Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New series Vol. 4, 1990, Fasc. 1

Theorems and Counterexamples on Contractive Mappings

J. Kincses, V. Totik

Presented by Bl. Sendov

Theorems and counterexamples are proved about fixed points of contractive mappings in complete metric spaces. These cover all known results of this kind known to us and solve many open problems.

1. Introduction

In the last 20 years a relatively wide study of contractive type mappings was carried out by different authors (a good summary and detailed reference can be found in [1]). Thus, it seems to be worthwhile examining the subject systematically, which we do in the present paper.

One of the most comprehensible articles about contractive mappings is [1] which contains several results as well as open problems for further study (although we have to remark that [1] contains some mistakes: Theorems 9 and 15 are not true — see Exaple 2 below). We shall keep the definitions of [1] and prove 7 theorems and present 7 counterexamples which cover most of the possible cases (see the "truth table" below). Some spacial cases of our theorems coincide with earlier results, but we do not give exact references, only refer to the summary in [1].

The paper is organized as follows.

Section 2 contains the definitions, section 3 the "truth table" and some comments on it, finally in sections 4 and 5 we prove our theorems and give our counterexamples.

2. Definitions

In what follows (X, d) will always be a complete metric space and $T: X \to X$ a mapping of X into itself. Recall that $x \in X$ is said to be a fixed point of T if $Tx = x$.

Following B. E. Rhoades [1] we define the following contractive type mappings.

(1) (Banach) There exists a number a, $0 \leqq a < 1$ such that, for each $x, y \in X$

$$d(Tx, Ty) \leqq ad(x, y).$$

(2) (Rakotch) There exists a monotone decreasing function α: $(0, \infty) \to [0, 1)$ such that, for each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) \leqq \alpha(d(x, y))d(x, y).$$

(3) (Edelstein) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < d(x, y).$$

(4) (Kannan) There exists a number a, $0 < a < \frac{1}{2}$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq a[d(x, Tx) + d(y, Ty)].$$

(5) (Bianchini) There exists a number h, $0 \leqq h < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq h \max \{d(x, Tx), d(y, Ty)\}.$$

(6) (Rhoades) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < \max \{d(x, Tx), d(y, Ty)\}.$$

(7) (Reich) There exist nonnegative numbers a, b, c satisfying $a + b + c < 1$ such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq ad(x, Tx) + bd(y, Ty) + cd(x, y).$$

(8) (Reich) There exist monotonically decreasing functions a, b, c from $(0, \infty)$ into $[0, 1)$ satisfying $a(t) + b(t) + c(t) < 1$ such that, for each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) \leqq a(d(x, y))d(x, Tx) + b(d(x, y))d(y, Ty) + c(d(x, y))d(x, y).$$

(9) (Rhoades) There exist nonnegative functions a, b, c on $X \times X$ satisfying

$$\sup_{x,y \in X} \{a(x, y) + b(x, y) + c(x, y)\} \leqq \lambda < 1$$

such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq a(x, y)d(x, Tx) + b(x, y)d(y, Ty) + c(x, y)d(x, y).$$

(10) (Sehgal) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < \max \{d(x, Tx), d(y, Ty), d(x, y)\}.$$

(11) (Chatterjea) There exists a number a, $0 < a < \frac{1}{2}$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq a[d(x, Ty) + d(y, Tx)].$$

(12) (Rhoades) There exists a number h, $0 \leqq h < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq h \max\{d(x, Ty), d(y, Tx)\}.$$

(13) (Rhoades) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

(14) (Rhoades) There exist nonnegative numbers a, b, c satisfying $a+b+c<1$ such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq ad(x, Ty) + bd(y, Tx) + cd(x, y).$$

(15) (Rhoades) There exist monotone decreasing functions a, b, c from $(0, \infty)$ into $[0, 1)$ satisfying $a(t)+b(t)+c(t)<1$ such that, for each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) \leqq a(d(x, y))d(x, Ty) + b(d(x, y))d(y, Tx) + c(d(x, y))d(x, y).$$

(16) (Rhoades) There exist nonnegative functions a, b, c on $X \times X$ satisfying

$$\sup_{x,y \in X} \{a(x, y) + b(x, y) + c(x, y)\} \leqq \lambda < 1$$

such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq a(x, y)d(x, Ty) + b(x, y)d(y, Tx) + c(x, y)d(x, y).$$

(17) (Rhoades) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx), d(x, y)\}.$$

(18) (Hardy and Rogers) There exist nonnegative constants a_i satisfying $\Sigma_{i=1}^{5} a_i < 1$ such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx).$$

(19) (Zamfirescu) There exist real numbers α, β, γ, $0 \leqq \alpha < 1$, $0 \leqq \beta, \gamma < \frac{1}{2}$ such that, for each $x, y \in X$, at least one of the following is true:

(i) $d(Tx, Ty) \leqq \alpha d(x, y)$,
(ii) $d(Tx, Ty) \leqq \beta[d(x, Tx) + d(y, Ty)]$,
(iii) $d(Tx, Ty) \leqq \gamma[d(x, Ty) + d(y, Tx)]$.

(20) (Rhoades) For each $x, y \in X$, $x \neq y$,

$$d(Tx, Ty) < \max\{d(x, y), [d(x, Tx) + d(y, Ty)]/2, [d(x, Ty) + d(y, Tx)]/2\}.$$

(21) (Ciric) There exist nonnegative functions q, r, s, t on $X \times X$ satisfying

$$\sup_{x,y \in X} \{q(x, y) + r(x, y) + s(x, y) + 2t(x, y)\} \leqq \lambda < 1$$

such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq q(x, y)d(x, y) + r(x, y)d(x, Tx) + s(x, y)d(y, Ty) + t(x, y)[d(x, Ty) + d(y, Tx)].$$

(22) (Rhoades) For each $x, y \in X, x \neq y$,

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty)+d(y, Tx)]/2\}.$$

(23) (Rhoades) There exist monotonically decreasing functions α_i: $(0, \infty)\to[0, 1)$ satisfying $\sum_{i=1}^{5} \alpha_i(t)<1$ such that, for each $x, y \in X, x \neq y$,

$$d(Tx, Ty) \leqq \alpha_1(d(x, y))d(x, Tx)+\alpha_2(d(x, y))d(y, Ty)+\alpha_3(d(x, y))d(x, Ty) + \alpha_4(d(x, y))d(y, Tx)+\alpha_5(d(x, y))d(x, y).$$

(24) (Ciric) There exists a constant h, $0 \leqq h < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leqq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

(25) (Rhoades) For each $x, y \in X, x \neq y$,

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

It may happen that some iterate of T satisfies one of the preceding definitions, thereby giving rise to an additional twenty-five definitions, which we shall number (26)—(50).

For example,

(29) (Singh) There exists a positive integer p and a number a, $0<a<\frac{1}{2}$, such that, for each $x, y \in X$,

$$d(T^p x, T^p y) \leqq a[d(x, T^p x)+d(y, T^p y)].$$

Let p, q be fixed positive integers. We shall use (51)—(75) to identify those mappings T which have the property that $T^p x$ and $T^q y$ satisfy the corresponding contractive condition from (1) to (25).

For example,

(64) There exist positive integers p, q and nonnegative numbers a, b, c satisfying $a+b+c<1$ such that, for each $x, y \in X$,

$$d(T^p x, T^q y) \leqq ad(x, T^q y)+bd(y, T^p x)+cd(x, y).$$

This was considered by Gupta and Srivistava with $c=0$ (see [1]).

It may happen that the particular iterate of T in (26)—(50) depends on the point in the space. These are Definitions (76)—(100) which refer to mappings that are locally contractive. For example,

(76) (Guseman) There exists a number α, $0<\alpha<1$, such that for each $x \in X$ there exists an integer $p(x)$, such that for each $y \in X$,

$$d(T^{p(x)}x, T^{p(x)}y) \leqq \alpha d(x, y).$$

Finally, the iterate of T may depend on both x and y, giving us definitions (101)—(125). For example,

(103) (Bailey) For each $x, y \in X$, $x \neq y$ implies that there exists an integer $p=p(x, y)$ such that

$$d(T^p x, T^p y) < d(x, y).$$

Beyond these conditions we introduce some further ones. The first is the "empty assumption":

A_0: *There is no further condition.*

The next two might be

B_0: *T is continuous,*

B_1: *T is uniformly continuous.*

It turns out, however, that B_0 and B_1 do not help anything (except possibly at the unsolved (90)): If any of the conditions (i) ($i \neq 90$; $1 \leqq i \leqq 125$) plus B_0 or B_1 ensure a fixed point then condition (i) alone ensures a fixed point, as well. In fact, we have to remark only that in all of our counterexamples used at a condition $(i)+A_0$ the mapping T is uniformly continuous and thus it works also for the condition $(i)+B_1$.

