \alpha^*$, where $\alpha^* = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. On the other hand, the map j induces the map $j^* : A^* \to A^*$ by the formula $j^*(\alpha^*) = j^*(\psi(\alpha)) = \psi(j(\alpha)) = (j(\alpha_1), \ldots, j(\alpha_p))$.

Then we define

$$\bar{\rho}_{\alpha^*}(x,\ y) = \max\{\bar{\rho}_{\alpha_i}(x,\ y) : i = 1,\ 2, \ldots,\ p\},$$

$$k_{\alpha^*} = \max\{k_{\alpha_1},\ k_{\alpha_2}, \ldots,\ k_{\alpha_p}\},$$

$$\bar{\rho}_{\alpha^*}(x,\ y) = k_{\alpha^*}^{\nu(\alpha^*)(x,y)}\bar{\rho}_{\alpha^*}(x,\ y),$$

(where $\nu(\alpha^*)\ (x,\ y) \equiv \nu(\alpha_1, \ldots,\ \alpha_p)(x,\ y)$)
for $-\infty < \nu(\alpha_1, \ldots,\ \alpha_p)(x,\ y) < \infty$. When $\nu(\alpha_1, \ldots,\ \alpha_p)(x,\ y) = \infty$ we put

$$\bar{\rho}_{\alpha^*}(x,\ y) = \lim_{\tau \to \infty} k_{\alpha^*}^{\tau}\bar{\rho}_{\alpha^*}(x,\ y) = 0\ (0 < k_{\alpha^*}^{\tau} < 1).$$

Obviously $\bar{\rho}_{\alpha^*}(\xi,\ \xi) = 0$ for every $\alpha^* \in A^*$ and $\bar{\rho}_{\alpha^*}(x, y)$ is positive definite and symmetric. We shall show the contraction property:

$$\bar{\rho}_{j^*(\alpha^*)}(T_t^{\Gamma}(x),\ T_t^{\Gamma}(y))$$

$$= k_{j^*(\alpha^*)}^{\nu(j^*(\alpha^*))\ (T_t^{\Gamma}(x),\ (T_t^{\Gamma}(y)))}\bar{\rho}_{j^*(\alpha^*)}\ (T_t^{\Gamma}(x),\ T_t^{\Gamma}(y))$$

$$\leqq k_{j^*(\alpha^*)}^{\nu(\alpha^*)(T_t^{\Gamma}(x),T_t^{\Gamma}(y))}\bar{\rho}_{\alpha^*}(x,\ y)$$

$$\leqq k_{j^*(\alpha^*)}^{\nu(\alpha^*)(x,y)+t}\bar{\rho}_{\alpha^*}(x,\ y) \leqq k_{\alpha^*}^{\nu(\alpha^*)(x,y)+t}\bar{\rho}_{\alpha^*}(x,\ y)$$

$$\leqq k_{\alpha^*}^{t}\, k_{\alpha^*}^{\nu(\alpha^*)(x,y)}\bar{\rho}_{\alpha^*}(x,\ y) = k_{\alpha^*}^{t}\bar{\rho}_{\alpha^*}(x,\ y).$$

(Let us recall that $k_{j(\alpha)} \leqq k_{\alpha}$ for all $\alpha \in A$.)

If $\bar{\rho}_{\alpha^*}(x, y)$ does not satisfy the triangle inequality, then we can proceed as in [3]. Denote by Σ_{xy} the set of all chains $\sigma_{xy} = \{x_0 = x, x_1, x_2, \ldots, x_m = y\}$ from x to y with lengths

$$L_{\alpha^*}(\sigma_{xy}) = \sum_{i=1}^{m}\bar{\rho}_{\alpha^*}(x_{i-1},\ x_i)$$

and put

$$\rho_{\alpha^*}^{*}(x,\ y) = \inf\{L_{\alpha^*}(\sigma_{xy}) : \sigma_{xy} \in \Sigma_{xy}\},\ \alpha^* \in A^*.$$

It is easy to verify that $\rho^*_{\alpha^*}(x, y)$ is symmetric and $\rho^*_{\alpha^*}(x, x)=0$, $\alpha^* \in A^*$. The triangle inequality holds since following a σ_{xy} with a σ_{yz} yields a σ_{xz}.

