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On the Behavior of Chebyshev Rational Approximants with a Fixed Number of Poles

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Let E be a regular set in the complex plane C, and G be the unbounded component of the complement of E. We set $D=\bar{C}-G$. Let f be a function, continuous on E and meromorphic in D. In the case, when f is not of the form φ/q, φ — an entire function, q — a polynomial, we investigate the limiting distribution of the zeros of the sequence of rational uniform approximants (Chebyshev approximants) to f on E with a fixed number of the poles and of a sequence of "near best rational uniform approximants".

Let E be a compact set in the complex plane C, and let G be the unbounded component of the complement E^C of E with respect to $\bar{C}$. It will be assumed throughout this paper that E is regular. This means that the domain G is regular with respect to the Dirichlet problem. We denote by $d(=d(E))$ the logarythmic (Green's) capacity (Cap (E)) of E; since G is regular, d is positive (see [1]). Let $g_E(z, \infty)$ be Green's function for G with pole at infinity; it is known (see [1]) that

$$\lim(\log|z|-g_E(z, \infty))=\log d, \quad \text{as} \quad z\to\infty.$$

We set $D=\bar{C}-G$ and $g_E(z, \infty)=0$ everywhere in D. The extended function $g_E(z, \infty)$ is continuous everywhere in C. For each R, $R>1$, we set

$$E_R=\{z, \ g_E(z, \infty)<\log R\}$$

and

$$\Gamma_R=\partial E_R.$$

It is known (see [2]) that Γ_R consists of a finite number of pairwise disjoint analytical Jordan arcs except (eventually) a finite number of points belonging to several arcs.

We set $E_R=D(=\bar{C}-G)$ and $\Gamma_R=\partial G$ for $R=1$.

Let the function f be continuous on E ($f\in C(E)$). Let B be an open set, $B\supseteq D$. For each nonnegative integer m, ($m\in N$), we shall write $f\in M_m(B, E)$, if f admits in B a continuation as a meromorphic function with not more than m poles (as usually, the poles are counted with regard to their multiplicities). We set

$$R_m=\sup\{R, \ f\in M_m(E_R, E)\}.$$

$R_m = R_m(f, E)$ is called the radius of m-meromorphy of the function $f(\in C(E))$ with respect to E. We set $R_m = 1$, if $\bar{\in} M_m(\bar{D}, E)$. Obviously $R_m <= R_{m+1}$ and f is holomorphic (analytic and single valued) on E, if $R_m > 1$ for some $m \in N$.

Everywhere in our further considerations we shall assume that m is a fixed integer.

For each integer n we set

$$\rho_{n,m} = \rho_{n,m}(f) = \inf \| f - r \|_E,$$

where the inf is taken in the class $r_{n,m}$ and $\| x \|_E$ is the uniform (Chebyshev) norm on E.

We remark that if $f \in M_m(D, E)$ then

$$\text{(1)} \qquad \rho_{n,m} \to 0, \quad \text{as} \quad n \to \infty.$$

Indeed, let $\alpha_1, \ldots, \alpha_\mu$ be the poles of the function f in D; $\mu \leqq m$; since $f \in C(E)$, they are situated in $\operatorname{int}(D - E)$. We set $q(z) = \Pi_{1 \leqq k \leqq \mu}(z - \alpha_k)$ and $F = fq$. Applying the theorem of Mergelyan to the function F and the compact set $\bar{D}$, we obtain that $\rho_{n,0}(F) \to 0$, as $n \to \infty$. Dividing the approximating polynomials by q, we come to (1).

Let $f \in C(E)$. By the theorem of Saff–Gonchar (see [3])

$$\limsup_{n \to \infty} \rho_{n,m}^{1/n} = 1/R_m.$$

In the case of polynomial approximation ($m = 0$), when $f \in \mathscr{H}(E)$, this theorem was proved by Bernstein (when E is a real finite segment) and by Walsh (in the general case, when E is regular and E^C is connected).