A significantly stronger assumption is

A_1: *T is continuous and $\{T^n x_0\}_{n=1}^{\infty}$ has a cluster point for an $x_0 \in X$.*

Finally, let

A_2: *X is compact and T is continuous.*

Clearly, this is the strongest one of our additional assumptions.

3. The "truth table"

This is the table shown by Fig. 1. It contains 25 rows, one for each of the basic conditions; 3 columns corresponding to assumptions A_0, A_1, A_2 and in each column 5-5 subcolumns. Thus, an element or box of our table is determined by a triplet (i, A_j, k), where $1 \leqq i \leqq 25$, $0 \leqq j \leqq 2$, $0 \leqq k \leqq 4$ and refers to the condition $(k \cdot 25+i)+A_j$; e.g. (1, A_2, 3) refers to the condition: X is compact, T is continuous and for each $x \in X$ there is a $p=p(x)$ such that for all $y \in X$ we have $d(T^p x, T^p y) \leqq \alpha d(x, y)$ where $0<\alpha<1$ is a fixed number.

Every box of the table contains a letter and a number. The letter is always f or t according to the falsity or truth of the following statement: the corresponding condition ensures the existence of a fixed point. The number indicates which of our theorems or examples cover the case in question. E. g. the element (1, A_0, 3) is t^4 and this means that condition $(3 \cdot 25+12)+A_0$, i.e. $(87)+A_0$ ensures a fixed point and this follows from Theorem 4. Similarly, since the element (23, A_2, 2) is f^2 we get that conditions $(73)+A_2$ does not guarantee any fixed point, and a counterexample is exactly Example 2.

Naturally, many cases are covered by more than one theorem or example. We also remark that our theorems and examples, as well as their methods can be used to settle many other problems concerning contractive type mappings.

There are 9 question marks in our table denoting those cases which we could not solve. The most interesting one is the following problem:

If $\{T^n x_0\}_{n=1}^{\infty}$ has a cluster point for some $x_0 \in X$ and if for all, x, $y \in X$, $x \neq y$ we have

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

then does T have necessarily a fixed point?

(We conjecture that the answer is negative.)

Finally, we mention that the answers to the questions of B. E. Rhoades [1] concerning conditions (25, A_1, 0), (95)–(100), (104)–(125) (as well as to other so far not considered problems) can be readily read out of the above table.

	A_0					A_1					A_2				
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	t^1	t^1	t^1	t^4	f^7	t^1	t^1	t^1	t^4	f^4	t^1	t^1	t^1	t^4	t^5
2	t^2	t^2	f^2	t^6	f^7	t^2	t^2	f^2	t^6	f^4	t^2	t^2	f^2	t^3	t^5
3	f^5	f^5	f^2	f^5	f^7	t^7	t^7	f^2	?	f^4	t^7	t^7	f^2	t^3	t^5
4	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
5	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
6	f^5	f^5	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1
7	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
8	t^2	t^2	f^2	f^1	f^1	t^2	t^2	f^2	f^1	f^1	t^2	t^2	f^2	f^1	f^1
9	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
10	f^5	f^5	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1
11	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^3	f^3
12	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^3	f^3
13	f^5	f^5	f^2	f^2	f^3	?	?	f^2	?	f^3	t^8	t^8	f^2	t^3	f^3
14	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^3	f^3
15	t^2	t^2	f^2	?	f^3	t^2	t^2	f^2	?	f^3	t^2	t^2	f^2	t^3	f^3
16	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^4	f^3	t^1	t^1	t^1	t^3	f^3
17	f^5	f^5	f^2	f^5	f^3	?	?	f^2	?	f^3	t^8	t^8	f^2	t^3	f^3
18	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
19	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
20	f^5	f^5	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1
21	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
22	f^5	f^5	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1	t^7	t^7	f^2	f^1	f^1
23	t^2	t^2	f^2	f^1	f^1	t^2	t^2	f^2	f^1	f^1	t^2	t^2	f^2	f^1	f^1
24	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1	t^1	t^1	t^1	f^1	f^1
25	f^5	f^5	f^2	f^1	f^1	f^6	f^6	f^2	f^1	f^1	t^8	t^8	f^2	f^1	f^1

Fig. 1

4. Theorems

Before giving our theorems and their proofs we remark that if below there is a fixed point at all, then this fixed point is always unique. The proof of this requires very simple computation in each cases, we omit the details (see also [1, p. 267]).

Theorem 1. *Let* α: $[0, \infty)\to[0; 1]$ *be monotonically decreasing with* $\alpha(t)<1$ $(t>0)$. *If for some positive integers* p, q *we have*

(1) $d(T^p x, T^q y)\leqq\alpha(\operatorname{diam}\{x, T^p x, y, T^q y\})\operatorname{diam}\{x, T^p x, y, T^q y\}$,

then T *has a fixed point* z *and* $T^n x_0\to z$ *for every* $x_0\in X$.

In particular, if $0\leqq\alpha<1$ *and*

$$d(T^p x, T^q y)\leqq\alpha \operatorname{diam}\{x, T^p x, y, T^q y\},$$

then T *has a fixed point (see* [1, *Theorem* 11]).

Proof. Let $x_0\in X$ and let $x_{n+1}=Tx_n$ $(n=0, 1, 2,\dots)$. If $Tx_0=x_0$ we are over, so let $Tx_0\neq x_0$. First we prove that

$$D_n=\operatorname{diam}\{x_0, x_1,\dots, x_n\}\leqq\frac{D_{p+q}}{1-\alpha(D_1)} \tag{2}$$

for every n. (2) is certainly satisfied for $n\leqq p+q$, thus it is enough to prove (2) for $n+1\geqq p+q$ assuming its validity for n. If $D_{n+1}\leqq D_n$, then we can use the induction hypothesis. If, however, $D_{n+1}>D_n$, then $D_{n+1}=d(x_{n+1}, x_i)$ for some $i\leqq n$. We state that here $i<p$. In fact, from $i\geqq p$ we get

$$d(x_{n+1}, x_i)\leqq\alpha(D_1)\max\{d(x_{n+1}, x_{n+1-q}), d(x_{n+1}, x_{i-p}), d(x_{n+1-q}, x_i),$$
$$d(x_{n+1-q}, x_{i-p}), d(x_{i-p}, x_i)\},$$

and so, by $D_{n+1}>D_n$,

$$d(x_{n+1}, x_i)<\max\{d(x_{n+1}, x_{n+1-q}), d(x_{n+1}, x_{i-p})\},$$

which contradicts the choice of i.

Thus, $i<p$ and we get

$$D_{n+1}\leqq d(x_i, x_{i+q})+d(x_{n+1}, x_{i+q})\leqq D_{p+q}+\alpha(D_1)\operatorname{diam}\{x_{n+1}, x_{n+1-p}, x_i, x_{i+q}\}$$
$$\leqq D_{p+q}+\alpha(D_1)D_{n+1}$$

by which

$$D_{n+1}\leqq\frac{D_{p+q}}{1-\alpha(D_1)},$$

and this completes the proof of (2).

Our next aim is to prove that $\{x_n\}$ is a Cauchy-sequence. Let $\varepsilon>0$ be arbitrary, k, l, n natural numbers, $n_1=n(p+q)+k$, $m_1=n(p+q)+l$ and for $1\leqq i\leqq n$ we define n_i and m_i inductively by

$$n_{i+1}, m_{i+1}\in\{n_i, n_i-p, m_i, m_i-q\},$$
$$d(x_{n_{i+1}}, x_{m_{i+1}})=\operatorname{diam}\{x_{n_i}, x_{n_i-p}, x_{m_i}, x_{m_i-q}\}=:d_i.$$

Using (1) we get

$$d(x_{n(p+q)+k}, x_{n(p+q)+l}) \leqq \alpha(d_1)d_1 \leqq \alpha(d_1)\alpha(d_2)d_2$$
$$\leqq \alpha(d_1)\alpha(d_2)\alpha(d_3)d_3 \leqq \dots \leqq \left(\prod_{j=1}^{i} \alpha(d_j)\right) d_i .$$

If, here, we have $d_i < \varepsilon$ for some $1 \leqq i \leqq n$, then we have also

$$d(x_{n(p+q)+k},\ x_{n(p+q)+l}) < \varepsilon.$$

If, however, for every $1 \leqq i \leqq n$ we have $d_i \geqq \varepsilon$, then

$$d(x_{n(p+q)+k},\ x_{n(p+q)+l}) \leqq (\alpha(\varepsilon))^n d_n \leqq K(\alpha(\varepsilon))^n$$

(see (2)) which is less than ε if $n \geqq N = N(\varepsilon)$. Since k and l were arbitrary, $\{x_n\}$ is indeed a Cauchy-sequence.