Let us underline that for $x \neq y$ there exists $\alpha_0 \in A$ such that $\rho_{\alpha_0}(x, y)>0$. This implies $\bar{\rho}_{\alpha^*}(x, y)>0$ for $\alpha^* \in A^*$, where $\alpha^* \equiv (\alpha_1, \ldots, \alpha_p)$, $\alpha_0 \in \{\alpha_1, \ldots, \alpha_p\}$.

Now we shall verify the contraction property and positive definiteness of $\rho^*_{\alpha^*}(x, y)$. Indeed,

$$\rho^*_{j^*(\alpha^*)}(T^\Gamma_t(x), T^\Gamma_t(y)) = \inf_{\substack{z_0 = T^\Gamma(x) \\ z_m = T^\Gamma(y)}} \sum_{i=1}^{m} \bar{\rho}_{j^*(\alpha^*)}(z_{i-1}, z_i)$$

$$\leqq \inf_{\substack{x_0 = x \\ x_m = y}} \sum_{i=1}^{m} \bar{\rho}_{j^*(\alpha^*)}(T^\Gamma_t(x_{i-1}), T^\Gamma_t(x_i))$$

$$\leqq \inf_{\substack{x_0 = x \\ x_m = y}} \sum_{i=1}^{m} k^t_{\alpha^*} \bar{\rho}_{\alpha^*}(x_{i-1}, x_i) \leqq k^t_{\alpha^*} \inf_{\substack{x_0 = x \\ x_m = y}} \sum_{i=1}^{m} \bar{\rho}_{\alpha^*}(x_{i-1}, x_i)$$

$$\leqq k^t_{\alpha^*} \rho^*_{\alpha^*}(x, y).$$

Let us show that for every $x \neq y$ there exists $\alpha^*_0 \in A^*$ such that $\rho^*_{\alpha^*_0}(x, y)>0$.

For every $\nu \in (-\infty, \infty)$ and every finite collection $\alpha_1, \ldots, \alpha_p \in A$ we define the closed sets

$$K_\nu(\alpha^*) \equiv K_\nu(\alpha_1, \ldots, \alpha_p) = \{x \in X : \nu(\alpha^*) \equiv \nu(\alpha_1, \ldots, \alpha_p)(x) \geqq \nu\}.$$

Obviously $\nu_1 \geqq \nu_2$ implies $K_{\nu_1}(\alpha_1, \ldots, \alpha_p) \subset K_{\nu_2}(\alpha_1, \ldots, \alpha_p)$ and $K_\nu \to \{\xi\}$ as $\nu \to \infty$ in $\rho_{\alpha_i}(x, y)$ $(i=1, \ldots, p)$.

Consider any $x \in X$ different from $y \in X$ in the pseudometrics $\rho_{\alpha_i}(i=1, \ldots, p)$, that is $\rho_{\alpha_i}(x, y)>0$ and suppose that $\rho_{\alpha_i}(y, \xi)>0$.

We may assume without loss of generality that $\nu(\alpha_1, \alpha_2, \ldots, \alpha_p)(x) \leqq \nu(\alpha_1, \alpha_2, \ldots, \alpha_p)(y)$.