For each $n \in N$ we denote by R_n the rational function of order (n, m) of best Chebyshev approximation to f on E, that is

$$\| f - R_n \|_E = \rho_{n,m}.$$

Let $R_n = P_n / Q_n$, where both polynomials have no common divisor. We set

$$P_n(z) = a_n z^n + \ldots .$$

Let C be an arbitrarily chosen and fixed positive constant such that $C > 2 \operatorname{diam}(E)$. Everywhere troughout this paper it will be assumed that the polynomial Q, $n \in N$, is normalized in the following way:

$$\text{(2)} \qquad Q_n(z) = \prod_{k=1}^{l_n} (z - \beta_{n,k}) \prod_{k=l_n+1}^{m_n} (z/\beta_{n,k} - 1), \quad m_n \leqq m,$$

where $|\beta_{n,1}| \leqq \ldots \leqq |\beta_{n,l_n}| < C \leqq |\beta_{n,l_n+1}| \leqq \ldots \leqq |\beta_{n,m_n}|$.

In the case of such a normalization of Q_n, $n \in N$, for every compact set K, we have

$$\|Q_n\|_K \leqq C_1 = C_1(K); \tag{3}$$

(here and everywhere afterwards C_i are positive constants which do not depend on n; $i=1, 2, \ldots$).

It is shown in the present work that the coefficients a_n characterize as $n \to \infty$ the radius R_m of m-meromorphy of f. We have, namely

Theorem 1. *Let E be a regular compact set in C, $d = \mathrm{Cap}(E)$, and let $m \in N$ be fixed. Suppose $f \in C(E)$. Let $R_n(z) = (a_n z^n + \ldots)/Q_n(z)$ be the Chebyshev rational approximant to f on E of order (n, m), $n \in N$, where Q_n are normalized as in (2). Suppose that*

$$\limsup_{n \to \infty} |a_n|^{1/n} \leqq 1/dR, \quad R > 1. \tag{4}$$

Then $f \in M_m(E_R, E)$.

On the other side, we have

Theorem 2. *Suppose, $f \in M_m(E_R, E)$, $R > 1$. Then*

$$\limsup_{n \to \infty} |a_n|^{1/n} \leqq 1/dR.$$

In the case of polynomial approximation ($m=0$) these results were obtained by A. P. Wojcik (see [4]).

In the case, when the function f is not m-meromorphic on E, (i.e. $R_m = 1$), Theorem 1 and Theorem 2 lead to

Theorem 3. *Let E, m and f be as in Theorem 1. Then the following statements are equivalent:*

1. $f \bar{\in} M_m(\bar{D}, E)$;
2. $\limsup |a_n|^{1/n} = 1/d$.

The last result generalizes the analogous theorem of E. B. Saff and H. P. Blatt (see [6]) concerning the polynomial approximation to a function $f \in H(\mathrm{int}\, E)$, but $f \bar{\in} H(E)$, where E^C is connected.

Consequently, the relation between the behavior of a_n as $n \to \infty$ and the radius R_m of m-meromorphy may be expressed as follows:

Corollary 1. *Let E, f, m and a_n, $n \in N$, be as in Theorem 1. Then*

$$\limsup_{n \to \infty} |a_n|^{1/n} = 1/dR_m.$$

This result is analogous to Cauchy-Hadamard formula for the radius of convergence of a power series.

Using the same method, as in the proof of these results, we can show that the conclusion of Corollary 1 also holds for sequences of "near best approximation" of order (n, m) (about the definition see [6]); that is: $\tilde{R}_n \in r_{n,m}$, $\tilde{R}_n = \tilde{P}_n/\tilde{Q}_n$, $\tilde{Q}_n$ are normalized as in (2), where

$$\| f - \tilde{R}_n \|_E = \rho_{n,m} + \varepsilon_n$$

and

$$\limsup (\varepsilon_n)^{1/n} < 1/R_m.$$

We have namely

Corollary 2. *Let E, f and m be as in Theorem 1; suppose that $\tilde{R}_n$, $n = 1, 2, \ldots$ are rational approximants of "near best approximation" of order (n, m); $R_n(z) = (\tilde{A}_n z^n + \ldots)/\tilde{Q}_n(z)$. Then*

$$\limsup |\tilde{A}_n|^{1/n} = 1/dR_m.$$

Before we continue we notice that if $f \in M_m(D, E)$ then

$$\| f - \tilde{R}_n \|_E \to 0, \quad \text{as} \quad n \to \infty. \tag{1'}$$

The requirement about the regularity of the compact set E in the proof of the formulated theorems is not essential. It is necessary for E to be of positive Green's capacity (see [2, Ch. 4] and [3]). This statement is equivalent to the assertion that G possesses Green's function in the extended sense (see [1]).