By completeness, there is a $z \in X$ with $x_n \to z$. As

$$d(T^p z,\ z) = \lim_{n\to\infty} d(T^p z,\ x_{n+q}) \leqq \alpha(d(z,\ T^p z)) \limsup_{n\to\infty} \operatorname{diam}\{z,\ T^p z,\ x_n,\ x_{n+q}\}$$
$$= \alpha(d(z,\ T^p z)) d(z,\ T^p z),$$

we have necessarily $T^p z = z$. If it were $Tz \neq z$ then for some $i<j$ we would have

$$0 < \operatorname{diam}\{z,\ Tz,\ T^2 z, \dots,\ T^{p-1} z\} = d(T^i z,\ T^j z) = d(T^p(T^i z),\ T^q(T^{(p-1)q} T^j z))$$
$$\leqq \alpha(d(T^i z,\ T^j z)) \cdot \operatorname{diam}\{z,\ Tz, \dots,\ T^{p-1} z\},$$

which is clearly impossible unless $z = Tz$; thus $Tz = z$ and the proof is complete.

Theorem 2. *Let* α_i: $(0,\ \infty) \to [0,\ 1)$ $(i = 1, \dots, 5)$, $\Sigma_1^5 \alpha_i(t) < 1$ *be monotonically decreasing functions and p a natural number. If for each* $x,\ y \in X$, $x \neq y$ *we have with* $d = d(x,\ y)$

$$d(T^p x,\ T^p y) \leqq \alpha_1(d) d(x,\ T^p x) + \alpha_2(d) d(y,\ T^p y) + \alpha_3(d) d(x,\ T^p y)$$
$$+ \alpha_4(d) d(y,\ T^p x) + \alpha_5(d) d(x,\ y),$$

then T has a fixed point z, and $T^n x_0 \to z$ *for every* $x_0 \in X$.

This is Theorem 7 in [1].

Theorem 3. *Let X be compact, T continuous. If for each* $x \in X$ *there is a natural number* $p(x)$ *such that for every* $y \in X$, $y \neq x$ *we have*

(3) $$d(T^{p(x)} x,\ T^{p(x)} y) < \max\{d(x,\ T^{p(x)} y),\ d(y,\ T^{p(x)} x),\ d(x,\ y)\},$$

then T has a fixed point z, and $T^n x_0 \to z$ *for every* $x_0 \in X$.

Proof. We prove Theorem 3 in some steps.

I. First we claim that Theorem 3 holds for finite X. In fact, in this case there is a point $x \in X$ and an n with $T^n x = x$, so — as we can restrict ourselves to the powers of x — we may assume that $X = \{x,\ Tx, \dots,\ T^{n-1} x\}$. If x was not a fixed point of T then we would have $n \geqq 2$. There is a point $x_0 \in X$ such that one of the members, say x_m, of the sequence $x_{r+1} := T^{p(x_r)}x_r$ $(r=0, 1, 2, \ldots)$ coincides with x_0. Let $d = \max\{d(y, x_i) | y \in X, 0 \leqq i \leqq m\}$. For some $y \in X$ and $0 \leqq i < m$, $d = d(y, x_i)$ and in the case $n \geqq 2$ clearly $d(y, x_i) > 0$. By the choice of the sequence $\{x_r\}_{r=0}^{m-1}$ $x_i = T^{p(x_j)}x_j$ for some x_j and there is also a $z \in X$ with $T^{(p(x_j)}z = y$. Since $y \neq x_i$, $z \neq x_j$ is satisfied as well, so (3) can be applied and we obtain

$$d = d(T^{(x_j)}x_j,\ T^{p(x_j)}z) < \max\{d(x_j, y),\ d(z, x_i),\ d(z, x_j)\}$$

which is a contradiction, since the right hand side is at most d. This contradiction arose from $n \geqq 2$, thus $n = 1$, i.e. x is a fixed point of T.

II. Let us turn to the general case. If $T^p x = x$ for some $x \in X$ and $p > 0$ then I can be applied, thus in the following we shall assume $T^p x \neq x$ for all $x \in X$ and $p > 0$ and we shall arrive at a contradiction.

Let us fix an x in (3), and let $\eta < 1/6 d(x, T^{p(x)}x)$. By the continuity of T there is a $0 < \delta < \eta$ such that for $d(x, z) < \delta$ we have $d(T^{p(x)}x, T^{p(x)}z) < \eta$ and so for z, $u \in U_\delta(x) := \{y \in X | d(y, z) < \delta\}$ we get

$$d(T^{p(x)}z,\ T^{p(x)}u) \leqq d(T^{p(x)}x,\ T^{p(x)}z) + d(T^{p(x)}x,\ T^{p(x)}u) < 2\eta$$

$$(4) \qquad = \frac{1}{2}4\eta \leqq \frac{1}{2}(d(x,\ T^{p(x)}x) - \eta - \eta) \leqq \frac{1}{2}(d(x,\ T^{p(x)}x) - d(x,\ u)$$

$$- d(T^{p(x)}x,\ T^{p(x)}z)) \leqq \frac{1}{2}d(u,\ T^{p(x)}z) \leqq \frac{1}{2}\max\{d(z,\ T^{p(x)}u),\ d(u,\ T^{p(x)}z),\ d(z,\ u)\}.$$

By (3) and by the continuity of T to each $y \neq x$ there is a neighbourhood U_y of y and a neighbourhood V_y of x as well as an $0 \leqq \alpha_y < 1$ such that for $z \in V_y$ and $u \in U_y$ we have

$$(5) \qquad d(T^{p(x)}z,\ T^{p(x)}u) \leqq \alpha_y \max\{d(z,\ T^{p(x)}u),\ d(u,\ T^{p(x)}z),\ d(u,\ z)\}.$$

The open sets U_y $(y \neq x)$ cover the compact set $X \backslash U_\delta(x)$, so we can choose a finite covering U_{y_i}, $1 \leqq i \leqq n$ of $X \backslash U_\delta(x)$. If $z \in U_\delta(x) \cap (\cap_{i=1}^n V_{y_i}) =: W_x$ then for $u \in U_\delta(x)$ by (4), and for $u \in X \backslash U_\delta(x)$ by (5) the inequality

$$d(T^{p(x)}z,\ T^{p(x)}u) \leqq \beta \max\{d(z,\ T^{p(x)}u),\ d(u,\ T^{p(x)}z),\ d(u,\ z)\}$$

holds with $\beta = \max\{1/2, \alpha_{y_1}, \alpha_{y_2}, \ldots, \alpha_{y_n}\} < 1$, i.e. we can choose as $p(z)$ the value $p(x)$ for all z in the neighbourhood W_x of x. The compactness of X guarantees that there is a finite covering $\cup_{i=1}^m W_{x_i}$ of X, and this yields at once: There is a finite set $P = \{p_1, p_2, \ldots, p_k\}$ of integers and an $\alpha < 1$ such that for every $x \in X$ there is a $p(x) \in P$ for which

$$(6) \qquad d(T^{p(x)}x,\ T^{p(x)}y) \leqq \alpha \max\{d(x,\ T^{p(x)}y),\ d(y,\ T^{p(x)}x),\ d(x,\ y)\}$$

holds true for every $y \in X$.

III. Now we proceed on by showing that (under the assumed hypothesis $T^p x \neq x$ for $x \in X$, $p \geqq 1$) a finite X^* and a $T^*: X^* \to X^*$ can be constructed so that they satisfy (3) and T^* has no fixed point. This is in contradiction with I, and this contradiction proves Theorem 3.

Let $\varepsilon>0$ and X_ε be an ε-net in X, i. e. $d(x, y)>\varepsilon$ for $x, y\in X_\varepsilon$, $x\neq y$ but for every $u\in X$ there is an $x_u\in X_\varepsilon$ with $d(x_u, u)\leqq\varepsilon$. Since X is compact, X_ε is finite. Let $T_\varepsilon: X_\varepsilon\to X_\varepsilon$ be any of the mappings which satisfy $T_\varepsilon y=x_{T_y}$, i. e. $T_\varepsilon y$ is one of the nearest points of X_ε to Ty. Clearly, $d(T_\varepsilon y, Ty)\leqq\varepsilon$ for all $y\in X_\varepsilon$. Since x is compact and T is continuous on X, T is also uniformly continuous and this gives easily the following: to every $\eta>0$ there is an $\varepsilon>0$ such that with X_ε, T_ε as above

$$d(T^{p_i}x, (T_\varepsilon)^{p_i}x)<\eta \tag{7}$$

for all $x\in X_\varepsilon$ and $p_i\in P$.