Let us choose arbitrary $s>0$ so that $s+\nu(\alpha_1, \ldots, \alpha_p)(y) \equiv s+\nu(\alpha^*)(y)>0$. Then $y \in X \setminus K_{\nu(\alpha^*)(y)+s}(\alpha_1, \ldots, \alpha_p)$. For any chain σ_{xy} two cases are possible: 1) all its points belong to $X \setminus K_{\nu(\alpha^*)(y)+s}(\alpha_1, \ldots, \alpha_p)$ and then because of $\nu(\alpha^*)(x_{i-1}, x_i) \leqq \nu(\alpha^*)(y)+s$ $(i=1, \ldots, m)$ we obtain

$$L_{\alpha^*}(\sigma_{xy}) = \sum_{i=1}^{m} k^{\nu(\alpha^*)(x_{i-1}, x_i)}_{\alpha^*} \bar{\rho}_{\alpha^*}(x_{i-1}, x_i)$$

$$\geqq \sum k^{\nu(\alpha^*)(y)+s}_{\alpha^*} \bar{\rho}_{\alpha^*}(x_{i-1}, x_i) \geqq k^{\nu(\alpha^*)(y)+s}_{\alpha^*} \bar{\rho}_{\alpha^*}(x, y)>0$$

(the triangle inequality for $\bar{\rho}_{\alpha^*}$ is valid); 2) there are elements of σ_{xy} belonging to $K_{\nu(\alpha^*)(y)+s}(\alpha_1, \ldots, \alpha_p)$ and then

$$L_{\alpha^*}(\sigma_{xy}) \geqq k_{\alpha^*}^{\nu(\alpha^*)(y)+s} \bar{\rho}_{\alpha^*}(y, K_{\nu(\alpha^*)(y)+s}(\alpha^*)) > 0.$$

Here $\bar{\rho}_{\alpha^*}(y, K_\nu)$ is the distance between y and the set K_ν. It follows that

$$\begin{aligned} \rho^*_{\alpha^*}(x, y) \geqq \min \{ & k_{\alpha^*}^{\nu(\alpha^*)(y)+s} \bar{\rho}_{\alpha^*}(x, y), \\ & k_{\alpha^*}^{\nu(\alpha^*)(y)+s} \bar{\rho}_{\alpha^*}(y, K_{\nu(\alpha^*)(y)+s}(\alpha^*))\}. \end{aligned} \tag{9}$$

Let $\rho_{\alpha_i}(y, \xi) = 0$ $(i = 1, \ldots, p)$. Then applying analogous reasoning we have

$$\rho^*_{\alpha^*}(x, y) = \rho^*_{\alpha^*}(x, \xi) \geqq k_{\alpha^*}^{\nu(\alpha^*)(x)+s} \bar{\rho}_{\alpha^*}(x, K_{\nu(\alpha^*)(x)+s}(\alpha^*)). \tag{10}$$

Consequently there exists $\alpha^* \in A^*$ such that $\rho^*_{\alpha^*}(x, y) > 0$.

3. In the third step we shall show that $\mathfrak{U}^* = \{\rho^*_{\alpha^*}(x, y) : \alpha^* \in A^*\}$ is j-topologically equivalent to $\mathfrak{U}$.

Let us define

$$\mu(\alpha^*)(x) \equiv \mu(\alpha_1, \ldots, \alpha_p)(x) = \inf \{\mu \geqq 0 : \mathbf{T}_\mu(x) \subset U(\alpha^*)(\xi, \eta)\}.$$

Obviously, $\nu(\alpha^*)(x) \geqq -\mu(\alpha^*)(x)$. Putting

$$D_{\mu(\alpha^*)(x)} = X \setminus \mathbf{T}^{-1}_{\mu(\alpha^*)(x)}(U(\alpha^*)(\xi, \eta))$$

we shall establish that

$$\bar{\rho}_{\alpha^*}(x, D_{\mu(\alpha^*)(x)+s}) > 0$$

for all $s > 0$. Indeed, if we suppose that $\bar{\rho}_{\alpha_i}(x, D_{\mu(\alpha^*)(x)+s}) = 0$ for all $i = 1, \ldots, p$ and $s > 0$, then we obtain

$$\begin{aligned} x \bar{\in} \mathbf{T}^{-1}_{\mu(\alpha^*)(x)+s}(U(\alpha^*)(\xi, \eta)) &\Rightarrow \mathbf{T}_{\mu(\alpha^*)(x)+s}(x) \bigcap U(\alpha^*)(\xi, \eta) \\ &= \emptyset \Rightarrow T^\Gamma_{\mu(\alpha^*)(x)+s}(x) \in X \setminus U(\alpha^*)(\xi, \eta) \end{aligned}$$