The next results concern the uniform distribution of the zeros of the Chebyshev rational approximants to f on E. The compact set E is assumed to be regular.

Let $P_n(z) = a \prod_{k=1}^{n} (z - \zeta_k)$. We shall denote by $\nu(p_n)$ the measure associated with the polynomial p_n, i.e.

$$\nu(p_n) = \sum_{k=1}^{n} \delta_{\zeta_k},$$

where δ_{ζ_k} is the unit measure at the point ζ_k, $k = 1, \ldots, n$.

For each compact set K, $K \subset C$, we shall denote by λ_K the equillibrium measure for K.

Using essentially the previous results, we obtain

Theorem 4. *Let E be a regular compact set in C, $m \in N$, $f \in M_m(D, E)$ and let $R_n = P_n/Q_n$, $n \in N$, be the Chebyshev rational approximant to E of order (n, m). Then, if $R < \infty$, there is a sequence Λ, $\Lambda \subset N$, such that*

$$\nu(P_n)/n \to \lambda_{\Gamma_{R_m}}, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda;$$

Λ is the sequence, for which $\lim |a_n|^{1/n} = 1/dR_m$. (The convergence here is to be understood as a weak convergence of measures.)

In the case, when E is the finite real segment and $f \in \mathscr{H}(E)$ is real-valued, Theorem 4 was proved by V. V. Prokhorov (see [8]).

In the case of polynomial approximation ($m=0$) and in the case when $m>0$, $f \in \mathscr{H}(\operatorname{Int} E)$, but $f \bar{\in} \mathscr{H}(E)$ and E^C is connected, this result was proved by E. B. Saff, H. P. Blatt and M. Simkani (see [7]). In essence, Theorem 4 is proved in [7]. The author received the cited paper a short time before presenting the present work. However, the idea of the proof is different from the one in [7].

For each compact set K in C and for each n, $n \in N$, we denote by $\tau(P_n)(K)$ the number of the zeros of P_n on K.

From Theorem 4 we have

Corollary 3. *Let E, f and P_n, $n \in N$ be as in Theorem 4. Suppose that there is an open set B, $B \cap E \neq 0$ such that*

$$\tau(P_n)(B) = o(n), \quad \text{as} \quad n \to \infty.$$

Then $B \subset E_{R_m}$ (i.e. the function f admits m-meromorphic continuation in B).

In the case of polynomial approximation Corollary 3 was proved in [4].

Applying the same considerations to the near best rational approximants as in the proof of Theorem 4 we obtain

Theorem 5. *Let E, f and m be as in Theorem 4.* Let $\tilde{R}_n = \tilde{P}_n / \tilde{Q}_n$, $n \in N$, *be "near best" rational approximants to f on E of order (n, m) $\tilde{P}_n(z) = \tilde{A}_n z^n + \dots$. Then, if $R_m < \infty$, there is a sequence Λ, $\Lambda \subset N$ such that*

$$\nu(\tilde{P}_n)/n \to \lambda_{\Gamma_{R_m}}, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

(Λ is the sequence, for which $\lim |\tilde{A}_n|^{1/n} = 1/dR_m$.)

Proofs of the theorems.

Preliminary results.