Our next aim is to choose η suitably. Let

$$\begin{aligned} d &= \inf_{x,y\in X} \max\{d(x,T^{p(x)}y),\ d(y,\ T^{p(x)}x),\ d(x,\ y)\},\\ d_1 &= \inf_{x\in X} d(x,\ Tx). \end{aligned} \tag{8}$$

There is a sequence $\{x_n, y_n\}$ with

$$d=\lim_{n\to\infty} \max\{d(x_n,\ T^{p(x_n)}y_n),\ d(y_n,\ T^{p(x_n)}x_n),\ d(x_n,\ y_n)\}$$

and, by compactness, we may assume $x_n\to x\in X$. Since P is finite, we may assume also that $p(x_n)=p\in P$ for all n. Now $d=0$ would imply $d(x, T^px)=0$, but we assumed $x\neq T^px$; thus $d>0$. A similar argument gives $d_1>0$.

After that let $\alpha<\beta<1$ and let $\eta>0$ be so small that

$$(\beta-\alpha)d-\eta\beta-2\eta>0 \tag{9}$$

be satisfied. To this η let $0<\varepsilon<(1/2)d_1$ be chosen according to (7) and let us consider $X^*=X_\varepsilon$, $T^*=T_\varepsilon$. by (6), (7), (8) and (9)

$$\begin{aligned} d((T_\varepsilon)^{p(x)}x,\ (T_\varepsilon)^{p(x)}y) &\leqq d(T^{p(x)}x,\ T^{p(x)}y)+2\eta\\ &\leqq\alpha \max\{d(x,\ T^{p(x)}y),\ d(y,\ T^{p(x)}x),\ d(x,y)\}+2\eta\\ &\leqq\beta \max\{d(x,\ (T)^{p(x)}y)-\eta,\ d(y,\ (T)^{p(x)}x)-\eta,\ d(x,\ y)\}\\ &\leqq\beta \max\{d(x,\ (T_\varepsilon)^{p(x)}y),\ d(y,\ (T_\varepsilon)^{p(x)}x),\ d(x,\ y)\} \quad (x,\ y\in X_\varepsilon), \end{aligned}$$

i. e. X_ε, T_ε satisfy the hypothesis of Theorem 3. We have to remark only that since

$$d(x,\ T_\varepsilon x)\geqq d(x,\ Tx)-d(Tx,\ T_\varepsilon x)\geqq d_1-\varepsilon>0 \quad (x\in X_\varepsilon),$$

T_ε does not have a fixed point.

Finally, let us prove that $T^nx_0\to z$ for every $x_0\in X$ where z is the fixed point of T. Applying (3) to $x=z$ we get with $p=p(z)$ $d(z, T^py)<d(z, y)$ for every $y\neq z$. Clearly, it is enough to prove that if $x_n=T^{n\cdot p}x_0$ then $x_n\to z$ as $n\to\infty$ (apply then this to $x_0, Tx_0, \dots, T^{p-1}x_0$ separately). By what we said above $d(z, x_{n+1}) < d(z, x_n)$. Let $a=\inf_n d(z, x_n)=\lim_{n\to\infty} d(z, x_n)$. We have to prove $a=0$. If we had $a>0$ then, by compactness, we would get a $z'\neq z$ and a sequence x_{n_k} with $x_{n_k}\to z'$ $(k\to\infty)$. The monotonicity of $\{d(z, x_n)\}_{n=0}^{\infty}$ gives then $a=d(z, z')=d(z, T^pz')$ which contradicts the above mentioned inequality.

Theorem 4. *If $\alpha<1$ and to every $x\in X$ there is a natural number $p(x)$ such that*

(10) $d(T^{p(x)}x,\ T^{p(x)}y)\leqq\alpha\max\{d(x,\ T^{p(x)}y),\ d(y,\ T^{p(x)}x),\ d(x,\ y)\}$

holds for every $y\in X$ then T has a fixed point z, and $T^n x_0\to z$ for all $x_0\in X$.

Proof. Let $x_0\in X$ and $x_n=T^n x_0$ $(n=1,\ 2,\dots)$. First we prove that x_n is a bounded sequence. Let $p=p(x_0)$ and

$$K=\max_{1\leqq i\leqq p} d(x_0,\ x_i)+\frac{2d(x_0,\ T^p x_0)}{1-\alpha}.$$

We prove that $d(x_0,\ x_n)\leqq K$. This is true for $n=1,\dots,p$ and we can proceed on by induction: assuming $n+1>p$, we have

$$d(x_0,\ x_{n+1})\leqq d(x_0,\ T^{p(x_0)}x_0)+d(T^{p(x_0)}x_0,\ T^{p(x_0)}x_{n+1-p})$$
$$\leqq d(x_0,\ T^p x_0)+\alpha\max\{d(x_0,\ T^p x_{n+1-p}),\ d(x_{n+1-p},\ T^p x_0),\ d(x_0,\ x_{n+1-p})\}$$
$$\leqq 2d(x_0,\ T^p x_0)+\alpha\max\{d(x_0,\ T^p x_{n+1-p}),\ d(x_0,\ x_{n+1-p})\}.$$

If here $d(x_0,\ T^p x_{n+1-p})\geqq d(x_0,\ x_{n+1-p})$ we get

$$d(x_0,\ x_{n+1})\leqq\frac{2d(x_0,\ T^p x_0)}{1-\alpha}\leqq K.$$

If, however, $d(x_0,\ x_{n+1-p})>d(x_0,\ T^p x_{n+1-p})$ then we can use the induction hypothesis and obtain

$$d(x_0,\ x_{n+1})\leqq 2d(x_0,\ T^p x_0)+\alpha d(x_0,\ x_{n+1-p})\leqq 2d(x_0,\ T^p x_0)+\alpha K\leqq K,$$

i.e. in any case $d(x_0,\ x_n)\leqq K$.

Next, we prove that $\{x_n\}$ is a Cauchy-sequence. Let $y_0=x_0$ and $y_{m+1}=T^{p(y_m)}y_m$ $(m=0,\ 1,\dots)$. Clearly, $\{y_m\}\subseteq\{x_n\}$, and by (10)

$$\limsup_{n\to\infty} d(y_{m+1},\ x_n)=\limsup_{n\to\infty} d(T^{p(y_m)}y_m,\ T^{p(y_m)}x_{n-p(y_m)})$$
$$\leqq\alpha\max\left\{\limsup_{n\to\infty} d(y_{m+1},\ x_n),\ \limsup_{n\to\infty} d(y_m,\ x_n)\right\}$$

by which

$$\limsup_{n\to\infty} d(y_{m+1},\ x_n)\leqq\alpha^m\limsup d(y_0,\ x_n)\leqq K\cdot\alpha^m<\varepsilon$$

if m is large enough; and this easily implies the Cauchy-character of $\{x_n\}$.

By completeness, $x_n\to z$ $(n\to\infty)$ for a $z\in X$. Using (10) we get

$$d(T^{p(z)}z,\ T^{p(z)}x_n)\leqq\alpha\max\{d(z,\ x_{n+p(z)}),\ d(x_n,\ T^{p(z)}z),\ d(x_n,\ z)\}$$

and letting here $n\to\infty$ we get that $T^{p(z)}z=z$. Applying Theorem 3 to the subspace $\{z,\ Tz,\ T^2z,\dots,\ T^{p(z)-1}z\}$ and to the restriction of T to this T-invariant set we get that z is a fixed point.

Finally, the last statement of Theorem 4 follows very easily from (10) (see also the end of the previous proof).

Theorem 5. *Let X be compact and T continuous. If for each $x, y\in X$, $x\neq y$ we have*

$$d(T^p x,\ T^p y)<d(x,\ y)$$

for a certain $p=p(x, y)$ then T has a fixed point.

Curiously enough, here $\{T^n x_0\}$ need not converge to the fixed point.

Proof. From the compactness of X, and from the continuity of T it follows that there exists a $z\in X$ with

$$a:=d(z,\ Tz)=\inf_{x\in X}(x,\ Tx).$$

If $a\neq 0$, then

$$a\leqq d(T^{p(z,Tz)}z,\ TT^{p(z,Tz)}z)=d(T^{p(z,Tz)}z,\ T^{p(z,Tz)}Tz)<d(z,\ Tz)=a$$

and this is a contradiction. So $a=0$, and z is a fixed point of T.

Theorem 6. *Suppose that there is a monotone decreasing function α: $(0, \infty)\to[0, 1)$ and for each $x\in X$ an integer $p(x)$ such that for all $y\in X$, $x\neq y$ we have*

$$d(T^{p(x)}x,\ T^{p(x)}y)\leqq\alpha(d(x,\ y))\cdot d(x,\ y).$$

Then T has a fixed point z and $T^n x_0\to z$ for every $x_0\in X$.