for every $\Gamma \in \Omega$. The last set is closed. In view of the uniform continuity of $\{T^\Gamma_t\}$ we have

$$T^\Gamma_{\mu(\alpha^*)(x)} \in X \setminus U(\alpha^*)(\xi, \eta)$$

implying

$$T^\Gamma_{\mu(\alpha^*)(x)}(x) \bar{\in} U(\alpha^*)(\xi, \eta)$$

which contradicts to the definition of $\mu(\alpha^*)(x)$.

For any $x \in X$ we shall show that

$$\lim_{\gamma \in S} \bar{\rho}_{\alpha}(x_\gamma, x) = 0 \Rightarrow \lim_{\gamma \in S} \rho^*_{\alpha^*}(x_\gamma, x) = 0, \tag{11}$$

where γ runs over the elements of the directed set S.

Suppose that $x \neq \xi$ and $s > 0$. Consider $y \in X$ for which

$$\bar{\rho}_{\alpha_i}(x, y) < \bar{\rho}_{\alpha_i}(x, D_{\mu(\alpha^*)(x)+s}) \quad (i = 1, 2, \ldots, p);\ \alpha^* \equiv (\alpha_1, \ldots, \alpha_p). \tag{12}$$

Then $y \in X \setminus D_{\mu(\alpha^*)(x)+s}$, that is, $T^{\Gamma}_{\mu(\alpha^*)(x)+s}(y) \in U(\alpha^*)(\xi, \eta)$ and hence $\mu(\alpha^*)(x) + s \geqq \mu(\alpha^*)(y)$. But $\nu(\alpha^*)(x) \geqq -\mu(\alpha^*)(x)$ and consequently $\nu(\alpha^*)(y) \geqq -\mu(\alpha^*)(x) - s$. Besides $\nu(\alpha^*)(x) \geqq -\mu(\alpha^*)(x) - s$. Then $\nu(\alpha^*)(x, y) \geqq -\mu(\alpha^*)(x) - s$, so that (11) implies the second inequality in the chain

$$\rho^*_{\alpha^*}(x, y) \leqq \bar{\rho}_{\alpha^*}(x, y) = k_{\alpha^*}^{\nu(\alpha^*)(x,y)} \bar{\rho}_{\alpha^*}(x, y) \leqq k_{\alpha^*}^{-\mu(\alpha^*)(x)-s} \bar{\rho}_{\alpha^*}(x, y) < \bar{\rho}_{\alpha^*}(x, y)$$

$$= \max\{\bar{\rho}_{\alpha_i}(x, y) : i = 1, 2, \ldots, p\} = \bar{\rho}_{\alpha_{i_0}}(x, y) \quad (i_0 \in \{1, 2, \ldots, p\}).$$

So we have obtained the implication (11) for $x \neq \xi$.

Let $x = \xi$. If $\bar{\rho}_{\alpha_i}(\xi, y) < \bar{\rho}_{\alpha_i}(\xi, D_0)$, then $\nu(\alpha_1, \ldots, \alpha_p)(y) \geqq 0$, so that $\nu(\alpha_1, \ldots, \alpha_p)(\xi, y) \geqq 0$ and

$$\rho^*_{\alpha^*}(\xi, y) \leqq \bar{\rho}_{\alpha^*}(\xi, y) \leqq \bar{\rho}_{\alpha^*}(\xi, y) = \bar{\rho}_{\alpha_{i_0}}(\xi, y),\ i_0 \in \{1, 2, \ldots, p\}.$$

Finally it remains to prove that

$$\lim_{\gamma \in S} \rho^*_{\alpha^*}(x_\gamma, x) = 0 \Rightarrow \lim_{\gamma \in S} \bar{\rho}_{\alpha}(x_\gamma, x) = 0. \tag{13}$$