Lemma 1 (Lemma of Bernstein—Walsh, see [2]). *Let E be a compact set of positive capacity and p — a polynomial of order n. Then, for each $z \in E^C$ there holds the inequality*

$$|p(z)| \leqq \|p\|_E \exp(n g_E(z, \infty)).$$

Lemma 2. *Let E be as in Lemma 1 and $p_n(z) = az^n + \dots = a p_n^*(z)$, with $a \neq 0$. Let θ be an arbitrary positive number. Then, if n is sufficiently large, the inequality*

$$|a| \leqq \|p_n\|_E (\operatorname{Cap}(E)^n) . \exp n\theta$$

holds.

Indeed, for each R, $R>1$ we have, from Lemma 1

$$|a| . \|p_n^*\|_{E_R} \leqq \|p_n\|_E . R^n. \tag{5}$$

On the other side, since the polynomial p_n^* is monic, it is valid (see [1]), for n sufficiently large

$$\exp(n\theta)\,.\,\|p_n^*\|_E \geqq (\operatorname{Cap}(E_R)^n) = (dR)^n.$$

Combining (5) and the last inequality, we obtain Lemma 2.

Lemma 3. *Let E be the same as in Lemma 1 and T_n, $n \in N$, be the Chebyshev polynomial to E of order n with zeros on E. Then* (see [1])

$$\|T_n\|_E^{1/n} \to \operatorname{Cap}(E), \quad \text{as} \quad n \to \infty \tag{6}$$

and

$$|T_n(z)|^{1/n} \to \operatorname{Cap}(E)\,.\exp(g_E(z, \infty)) \quad \text{as} \quad n \to \infty \tag{7}$$

uniformly on compact subsets of G.

Let now Ω be an open set in $\mathbf{C}$, let φ_n be a sequence of meromorphic functions in Ω, and let φ be a function, given in Ω, $\varphi(\Omega) \subset \mathbf{C}$. We say that $\varphi_n \to \varphi$, as $n \to \infty$, uniformly in Green's capacity on each compact subset K of Ω, if for each positive ε there is a subset K_ε of K such that $\operatorname{Cap}(K_\varepsilon) < \varepsilon$ and $\varphi_n \to \varphi$, as $n \to \infty$, uniformly on $K - K_\varepsilon$ in the sense of Chebyshev. As it is known, the convergence in capacity yields an uniform one, if the functions φ_n are holomorphic in Ω. On the other side, if the limit function φ is meromorphic in Ω and if it has exactly μ poles there, then the functions φ_n have, for n sufficiently large, necessarily at least μ poles in Ω and each pole of φ attracts at least as many poles of φ_n as its multiplicity.

All details about the convergence in capacity are to be found in [9].

Lemma 4 (see [3]): *Let E be a compact set in $\mathbf{C}$ of positive capacity and suppose the function $f \in C(E)$ is m-meromorphic in E_R, $R > 1$. Let $\tilde{r}_n$, $n = 1, 2, \ldots$ be rational functions, $\tilde{r}_n \in r_{n,m}$ such that $\limsup \|f - \tilde{r}_n\|_E^{1/n} < = 1/R$. Then $\tilde{r}_n \to f$, as $n \to \infty$, uniformly in capacity on compact subsets of E. For each compact set K, $K \subset E_R$ which does not intersect neither the poles of f in E_R, nor the set of the poles of $\{\tilde{r}_n\}_{n=1}^{\infty}$, the speed of convergence is given by the inequality*

$$\limsup_{n \to \infty} \|f - \tilde{r}_n\|_K^{1/n} \leqq \exp \|g_E(z, \infty)\|_K / R.$$

Proof of Theorem 1.

We set, as in the introduction, $d = \operatorname{Cap}(E)$.