Proof. Let $x_0\in X$ and consider the sequence $\{x_0, Tx_0,\dots, T^n x_0,\dots\}$. If there exist integers i, j, such that $i\neq j$ and $T^i x_0=T^j x_0$, then this sequence is periodic. Let the period be $Z=\{x_0^*, x_1^*,\dots, x_n^*\}$, where

$$x_i^*=T^i x_0^* \quad \text{if} \quad i\neq n+1 \quad \text{and} \quad T^{n+1}x_0^*=x_0^*.$$

In this case the statement follows from Theorem 3.

a.) We claim that the sequence $\{T^n x_0\}$ is bounded. Let s be any fixed integer satisfying $0\leqq s<p_0=p(x_0)$, and let $d_0=d(x_0,\ T^{p_0}x_0)$. Then

$$d(T^{kp_0+s}x_0,\ T^{p_0}x_0)<\alpha(d(T^{(k-1)p_0+s}x_0,\ x_0))\cdot d(T^{(k-1)p_0+s}x_0,\ x_0).$$

If $d(T^{(k-1)p_0+s}x_0,\ x_0)<d_0$, then

$$(11)\qquad d(T^{kp_0+s}x_0,\ x_0)\leqq d(T^{kp_0+s}x_0,\ T^{p_0}x_0)+d(T^{p_0}x_0,\ x_0)<d_0+d_0=2d_0.$$

If, however, $d(T^{(k-1)p_0+s}x_0,\ x_0)\geqq d_0$, then

$$(12)\qquad \begin{aligned} d(T^{kp_0+s}x_0,\ x_0)&\leqq d(T^{kp_0+s}x_0,\ T^{p_0}x_0)+d(T^{p_0}x_0,\ x_0)\\ &\leqq\alpha(d_0)d(T^{(k-1)p_0+s}x_0,\ x_0)+d_0.\end{aligned}$$

On the ground of (11) and (12) an easy induction gives

$$d(T^{kp_0+s}x_0,\ x_0)\leqq \max\left\{\frac{2d_0}{1-\alpha(d_0)},\ \frac{d_0}{1-\alpha(d_0)}+d(T^s x_0,\ x_0)\right\},$$

which in turn implies

$$d(T^n x_0,\ x_0)\leqq \max\left\{\frac{2d_0}{1-\alpha(d_0)},\ \max_{s=0,\dots,p_0-1}\left\{\frac{d_0}{1-\alpha(d_0)}+d(T^s x_0,\ x_0)\right\}\right\}:=K$$

and this is the stated boundedness.

b.) Define the sequence $x_0,\ x_1=T^{p(x_0)}x_0,\dots,\ x_{n+1}=T^{p(x_n)}x_n,\dots$. We prove that $\{x_n\}$ is a Cauchy-sequence. To show this let $\varepsilon>0$, and n_0 be an integer, so that for each $n\geqq n_0 K\cdot(\alpha(\varepsilon))^n<\varepsilon$ be true. Let $n>n_0$ and m arbitrary integers. Then

$$\begin{aligned}
d(x_{n+m},\ x_n) &= d(T^{p(x_0)+\dots+p(x_{n-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ T^{p(x_{n-1})}x_{n-1})\\
&< \alpha(d(T^{p(x_0)+\dots+p(x_{n-2})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-1}))\cdot\\
&\quad \cdot d(T^{p(x_0)+\dots+p(x_{n-2})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-1})\\
&< \alpha(d(T^{p(x_0)+\dots+p(x_{n-2})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-1}))\cdot\\
&\quad \cdot\alpha(d(T^{p(x_0)+\dots+p(x_{n-3})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-2}))\cdot\\
&\quad \cdot d(T^{p(x_0)+\dots+p(x_{n-3})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-2})\\
&< \prod_{i=1}^{k}\alpha(d(T^{p(x_0)+\dots+p(x_{n-i-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-i}))\cdot\\
&\quad \cdot d(T^{p(x_0)+\dots+p(x_{n-k-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-k})
\end{aligned}\tag{13}$$

for each $k=1,\ 2,\dots,\ n$. If there exists an integer $1\leqq j\leqq n$ for which

$$d(T^{p(x_0)+\dots+p(x_{n-j-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-j})<\varepsilon$$

then (13) gives with $k=j: d(x_{n+m},\ x_n)<\varepsilon$. If, however, we have

$$d(T^{p(x_0)+\dots+p(x_{n-j-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-j})\geqq\varepsilon$$

for each $1\leqq j\leqq n$ then (13) gives with $k=n$ that

$$\begin{aligned}
d(x_{n+m},\ x_n) &< \prod_{i=1}^{n}\alpha(d(T^{p(x_0)+\dots+p(x_{n-i-1})+p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_{n-i}))\cdot\\
&\quad \cdot d(T^{p(x_n)+\dots+p(x_{n+m-1})}x_0,\ x_0)\leqq[\alpha(\varepsilon)]^n\cdot K<\varepsilon.
\end{aligned}$$

Thus, if $n>n_0$ then in any case $d(x_{n+m},\ x_n)<\varepsilon$ and this means that $\{x_n\}$ is a Cauchy-sequence. Since X is complete, $\{x_n\}$ has a limit z. All we have to prove is that z is a fixed point of T, and that $T^n x_0\to z$. Exactly as before we get that $\lim_{n\to\infty} d(T^m x_n,\ x_n)=0$ for each fixed m, and so, for each fixed $m: T^m x_n\to z$ $(n\to\infty)$. Let now $m=p(z)$. Since

$$\begin{aligned}
d(T^{p(z)}z,\ z) &\leqq d(T^{p(z)}z,\ T^{p(z)}x_n)+d(T^{p(z)}x_n,\ z)\\
&\leqq d(z,\ x_n)+d(T^{p(z)}x_n,\ z)=o(1),
\end{aligned}$$

we get that $T^{p(z)}z=z$ and so $Tz\neq z$ would imply $d(Tz, z)=d(T^{p(z)}Tz, T^{p(z)}z) < d(Tz, z)$, which is impossible.

Finally very easy consideration gives that $T^n x_0 \to z$ for every $x_0\in X$ (see the idea of this proof).

Theorem 7. *If there is an integer p such that for all $x, y\in X$, $x\neq y$*

$$d(T^p x, T^p y)<\max\{d(x, y), d(x, T^p x), d(y, T^p y), \frac{1}{2}[d(x, T^p y)+d(y, T^p x)]\},$$

and if T is continuous and $\{T^n x_0\}_{n=1}^{\infty}$ has a cluster point for some $x_0\in X$, then T has a fixed point z and $T^n x_0\to z$ as $n\to\infty$.

Proof. First we prove that T^p has a fixed point. For each $x\in X$, if $T^p x\neq x$, then

$$d(T^{2p}x, T^p x)<\max\{d(T^p x, x), d(T^p x, T^{2p}x), d(x, T^p x), \frac{1}{2}[d(T^p x, T^p x)+d(x, T^{2p}x)]\}.$$

The right side of the inequality must be either $d(T^p x, x)$ or $1/2[d(T^p x, T^p x)+d(x, T^{2p}x)]=1/2d(x, T^{2p}x)$. In the second case

$$d(T^{2p}x, T^p x)<\frac{1}{2}d(x, T^{2p}x)\leqq\frac{1}{2}[d(x, T^p x)+d(T^p x, T^{2p}x)]$$

and from this

$$d(T^{2p}x, T^p x)<d(T^p x, x) \tag{14}$$

follows.

Thus, for each $x\in X$, $x\neq T^p x$, (14) holds. Since the sequence $\{T^n x_0\}$ has a cluster point, there exists an integer $0\leqq s\leqq p$, such that the sequence $\{y_0=T^s x_0, \dots, y_n=T^{np+s}x_0, \dots\}$ also has a cluster point. If there exists an integer k such that $y_{k+1}=y_k$, then y_k is a fixed point of T^p. If, however, for each n, $y_n\neq y_{n+1}$, then (14) gives

$$d(y_{n+1}, y_n)=d(T^{2p}y_{n-1}, T^p y_{n-1})<d(T^p y_{n-1}, y_{n-1})=d(y_n, y_{n-1}),$$

i.e. the sequence $\{d(y_{n+1}, y_n)\}$ is monotone decreasing. Call its limit d. Let z be the cluster point of $\{y_n\}$ and let $\{y_{n_k}\}$ be a subsequence which converges to z. Clearly

$$\left.\begin{aligned} d(y_{n_k+1}, y_{n_k})&\to d(T^p z, z)=d \\ d(y_{n_k+2}, y_{n_k+1})&\to d(T^{2p}z, T^p z)=d, \end{aligned}\right. \qquad (k\to\infty)$$

so taking into account (14), we get $d=0$, by which z is a fixed point of T^p.