First assume $x \neq \xi$ that is, there exist $\alpha_1, \ldots, \alpha_p \in A$ such that $\rho_{\alpha_i}(x, \xi) > 0$. Let us choose arbitrary $s > 0$. Since $K_\nu \to \{\xi\}$ as $\nu \to \infty$, there exists $l(\alpha^*)(x) > \max\{0, \nu(\alpha^*)(x) + s\}$ such that $z \in K_{l(\alpha^*)(x)}$ implies $\rho^*_{\alpha^*}(\xi, z) < \frac{1}{2}\rho^*_{\alpha^*}(x, \xi)$. Then

$$\rho^*_{\alpha^*}(x, K_{l(\alpha^*)(x)}) \geqq \frac{1}{2}\rho^*_{\alpha^*}(x, \xi). \tag{14}$$

Having in mind (14) and the definition of $\rho^*_{\alpha^*}$ (forming as an infimum), we consider only the chains disjoint from $K_{l(\alpha^*)(x)}$ so that

$$\rho^*_{\alpha^*}(x, y) \geqq k_{\alpha^*}^{l(\alpha^*)(x)} \bar{\rho}_{\alpha^*}(x, y). \tag{15}$$

Consider any $y \in X$ for which

$$\rho^*_{\alpha^*}(x, y) < \min\{\frac{1}{2}\rho^*_{\alpha^*}(x, \xi),\ k_{\alpha^*}^{l(\alpha^*)(x)} \bar{\rho}_{\alpha^*}(x, K_{\nu(\alpha^*)(x)+s})\}. \tag{16}$$

Then (15) and (16) imply

$$\bar{\rho}_{\alpha^*}(x,\ y) < \bar{\rho}_{\alpha^*}(x,\ K_{\nu(\alpha^*)(x)+s}). \tag{17}$$

We shall show that

$$\bar{\rho}_{\alpha^*}(x,\ y) \leqq k_{\alpha^*}^{-\nu(\alpha^*)(x)-s}\, \rho^*_{\alpha^*}(x,\ y). \tag{18}$$

If $\nu(\alpha^*)(y) \leqq \nu(\alpha^*)(x)$, then the inequality (18) can be obtained by applying (9) with x and y reversed and taking into account (17).

If $\nu(\alpha^*)(y) \geqq \nu(\alpha^*)(x)$, then(17) implies $y \in X \setminus K_{\nu(\alpha^*)(x)+s}$. Choose an arbitrary chain σ_{xy}. Two cases are possible:

1) σ_{xy} lies in $X \setminus K_{\nu(\alpha^*)(x)+s}$ and therefore

$$L_{\alpha^*}(\sigma_{xy}) \geqq k_{\alpha^*}^{\nu(\alpha^*)(x)+s}\, \bar{\rho}_{\alpha^*}(x,\ y).$$

2) σ_{xy} has a first among its links which enters $K_{\nu(\alpha^*)(x)+s}$ and therefore

$$L_{\alpha^*}(\sigma_{xy}) \geqq k_{\alpha^*}^{\nu(\alpha^*)(x)+s}\, \bar{\rho}_{\alpha^*}(x,\ K_{\nu(\alpha^*)(x)+s})$$

which in view of (17) implies

$$\rho^*_{\alpha^*}(x,\ y) \geqq k_{\alpha^*}^{\nu(\alpha^*)(x)+s}\, \bar{\rho}_{\alpha^*}(x,\ y).$$

The last inequality is equivalent to (18). Consequently the implication (13) is valid for $x \neq \xi$.