From the definition of Chebyshev rational approximants we have

$$\rho_n = \|f - R_n\|_E = \|f - \frac{a_n z^n + \ldots}{\Pi(z - \beta_{n,k})\,\Pi(z/\beta_{n,k} - 1)}\|_E$$

$$\leqq \|f(z) - (R_{n+1}(z) - \frac{a_{n+1} T_{n+1} - l_{n+1}(z)}{\prod\limits_{k=l_{n+1}+1}^{m_{n+1}} (z/\beta_{n+1,k} - 1)})\|_E \leqq \rho_{n+1} + C_2 |a_{n+1}|\,.\,\|T_{n+1} - l_{n+1}\|_E.$$

Let θ_1 be a positive number such that $\exp\theta_1 < R$. Then (4), (6) and (2) yield

$$\rho_n \leqq \rho_{n+1} + C_3 (\exp\theta_1/R)^n, \quad n \geqq n_1 = n_1(\theta_1).$$

This implies

$$0 < \rho_n - \rho_{n+1} \leqq C_3 (\exp\theta_1/R)^n, \quad n > n_1.$$

Consequently

$$\rho_n < = C_4 (\exp\theta_1/R)^n, \quad n > n_2 \geqq n_1.$$

Since θ_1 is arbitrary, we have

$$\limsup_{n\to\infty} \rho_n^{1/n} \leqq 1/R.$$

The assertion of Theorem 1 follows now from the theorem of Saff—Gonchar.

Proof of Theorem 2.

Let us choose $\varepsilon > 0$ such that $\varepsilon < \mathrm{dist}(E, \Gamma_R)$. For each $k = 1, \ldots, m_n$, $n \in N$, let $U_{n,k}(\varepsilon)$ be the disk of radius $< \varepsilon/4m_n n^2$ centered at $\beta_{n,k}$ and let

$$U(\varepsilon) = \bigcup_{n-1}^{\infty} \bigcup_{k=1}^{m_n} U_{n,k}(\varepsilon).$$

The sum of the diameters of $U_{n,k}$, $n = 1, 2, \ldots$ does not exceed $(\varepsilon/2)\Sigma n^{-2} < \varepsilon$. Therefore there is Γ_ρ, $1 < \rho < R$ which does not intersect U (ε). We set

$$\rho_\varepsilon = \sup \{\rho, \ \Gamma_\rho \cap U(\varepsilon) = \emptyset, \quad 1 < \rho < R\}.$$

It is not hard to see that for each z, $z \bar{\in} U(\varepsilon)$ the inequality

$$\text{(8)} \qquad \min |Q_n(z)| > (\varepsilon/4mn^2)^m$$

holds.

For each set B we set $B(\varepsilon) = B - U(\varepsilon)$. It follows from Lemma 4 that $f \in H(E_{\rho_\varepsilon}(\varepsilon))$.

Select positive numbers θ_2 and ρ such that $\exp\theta_2 < R/\rho$ and $\Gamma_\rho \subset E_{\rho_\varepsilon}(\varepsilon)$. From Lemma 4 we obtain

$$\| f - R_n \|_E < = ((\exp\theta_2)\rho/R)^n, \quad n > n_3 = n_3(\theta_2, \varepsilon).$$

Therefore (see (3))

(9) $$\|P_n\|_{\Gamma_\rho} < = C_6', \quad n \geqq n_4 \geqq n_3.$$

On the other hand the coefficient a_n, $n \in N$, is given by

$$a_n = (1/2\pi i) \int_{\Gamma_\rho} \frac{P_n(t)}{T_{n+1}(t)} dt.$$

From here, using (7) and (9), we obtain

$$|a_n| \leqq C_7 (\exp \theta_2 / d\rho)^n, \quad n > n_5 \geqq n_4.$$

Consequently

$$\limsup_{n \to \infty} |a_n|^{1/n} \leqq \exp \theta_2 / d\rho.$$

Letting $\rho \to \rho_\varepsilon$ and $\theta_2 \to 0$, we get

$$\limsup |a_n|^{1/n} \leqq 1/d\rho_\varepsilon.$$

Obviously $U(\varepsilon_1) \subset U(\varepsilon_2)$, if $\varepsilon_1 < \varepsilon_2$. Therefore $\rho_\varepsilon \to R$, as $\varepsilon \to 0$. This proves Theorem 2.

Proof of Theorem 3.
We note, first, that under the conditions of Theorem 3 the estimation

(10) $$\limsup_{n \to \infty} |a_n|^{1/n} \leqq 1/d$$

holds.