Since

$$d(z, y_n)=d(T^p z, T^p y_{n-1})<\max\{d(z, y_{n-1}), d(z, T^p z), d(y_{n-1}, T^p y_{n-1}), \frac{1}{2}[d(z, T^p y_{n-1})+d(y_{n-1}, T^p z)]\}$$

it follows that

$$d(z,\ y_n) < \max\{d(z,\ y_{n-1}),\ d(y_{n-1},\ y_n)\}.$$

By induction (take also into account the monotonicity of $\{d(y_{n-1},\ y_n)\}$) we get

$$d(z,\ y_n) < \max\{d(z,\ y_{n-k}),\ d(y_{n-k},\ y_{n-k+1})\}$$

for each $k=1, 2, \dots, n$, and so $d(z,\ y_{n_k}) \to 0$ $(k\to\infty)$ and $d(y_{n-1},\ y_n)\to 0$ $(n\to\infty)$ give $\lim_{n\to\infty} y_n = z$.

Now let $d_1 = \sup_{i,j=0,\dots,p-1} d(T^i z,\ T^j z)$. If $d_1 \neq 0$ then there exist integers i, j such that

$$d_1 = d(T^i z,\ T^j z) = d(T^{i+p} z,\ T^{j+p} z) < \max\{d(T^i z,\ T^j z),\ d(T^i z,\ T^{i+p} z),$$
$$d(T^j z,\ T^{j+p} z),\ \frac{1}{2}[d(T^i z,\ T^{j+p} z) + d(T^j z,\ T^{i+p} z)]\} = d(T^i z,\ T^j z) = d_1$$

which is a contradiction, so $d_1 = 0$ and $Tz = z$. Now $\lim_{n\to\infty} y_n = z$ and the continuity of T yield easily that $\lim_{n\to\infty} T^n x_0 = z$.

Theorem 8. *Let X be compact and T continuous. If there is an integer p such that for all $x,\ y \in X,\ x \neq y$*

$$d(T^p x,\ T^p y) < \operatorname{diam}\{x,\ T^p x,\ y,\ T^p y\}$$

then T has a fixed point z, and for every $x \in X$ $\lim_{n\to\infty} T^n x = z$.

Proof. Clearly, it is sufficient to show that there exists a fixed point z of T^p, and for every $x_0 \in X$ $\lim_{n\to\infty} T^{pn} x_0 = z$ (see Theorem 1).

Now, define the sequence of sets as follows

$$X_0 = X, \quad X_{n+1} = T^p X_n \ (n=1,\ 2, \dots).$$

From the continuity of T and from the compactness of X it follows that X_n is compact $(n=1, 2, \dots)$, $X_{n+1} \subseteq X_n$, thus $\emptyset \neq Z := \cap_{n=1}^{\infty} X_n$, $T^p Z \subseteq Z$ furthermore, Z is compact. Let $z_0 \in Z$ be arbitrary. There exists a sequence $\{x_n\}$, such that $x_n \in X_n$, and $z_0 = T^p x_n$ $n=1, 2, \dots$. Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$ and let $\lim_{k\to\infty} x_{n_k} = x_0$. Then, clearly, $x_0 \in Z$ and $T^p x_0 = z_0$. This proves

$$T^p Z = Z. \tag{15}$$

If $\operatorname{diam} Z \neq 0$, then there are points $z_1,\ z_2,\ z_3,\ z_4 \in Z$, such that

$$\operatorname{diam} Z = d(z_1,\ z_2), \quad z_1 = T^p z_3, \quad z_2 = T^p z_4,$$

but then

$$\operatorname{diam} Z = d(z_1,\ z_2) = d(T^p z_3,\ T^p z_4) < \operatorname{diam}\{z_3,\ T^p z_3,\ z_4,\ T^p z_4\} \leq \operatorname{diam} Z$$

would hold, which is a contradiction, so $\operatorname{diam} Z = 0$.

What we proved is that

$$Z = \{z\} = \cap_{n=1}^{\infty} X_n,$$

thus $\lim_{n\to\infty} T^{pn} x_0 = z$ for every $x_0 \in X$ and (15) gives that z is a fixed point of T^p.

5. Examples

Example 1. *There is a compact metric space X, a continuous mapping $T:X\to X$ and an $\alpha<1/2$ such that*

(i) *for every $x\in X$ there exists a $p(x)$ satisfying*

$$d(T^{p(x)}x,\ T^{p(x)}y)\leqq\alpha[d(x,\ T^{p(x)}x)+d(y,\ T^{p(x)}y)]$$

for all $y\in X$,

(ii) *T does not have any fixed point.*

Proof. Let $X=\{x_1,\dots,x_7\}$ be a set, and define the symmetric function (not metric!) $d_0(x, y)$ from $X\times X$ into the integers as follows. Let $d_0(x, x)=0$ for all $x\in X$; if x or y is contained in the set $\{x_1, x_2, x_3\}$, then let $d_0(x, y)$ be the integer shown by Fig. 2; finally, if x and y are not contained in $\{x_1, x_2, x_3\}$, then let $d_0(x, y)=10$. Define the map T from X into itself by

$$Tx_i=x_{i+1}\quad\text{if}\quad i\neq 7,\quad\text{and}\quad Tx_7=x_1.$$

Let $p(x_1)=1$, $p(x_2)=1$, $p(x_3)=5$, $p(x_4)=4$, $p(x_5)=5$, $p(x_6)=3$, $p(x_7)=2$. We claim that for each x_i, x_j

$$(1)\qquad d_0(T^{p(x_i)}x_i,\ T^{p(x_i)}x_j)<\max\{d_0(x_i,\ T^{p(x_i)}x_i),\ d_0(x_j,\ T^{p(x_i)}x_j)\}.$$

Remark that for each $i=1,\dots,7$ $T^{p(x_i)}x_i\in\{x_1, x_2, x_3\}$, so it is sufficient to show the inequality (1) for such x_j's, for which x_j or $T^{p(x_i)}x_j$ is contained in $\{x_1, x_2, x_3\}$, since the left side of (1) is always less than 10. We shall prove (1) only for $i=4$, the other cases can be similarly proved. By definition $T^{p(x_4)}x_4=x_1$, thus we have the table:

x_j	the left side of (1)	the right side of (1)
x_1	$d_0(x_1, x_5)=3$	$d_0(x_1, x_4)=4$
x_2	$d_0(x_1, x_6)=3$	$d_0(x_2, x_6)=6$
x_3	$d_0(x_1, x_7)=5$	$d_0(x_3, x_7)=6$
x_4	$d_0(x_1, x_1)=0$	$d_0(x_1, x_4)=4$
x_5	$d_0(x_1, x_2)=3$	$d_0(x_5, x_2)=5$
x_6	$d_0(x_1, x_3)=4$	$d_0(x_3, x_6)=7$
x_7	$d_0(x_1, x_4)=4$	$d_0(x_4, x_7)=10$

which proves our assertion. Now let

$$d_1(x, y)=\begin{cases}2^{d_0(x,y)} & \text{if } x\neq y\\ 0 & \text{if } x=y\end{cases}\qquad x, y\in X.$$