Now we are going to show the implication (13) for $x = \xi$. Let $\eta > 0$ and $\alpha_1, \ldots, \alpha_p \in A$ be arbitrary chosen. Then there exists $\nu > 0$ such that $\bar{\rho}_{\alpha^*}(z,\ \xi) < \frac{1}{2}\eta$ for all $z \in K_\nu$. Consider any $y \in X$ for which

$$\bar{\rho}_{\alpha_i}(\xi,\ y) > \eta \quad (i = 1,\ 2, \ldots,\ p). \tag{19}$$

It follows that $\nu(\alpha^*)(y) > \nu$ and $\bar{\rho}_{\alpha_i}(y,\ K_\nu) \geqq \frac{\eta}{2}$. Choose $s > 0$ so that $\nu(\alpha^*)(y) + s = \nu$. Then in view of (18) we obtain

$$\rho^*_{\alpha^*}(\xi,\ y) \geqq k_{\alpha^*}^{\nu}\, \bar{\rho}_{\alpha^*}(y,\ K_{\nu(\alpha^*)(y)+s}) \geqq k_{\alpha_i}^{\nu}\, \bar{\rho}_{\alpha_i}(y,\ K_\nu) \geqq k_{\alpha_i}^{\nu} \frac{\eta}{2} = \delta > 0$$

or $\bar{\rho}_{\alpha_i}(\xi,\ y) > \eta$ $(i = 1,\ 2, \ldots,\ p)$ implies $\rho^*_{\alpha^*}(\xi,\ y) > \delta$.

The contrapositive of the last implication is: for every $\eta > 0$ there is $\delta > 0$ such that $\rho^*_{\alpha^*}(\xi,\ y) < \delta$ implies $\bar{\rho}_{\alpha_i}(\xi,\ y) \leqq \eta$. So (13) is valid for $x = \xi$.

In order to complete the proof it suffices to show that T_t^{Γ} is j^*-regular for every $x \in X$ and X is j^*-bounded.

Indeed, suppose that $\{T_t^{\Gamma,n}(x)\}_{n=0}^{\infty}$ is not Cauchy sequence. Then there exists $\varepsilon > 0$ and $\alpha^* \in A^*$ such that

$$\bar{\rho}_{\alpha^*}(T_t^{\Gamma,m+s}(x),\ T_t^{\Gamma,m}(x)) \geqq \varepsilon,\ \alpha^* \equiv (\alpha_1, \ldots,\ \alpha_p)$$

that is,

$$\bar{\rho}_{\alpha^*}(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x)) \geqq \varepsilon.$$

Then we can find $\alpha_{i_0} \in \{\alpha_1, \ldots,\ \alpha_p\}$ such that

$$\bar{\rho}_{\alpha_{i_0}}(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x)) \geqq \varepsilon.$$

But T_t^{Γ} is j-regular and there is $\delta > 0$ such that

$$\bar{\rho}_{j(\alpha_{i_0})}(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x)) \geqq \delta.$$

Consequently

$$\bar{\rho}_{j^*(\alpha^*)}(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x)) \geqq \delta. \tag{20}$$

Then

$$\bar{\rho}_{\alpha^*}(T_t^{\Gamma,m+s}(x),\ T_t^{\Gamma,m}(x)) = \bar{\rho}_{\alpha^*}(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x))\, k_{\alpha^*}^{\nu(\alpha^*)(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x))} \geqq \varepsilon.$$

In view of (20) we have

$$k_{j^*(\alpha^*)}^{\nu(f_{(\alpha^*)}^{*}(\alpha^*))(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x))}\, \bar{\rho}_{j^*(\alpha^*)}(T_{(m+s)t}^{\Gamma},\ T_{mt}^{\Gamma}(x)) \geqq k_{j^*(\alpha^*)}^{\nu(f_{(\alpha^*)}^{*}(\alpha^*))(T_{(m+s)t}^{\Gamma}(x),\ T_{mt}^{\Gamma}(x))}\, \delta.$$

Choose $\delta' = k_{j^*(\alpha^*)}^{\nu}\, \delta$ we reach the desired conclusion.