Indeed, from the definition of the Chebyshev rational approximants to a continuous function on E we have for each n, $n \in N$

$$\|f - R_n\|_E \leqq \|f - R_{n-1}\|_E.$$

Then, by (3)

(11) $$\|P_n\|_E \leqq C_8, \quad n \in N, \quad C_8 = C_8(f, E).$$

Using Lemma 2 and the last inequality, we get (10).

Now the assertion of Theorem 3 follows immediately from Theorem 1 and Theorem 2. Indeed, let $f\in C(E)$. Then (10) is valid. Suppose, $f\bar{\in} M_m(\bar{D}, E)$. The assumption that statement 2. is not valid leads, after (10) and Theorem 1, to the conclusion that $f\in M_m(\bar{D}, E)$. Conversely, if statement 2. is valid, we obtain statement 1 from Theorem 2.

Now we set

$$P_n(z)=a_nP_n^*(z)=a_n\prod_{k=1}^{n}(z-\zeta_{n,k}),\quad n\in N.$$

Proving Theorem 4 we shall follow in general the considerations of Prokhorov (see [8]).

Basic for the proof of Theorem 4 is the following

Lemma 5. *Under the conditions of Theorem* 4 *there is a sequence* Λ, $\Lambda\subset N$ *such that*

1. $\|P_n^*(z)\|_{\Gamma_{R_m}}^{1/n}\to dR_m$, as $n\to\infty$, $n\in\Lambda$;
2. *For each compact set K in C which does not intersect the level curve*

$$\tau(P_n^*)(K)=o(n),\quad \text{as}\quad n\to\infty,\quad n\in\Lambda.$$

Proof of the lemma. We shall show first that under the conditions of Theorem 4

$$\|P_n\|_{\Gamma_{R_m}}^{1/n}\to 1,\quad \text{as}\quad n\to\infty. \tag{12}$$

Suppose, first, that $R>1$. Let ε be a positive number, $\varepsilon<\operatorname{dist}(E, \Gamma_{R_m})$. In the same way as in the proof of Theorem 2 we define the parameter ρ_ε, $\rho_\varepsilon<$R. Select positive numbers θ_3 and ρ such that $\Gamma_\rho\subset E_{\rho_\varepsilon}(\varepsilon)$ and $\exp\theta_3<R/\rho$. Using (3) and (8) and Lemma 4, we obtain

$$C_8'(\varepsilon/mn^2)^m\leqq\|P_n\|_{\Gamma_\rho}\leqq C_8'',\quad n>n_6=n_6(\theta_3).$$

Hence

$$\lim\|P_n\|_{\Gamma_\rho}^{1/n}=1,\quad \text{as}\quad n\to\infty. \tag{13}$$

The application of the lemma of Bernstein—Walsh to the polynomials P_n, the compact set $\bar{E}_\rho$ and the level curve Γ_{R_m} and the maximum principle lead to

$$\|P_n\|_{\Gamma_\rho}\leqq\|P_n\|_{\Gamma_{R_m}}\leqq\|P_n\|_{\Gamma_\rho}\cdot(R_m/\rho)^n.$$

Using (13) and letting $\rho\to\rho_\varepsilon$ and $\varepsilon\to 0$, we get (12).

Suppose, now, that $R_m=1$. Under the conditions of the theorem this means that $f\in M_m(D, E)$, but $f\bar{\in}M_m(\bar{D}, E)$. From the maximum principle and from (11) we have $\limsup \|P_n\|_D^{1/n} = \limsup \|P_n\|_E^{1/n} \leqq 1$. The assumption that $\liminf \|P_n\|_D^{1/n} < 1$ leads after (1) to the conclusion that $f \equiv 0$ which is impossible in the given conditions (namely that $R_m < \infty$). This proves (12) completely.

Now we remark that for each compact subset K of E_{R_m}, $R_m \geqq 1$ we have

$$\limsup_{n\to\infty} \tau(P_n)(K) \leqq C_9(K). \tag{14}$$

Indeed, if $R_m > 1$, (14) follows from Lemma 4 and from the argument principle. We have namely

$$\tau(P_n)(K) = \tau(Q_n)(K) + C_9,$$

where $C_9 = C_9(K, f)$. This equality is valid for all n sufficiently large, $n \in N$. Since $\tau(Q_n) \leqq m$, we obtain (14).