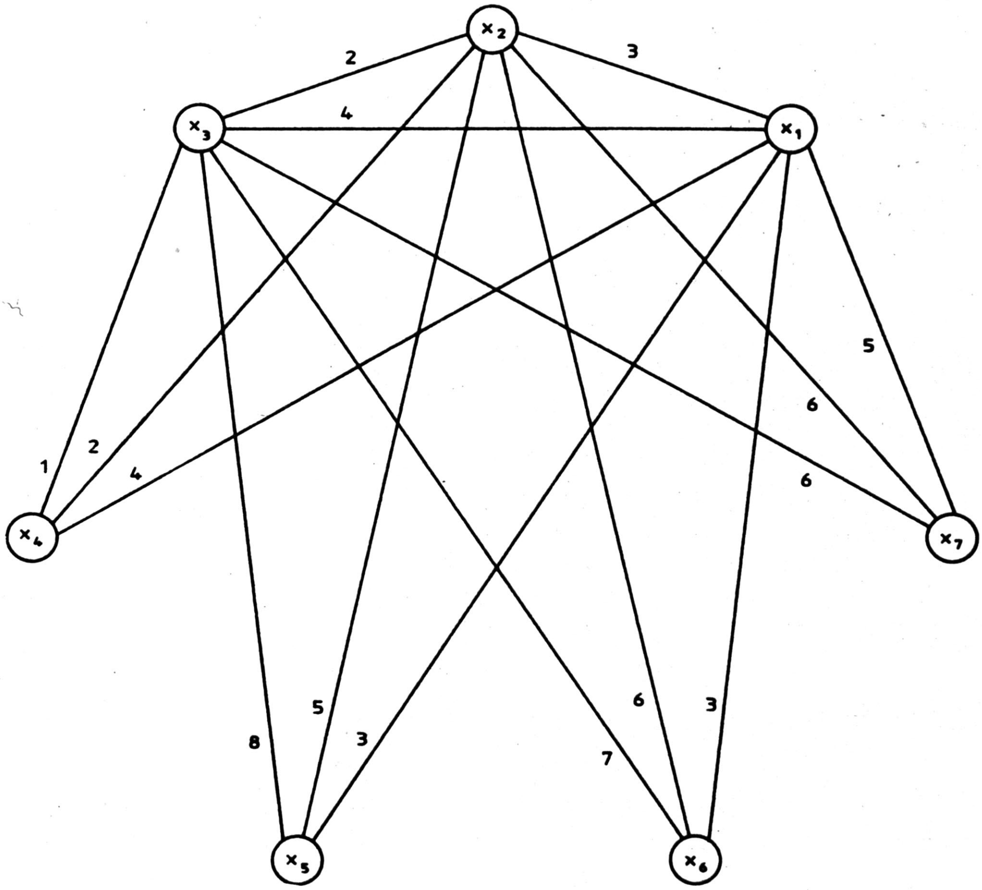

Fig. 2

For each i, j

$$d_1(T^{p(x_i)}x_i,\ T^{p(x_i)}x_j)=\begin{Bmatrix} 2^{d_0(T^{p(x_i)}x_i,\,T^{p(x_i)}x_j)} \\ 0 \end{Bmatrix} \leqq 2^{\max\{d_0(x_i,T^{p(x_i)}x_i),\,d_0(T^{p(x_i)}x_j,\,x_j)\}-1} \tag{2}$$

$$<\frac{1}{2}[d_1(x_i,\ T^{p(x_i)}x_i)+d_1(x_j,\ T^{p(x_i)}x_j)],$$

where we used (1) and the fact that $d_0(x, y)$ is an integer for all x, y. Let K be an integer with $K>\max_{x,y\in X} d_1(x, y)$ and let $d(x, y)=d_1(x, y)+K$ if $x\neq y$, and $d(x, x)=0$ $(x, y\in X)$. Then $d(x, y)$ is already a metric on X, and (2) gives for each i, j

$$d(T^{p(x_i)}x_i,\ T^{p(x_i)}x_j) < \frac{1}{2}[d(x_i,\ T^{p(x_i)}x_i) + d(x_j,\ T^{p(x_i)}x_j)]$$

(take into account that $1 \leqq p(x_i) < 7$ and thus $d_1(x_j,\ T^{p(x_i)}x_j) \neq 0$ for all i and j). Since X is finite, there exists a real number $0 < \alpha < 1/2$ such that for each i, j

$$d(T^{p(x_i)}x_i,\ T^{p(x_i)}x_j) \leqq \alpha[d(x_i,\ T^{p(x_i)}x_i) + d(x_j,\ T^{p(x_i)}x_j)].$$

Thus, T satisfies (i), and obviously T has no fixed point.

Example 2. This is a very simple one: $X = \{x_0,\ x_1\}$, $Tx_0 = x_1$, $Tx_1 = x_0$, $d(x_0,\ x_1) = 1$. Then T has no fixed point and if $p = 1$, $q = 2$, $x \neq y$ then we have $d(T^p x,\ T^q y) = 0$.

Example 3. *There is a compact metric space X, a continuous mapping $T: X \to X$ and an $0 < \alpha < 1/2$ such that*

(*i*) *for all x, $y \in X$ there exists a $p = p(x, y)$ such that*

$$d(T^p x,\ T^p y) \leqq \alpha[d(x,\ T^p y) + d(y,\ T^p x)],$$

(*ii*) *T does not have any fixed point.*

Proof. Let $X = \{x_0, \ldots, x_5\}$ be a set, and $d_0(x, y)$ be a function from $X \times X$ into the integers as is given in Fig. 3., with $d_0(x, x) = 0$ (this d_0 is not a metric yet!). In the following we use the index in x_i modulo 6. Let T be a mapping from X into X given by $Tx_i = x_{i+1}$ for each i, and let $p(x, y) = p(y, x)$, $p(x, x) = 1$,

$$p(x_0,\ x_1) = 3 \qquad p(x_1,\ x_2) = 2$$

$$p(x_0,\ x_2) = 3 \qquad p(x_1,\ x_3) = 1$$

$$p(x_0,\ x_3) = 2 \qquad p(x_1,\ x_4) = 2$$

$$p(x_i,\ x_{i+j}) = p(x_1,\ x_{1+j}) \text{ if } i \text{ is odd and } j = 1,\ 2,\ 3$$

$$p(x_i,\ x_{i+j}) = p(x_0,\ x_j) \text{ if } i \text{ is even and } j = 1,\ 2,\ 3.$$

One can easily see that this is a correct definition and for each i, j

$$(3)\ d_0(T^{p(x_i, x_j)}x_i,\ T^{p(x_i, x_j)}x_j) < \frac{1}{2}[d_0(x_j,\ T^{p(x_i, x_j)}x_i) + d_0(x_i,\ T^{p(x_i, x_j)}x_j)].$$

Notice that in each case neither term on the right side of (3) vanishes. Now let $d(x, y) = d_0(x, y) + 101$ if $x \neq y$, and $d(x, x) = 0$ $(x, y \in X)$, where $101 > \max_{x, y \in X} d_0(x, y)$. Then $d(x, y)$ is a metric on X, and for each i, j (3) holds as well for d. So there exists a real number $0 < \alpha < 1/2$, such that for every i, j

$$d(T^{p(x_i, x_j)}x_i,\ T^{p(x_i, x_j)}x_j) \leqq \alpha[d(x_j,\ T^{p(x_i, x_j)}x_i) + d(x_i,\ T^{p(x_i, x_j)}x_j)],$$

i.e. T satisfies (i), and since T does not have a fixed point, our proof is over.

Example 4. *There is a complete metric space X and a continuous mapping $T: X \to X$ such that*

(*i*) *$\{T^n x_0\}$ has a cluster point for some $x_0 \in X$,*

(*ii*) *to every x, $y \in X$ there is a $p = p(x, y)$ with*

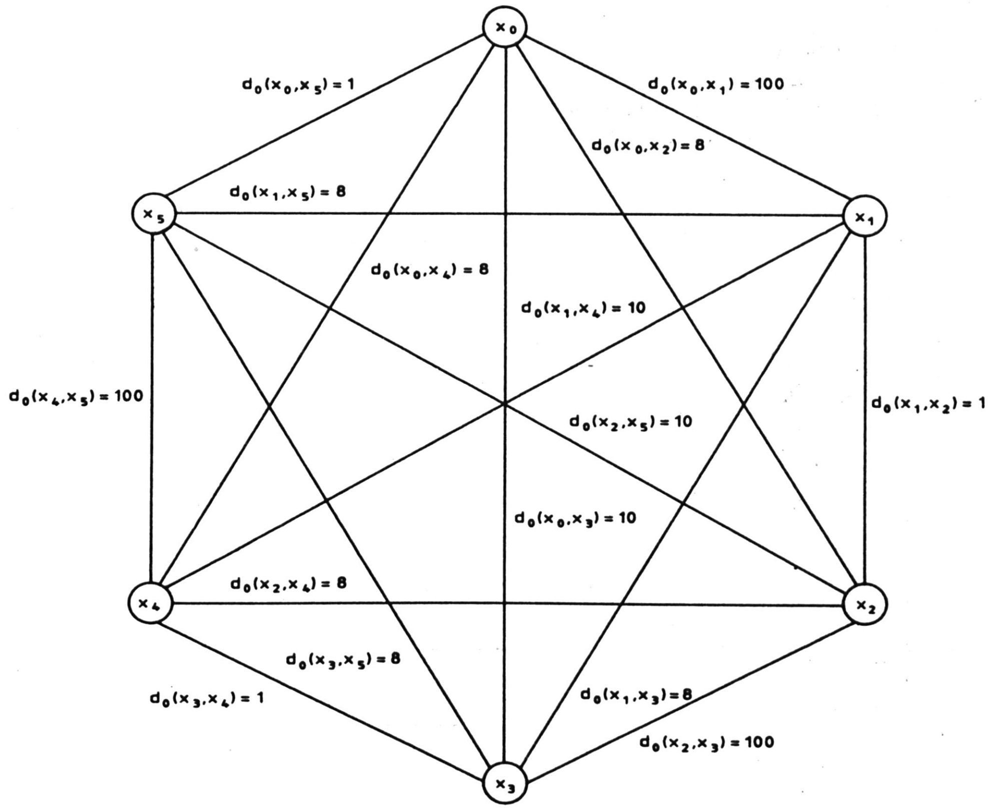

Fig. 3

$$d(T^p x,\ T^p y) \leqq \frac{1}{2} d(x,\ y),$$

(iii) T does not have any fixed point.