Let us consider an arbitrary chain $x_0,\ x_1, \ldots,\ x_n$ with

$$x_0 = T_{(m+s)t}^{\Gamma}(x),\ x_n = T_{mt}^{\Gamma}(x).\ \text{Then}\ \bar{\rho}_{\alpha^*}(x_i,\ x_{i+1}) \geqq \varepsilon$$

implies $\bar{\rho}_{j(\alpha^*)}(x_i,\ x_{i+1}) \geqq \delta$ and consequently

$$\sum_{i=0}^{n-1} \bar{\rho}_{\alpha^*}(x_i,\ x_{i+1}) \geqq n\varepsilon \geqq \varepsilon$$

implies

$$\sum_{i=0}^{n-1} \bar{\rho}_{j^*(\alpha^*)}(x_i,\ x_{i+1}) \geqq n\delta \geqq \delta.$$

Taking the infimum we obtain the j^*-regularity of T_t^{Γ}.

We shall show that X is j^*-bounded. Indeed,

$$\bar{\rho}_{j^*-n(\alpha^*)}(x,\ y)=k_{j^*-n(\alpha^*)}^{\nu(j^*-n(\alpha^*))(x,\ y)}\ \bar{\rho}_{j^*-n(\alpha^*)}(x,\ y)$$

$$<\bar{\rho}_{j^*-n(\alpha^*)}(x,\ y)\leqq Q(x,\ y,\ \alpha^*)<\infty$$

and

$$\rho^*_{j^*-n(\alpha^*)}(x,\ y)=\inf\sum_{i=0}^{n-1}\bar{\rho}_{j^*-n(\alpha^*)}(x_i,\ x_{i+1})\leqq\bar{\rho}_{j^*-n(\alpha^*)}(x,\ y)$$

$$\leqq Q(x,\ y,\ \alpha^*)<\infty\ \ (n=0,\ 1,\ 2,\ldots).$$

From the definitions of the pseudometrics we obtain $\rho^*_{j^*(\alpha^*)}(x,\ y)\leqq\rho^*_{\alpha^*}(x,\ y)$. Therefore X is j^*-bounded.

In order to complete the proof of sufficiency we shall show the completeness of X with respect to the family $\mathfrak{U}^*=\{\rho^*_{\alpha^*}(x,\ y):\alpha^*\in A^*\}$. Let $\{x_\gamma\}_{\gamma\in S}$ be a Cauchy net in $\mathfrak{U}^*$. If $\{x_\gamma\}_{\gamma\in S}$ is not convergent to ξ, then for every $\gamma\in S$ the number $\nu(\alpha^*)(x_\gamma)<\nu<\infty$ is finite, that is, $x_\gamma\bar{\in}K_\nu$ for some ν. Define $(i=1,\ 2,\ldots,\ p)$

$$\beta_{\alpha_i}=\inf\{\bar{\rho}_{\alpha_i}(x,\ K_{\nu+\Delta}):x\in K_\nu\}>0,\ \Delta>0,\ \alpha^*=\{\alpha_1,\ldots,\ \alpha_p\}$$

for $\alpha_1,\ldots,\ \alpha_p\in A$. Then

$$\rho^*_{\alpha^*}(x_{\gamma_1},\ x_{\gamma_2})<k_{\alpha^*}^{\nu+\Delta}\beta_{\alpha_i}$$

which implies in view of (8) the inequality

$$k_{\alpha^*}^{-\nu-\Delta}\rho^*_{\alpha^*}(x_{\gamma_1},\ x_{\gamma_2})\geqq\bar{\rho}_{\alpha^*}(x_{\gamma_1},x_{\gamma_2})\geqq\bar{\rho}_{\alpha_i}(x_{\gamma_1},\ x_{\gamma_2})\ \ (i=1,\ 2,\ldots,\ p).$$

Thus $\{x_\gamma\}$ is a Cauchy net in $(X,\ \mathfrak{U})$ and therefore is convergent in X because $(X,\ \mathfrak{U})$ is complete. But $(X,\ \mathfrak{U})$ and $(X,\ \mathfrak{U}^*)$ are topologically equivalent which implies that $\{x_\gamma\}_{\gamma\in S}$ is convergent in $(X,\ \mathfrak{U}^*)$.