If $R_m = 1$, (14) follows from (1) and from the theorem of Hurwitz.

Let now Λ be the sequence, for which

$$\lim |a_n|^{1/n} = 1/dR, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

(The existence of such a sequence follows from Corollary 1.)

From (12) we obtain

$$\|P_n^*\|_{\Gamma_{R_m}}^{1/n} \to dR_m, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

Now we note that $\operatorname{Cap}(E_{R_m}) = \operatorname{Cap}(E) . R_m = dR_m$. Following the results of [10], we conclude that for each compact set K, $K \cap E_{R_m} = \emptyset$

$$\lim \|P_n^*\|_R^{1/n} = dR_m \|\exp(g_{E_{R_m}}(z, \infty))\|_R = d\|\exp g_E(z, \infty)\|, \tag{15}$$

$$\text{as} \quad n \to \infty, \quad n \in \Lambda.$$

This provides further (see [10, Theorem 1]) that

$$\tau(P_n^*)(K) = 0(n), \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

The statements of the lemma follow now from here and from (14).

Proof of Theorem 4.

First we are going to prove that for each subsequence Λ_1 of Λ we have

$$\nu(P_n^*)/n \to \lambda_{\Gamma_{R_m}}, \quad \text{as } n \to \infty, \quad n \in \Lambda_1 \tag{16}$$

(the convergence is to be understood as a weak convergence).

Indeed, let Λ_1 be a subsequence of Λ. Then there is a measure γ (see [5]) such that

(17) $$\nu(P_n^*)/n \to \gamma, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda.$$

We shall show that $\gamma = \lambda_{\Gamma_{R_m}}$.

It follows from Lemma 5 that the support of γ belongs to Γ_{R_m} ($\operatorname{supp} \gamma \subset \Gamma_{R_m}$).

Let $V_\gamma(z)$ be the conductor potential, associated with the measure γ, that is (see [5])

$$V_\gamma(z) = \int \log \frac{1}{|\zeta - z|} d\gamma(\zeta).$$

As it is known, V_γ is harmonic in $C - \operatorname{supp}(\gamma)$, subharmonic in $\bar{C} - \operatorname{supp}(\gamma)$ and superharmonic and lower semi-continuous in C.

From (17) we obtain

$$V_{\frac{\nu(P_n^*)}{n}}(z) \to V_\gamma(z), \quad \text{as} \quad n \to \infty, \quad n \in \Lambda_1$$

uniformly on each compact set in C which does not intersect Γ_{R_m}.

We have, further (see [11])

$$\min_{z \in \Gamma_{R_m}} V_{\frac{\nu(P_n^*)}{n}}(z) + \frac{1}{n} \sum_{\zeta_{n,k} \bar{\in} E_{2R_m}} \log |\zeta_{n,k}| \to \min_{z \in \Gamma_{R_m}} V_\gamma(z).$$

From Lemma 5 (statement 2.) we see that (see (15))

$$\frac{1}{n} \sum_{\zeta_{n,k} \bar{\in} E_{2R_m}} \log |\zeta_{n,k}| \to 0, \quad \text{as} \quad n \to \infty, \quad n \in \Lambda_1.$$

On the other side, statement 1. Of the lemma provides

$$\min_{z \in \Gamma_{R_m}} V_{\frac{\nu(P_n^*)}{n}}(z) \to \log(dR_m), \quad \text{as} \quad n \to \infty, \quad n \in \Lambda_1.$$

Thus

(18) $$\min_{z \in \Gamma_{R_m}} V_\gamma(z) = \log(dR_m).$$

As it is known (see [5])

$$V_{\lambda_{\Gamma_{R_m}}}(z) = \log(dR_m).$$

Let G_{R_m} be the unbounded component of $E_{R_m}^c (= \bar{C} - E_{R_m})$. We consider now the function $V = (V_{\lambda_{\Gamma_{R_m}}} - V_\gamma)$. It is harmonic in G, $V(\infty) = 0$ and, as it follows from (18) from and the last equality

$$\min_{z\in\Gamma_{R_m}} V(z)=0.$$

The minimum principle for harmonic functions provides that

$$V_\gamma=V_{\lambda_{\Gamma_{R_m}}} \tag{19}$$

everywhere in G_{R_m}.

Let now z_0, $z_0\in\Gamma_{R_m}$, be a regular point. Then it is $V_{\lambda_{\Gamma_{R_m}}}(z_0)=V_\gamma(z_0)$ (see [5]).

Suppose now, $R_m>1$. Since E is a regular compact set in C, each point z, $z\in\Gamma_{R_m}$ (except eventually a finite number of points), is analytical (see [2]) and therefore regular. Since V_γ is semicontinuous in C, we obtain that (19) holds for each z, $z\in G$. Then $\int(V_{\lambda_{\Gamma_{R_m}}}-V_\gamma)(z)\,d(\lambda_{\Gamma_{R_m}}-\gamma)=0$. Thus (16) holds.

We consider now the case $R_m=1$. Since E is a regular compact set, each point z_0, $z\in\partial D$, is a regular point. In the same way as above we conclude that (16) holds.

Theorem 4 is completely proved.

Proof of Corollary 2.

First we notice that

$$\limsup_{n\to\infty}\|f-\tilde{R}_n\|_E^{1/n}=1/R_m. \tag{20}$$

Then Lemma 4 with respect to the rational functions $\tilde{R}_n$, $n\in N$, is valid.

Now it is easy to prove Corollary 2. Indeed, suppose that $\limsup|\tilde{A}_n|^{1/n}\leqq 1/d.R$, $R>1$. In the same way, as in the proof of Theorem 1 we obtain

$$\limsup_{n\to\infty}\rho_n^{1/n}\leqq\max(\limsup_{n\to\infty}\varepsilon_n^{1/n},\ 1/R). \tag{21}$$

The assumption that $R>R_m$ leads to the conclusion that $\limsup\rho_n^{1/n}<1/R_m$ which is a contradiction to the theorem of Saff—Gonchar. Thus $R_m\geqq R$ and $f\in M_m(E_R,\ E)$.

On the other side, repeating the considerations in the proof of Theorem 2, we get from (20) that $\limsup|\tilde{A}_n|^{1/n}\leqq 1/dR$, if $f\in M_m(E_R,\ E)$ with $R>1$.

Suppose, now, that $f\bar{\in} M_m(\bar{D},\ E)$. Then $R_m=1$ and $\limsup\varepsilon_n^{1/n}<1$. In the same way as in the proof of Theorem 3 we find that $\limsup|\tilde{A}_n|^{1/n}\leqq 1/d$. The assumption that $\limsup|A_n|^{1/n}<1/d$, leads (see (21)) to the inequalities

$$1/R_m=\limsup_{n\to\infty}\rho_n^{1/n}\leqq\max(\limsup_{n\to\infty}\varepsilon_n^{1/n},\ \limsup_{n\to\infty}|\tilde{A}_n|^{1/n})<1.$$

This is a contradiction to the assumption that $f\bar{\in} M_m(\bar{D},\ E)$.

Conversely, suppose that $\limsup|\tilde{A}_n|^{1/n}=1/d$. The assumption that $f\in M_m(\bar{D},\ E)$ yields $R_m>1$. Then, after Theorem 2 and Theorem 1 we have $\limsup_{n\to\infty}|\tilde{A}_n|^{1/n}=1/dR$ $(<1/d)$.

This proves corollary 2.

An analogue of Lemma 5 with respect to the "near best rational approximants" $\tilde{R}_n$, $n \in N$, is valid. Its proof is based on Lemma 4 and on (1′).

Now the proof of Theorem 5 is a repetition of the considerations in the proof of Theorem 4.

In the end we notice that the results of the present paper are announced in Comptes rendus de l'Academie bulgare des Sciences, Tome 42, No 2, 1988.

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