4. Necessity

Let $\mathfrak{U}^*$ be a family of pseudometrics j-topologically equivalent to $\mathfrak{U}$ and T_t^Γ for every fixed $\Gamma\in\Omega$ and $t>0$ is Φ-contractive with respect to $\mathfrak{U}^*$. Then Theorem 1 implies conditions (RC 1) and (RC 3).

To prove (RC 4) we shall consider the neighbourhood $U^*(\alpha_1^*,\ldots,\alpha_k^*)(\xi,\ \frac{1}{2})$ with an arbitrary finite collection $\alpha_1^*,\ldots,\ \alpha_k^*\in A^*$ and let $\varepsilon>0$ be arbitrary chosen. Then $U^*(\alpha_1^*,\ldots,\ \alpha_k^*)(\xi,\ \varepsilon)$ is open in $(X,\ \mathfrak{U})$.

Let us fix $t_0>0$. Then every $t>0$ can be written in the form $t=nt_0+\sigma$, where $\sigma\in[0,\ t_0)$. Consequently, for every $x,\ y\in U^*(\alpha_1^*,\ldots,\ \alpha_k^*)(\xi,\ \varepsilon)$ we have

$$\rho^*_{\alpha^*_1}(T^\Gamma_t(x),\ T^\Gamma_t(y))=\rho^*_{\alpha^*_1}(T^{\Gamma_1}_\sigma(T^{\Gamma 2\cdot n}_{t_0}(x)),\ T^{\Gamma_1}_\sigma(T^{\Gamma 2\cdot n}_{t_0}(y)))$$

$$\leqq k^\sigma_{\alpha^*_i}\rho^{*-1}_{j^*(\alpha^*_1)}(T^{\Gamma 2\cdot n}_{t_0}(x),\ T^{\Gamma 2\cdot n}_{t_0}(y))\leqq\ldots\leqq k^\sigma_{\alpha^*_i}k^{t_0}_{j^*-1(\alpha^*_i)}k^{t_0}_{j^*-2(\alpha^*_i)}\ldots$$

$$k^{t_0}_{j^*-n(\alpha^*_i)}\rho^*_{j^*-n(\alpha^*_i)}(x,\ y)\ (i=1,\ 2,\ldots,\ k),\ \text{where}$$

$$\alpha^*_1=\{\alpha_{11},\ \alpha_{12},\ldots,\ \alpha_{1p_1}\},\ \alpha^*_2=\{\alpha_{21},\ \alpha_{22},\ldots,\ \alpha_{2p_2}\},\ldots,$$

$$\alpha^*_k=\{\alpha_{k1},\ \alpha_{k2},\ldots,\ \alpha_{kp_k}\},\ \alpha_{il}\in A.$$

Then we can take $\bar{n}>0$ so lagre, such that for any $t>\bar{t}$ and $\Gamma\in\Omega$ $(i=1,\ 2,\ldots,\ k)$

$$\rho^*_{\alpha^*_i}(T^\Gamma_t(x),\ T^\Gamma_t(y))<\varepsilon \text{ and therefore } \rho_{\alpha_{i,l}}(T^\Gamma_t(x),\ T^\Gamma_t(y))<\varepsilon,$$

where $t=\bar{n}t_0+\sigma$.

Since the space is j^*-bounded, that is $\rho^*_{j^*(\alpha^*)}(x,\ y)\leqq\rho^*_{\alpha^*}(x,\ y)$ we have

$$\mathbf{T}_t(x)\bigcap\mathbf{cl}U(j(\alpha_1),\ldots,\ j(\alpha_p))(\xi,\ \eta)=\emptyset\Rightarrow\mathbf{T}_t(x)\bigcap\mathbf{cl}U(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta)=\emptyset;$$

$$x\in\mathbf{cl}\mathbf{T}_t(U(\alpha_1,\ldots,\ \alpha_p)(\xi,\ \eta))\Rightarrow x\in\mathbf{cl}\mathbf{T}_t(U(j(\alpha_1),\ldots,\ j(\alpha_p))(\xi,\ \eta)).$$

Thus the Theorem is proved.

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