Proof. Let $X = \cup_{i=0}^{\infty} Y_i$ be a subspace of R^2 where

$$Y_0 = \{y_1^{(0)},\ y_2^{(0)}, \dots\}, \qquad Y_i = \{y_1^{(i)},\ y_2^{(i)}, \dots, y_{m_i}^{(i)}\} \qquad (i = 1,\ 2, \dots),$$

$$y_k^{(0)} = \left(0;\ 1 + \frac{1}{2} + \dots + \frac{1}{k}\right),\quad y_k^{(i)} = \left(\frac{1}{2^{i+k}},\ 1 + \frac{1}{2} + \dots + \frac{1}{k}\right) \qquad (i,\ k = 1,\ 2, \dots),$$

and let $m_1 = 1$, $m_{r+1} = 2^{2^{m_r}}$, $r = 1, 2, \dots$ and d is the usual Euclidean distance on R^2 (see Fig. 4.). Let T be defined by

$$Ty_k^{(0)} = y_{k+1}^{(0)} \quad (k = 1,\ 2, \dots),$$

$$Ty_k^{(i)} = \begin{cases} y_{k+1}^{(i)} & \text{if } k < m_i \\ y_1^{(i+1)} & \text{if } k = m_i. \end{cases}$$

Obviously
1. X is complete;
2. T is continuous;
3. T does not have any fixed point;
4. y_1^0 is a cluster point of $\{T^n y_1^{(1)}\}$.

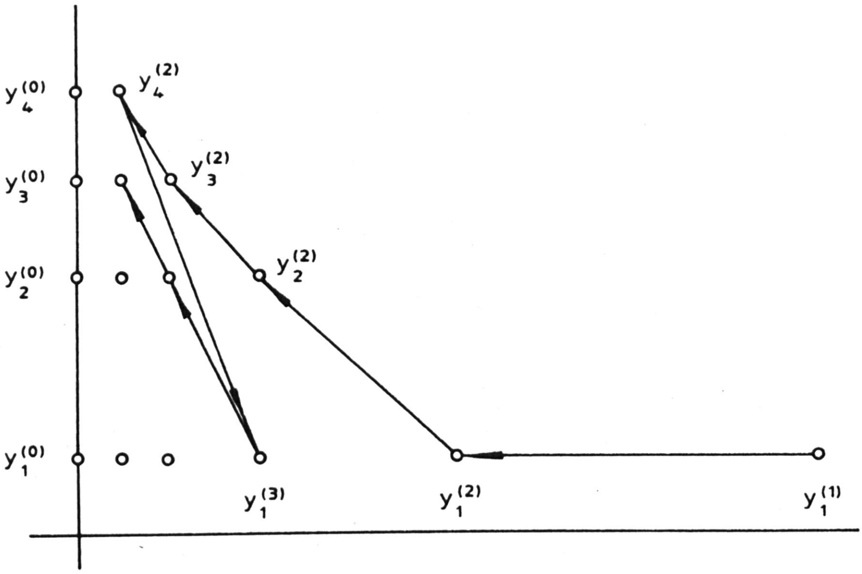

Fig. 4

Let now $x, y \in \cup_{i=1}^{\infty} Y_i$. Then for some r $x, y \in \cup_{i=1}^{r} Y_i$ and if r is the smallest such number then obviously

$$d(x, y) \geqq \frac{1}{2^{m_r}} \tag{4}$$

and we set $p(x, y) = m_{r+1} = 2^{2^{m_r}}$. Then with $p = p(x, y)$ we have with $m_r^* = \Sigma_{j=1}^{r} m_j \leqq 2m_r$

$$T^p x, \ T^p y \in \{y_i^{(r+1)}\}_{i=m_{r+1}-m_r^*-1}^{m_{r+1}}$$

and so

$$d(T^p x, \ T^p y) < \frac{1}{2^{m_{r+1}}} + \log \frac{m_{r+1}}{m_{r+1} - m_r^* - 1} < \frac{1}{2^{m_r+1}} \leqq \frac{1}{2} d(x, y)$$

(see (4)).

If one or two of x, y belongs to Y_0 then the argument is similar.

Example 5. *This is the following simple one*: $X = \{x_i\}_{i=1}^{\infty}$, $d(x_i, x_j) = 1 + 1/i \cdot j$ $(i \neq j)$ *and* $Tx_i = x_{i+1}$ $(i = 1, 2, \ldots)$.

Example 6. *There is a complete metric space X and a continuous mapping $T: X \to X$ such that*

(i) $\{T^n x_0\}_{n=1}^{\infty}$ *has a cluster point for some* $x_0 \in X$,

(ii) for each $x, y \in X$, $x \neq y$

$$d(Tx, Ty) < \operatorname{diam}\{x, y, Tx, Ty\},$$

(iii) T *does not have a fixed point.*

Proof. Let

$$X = W \cup (\cup_{i=1}^{\infty} Y_i) \cup (\cup_{i=1}^{\infty} Z_i) \cup Y_\infty \cup Z_\infty,$$

$$W = \{\dots, w_{-n}, \dots, w_{-2}, w_2, w_3, \dots, w_n, \dots\},$$

$$Y_\infty = \{y_j^{(\infty)}\}_{j=1}^{\infty},\ Z_\infty = \{z_j^{(\infty)}\}_{j=1}^{\infty},\ Y_i = \{y_j^{(i)}\}_{j=1}^{i},\ Z_i = \{z_j^{(i)}\}_{j=1}^{i}.$$

Let

$$S_j = \begin{cases} \{w_j\} \cup \{y_j^{(i)} | i \geqq j\} \cup \{y_j^{(\infty)}\} & \text{if } j \geqq 1 \\ \{w_j\} \cup \{z^{(i)}_{-j} | i \geqq -j\} \cup \{z^{\infty}_{-j}\} & \text{if } j \leqq -1, \end{cases}$$

where w_j for $j = \pm 1$ is understood to be the empty set. Clearly, $\cup_{j \neq 0} S_j = X$. S_j can be taught as the "j-th lelvel" of X (see Fig. 5.).

Before giving the metric d we define T. Let

$Ty_j^{(i)} = y_{j+1}^{(i)}$ $(j < i)$; $\quad Ty_i^{(i)} = w_{i+1}$ $(i = 1, 2, \dots)$;

$Tw_i = z_1^{(i-1)}$ $(i = 2, 3, \dots)$; $\quad Tz_j^{(i)} = z_{j+1}^{(i)}$ $(j < i)$;

$Tz_i^{(i)} = w_{-i-1}$ $(i = 1, 2, \dots)$; $\quad Tw_{-i} = y_1^{(i)}$ $(i = 2, 3, \dots)$;

$Ty_i^{(\infty)} = y_{i+1}^{(\infty)}$; $\quad Tz_i^{(\infty)} = z_{i+1}^{(\infty)}$ $(i = 1, 2, \dots)$.

The action of T is shown in Fig. 5.
After these we define $d(x, y)$ for each $x, y \in X$. Let $x_1 = y_1^{(1)}$, and $x_{n+1} = Tx_n$ for $n = 1, 2, \dots$.

Clearly, $\{x_n\}_{n=1}^{\infty} = X \setminus (Y_\infty \cup Z_\infty)$. Let now $x, y \in X$.

a.) If $x \in W$, $y \in Y_\infty \cup Z_\infty$, let $d(x, y) = 3$.

b.) If $x \in W$, $y \notin Y_\infty \cup Z_\infty$, say $x = x_n$, $y = x_m$, $m \neq n$, let $d(z, y) = 3 + 1/m \cdot n$.

c.) If $x \in S_i \setminus W$, $y \in S_j \setminus W$ and $i \neq j$, let

$$d(x, y) = 2 + \frac{1}{\max\{|i|, |j|\} + 1}.$$

d.) If $x, y \in S_k \setminus W$, say $x = y_k^{(i)}$, $y = y_k^{(j)}$ or $x \in z_k^{(i)}$, $y = z_k^{(j)}$ $(i, j = 1, 2, \dots, \infty\ i \neq j)$, let $d(x, y) = |1/i + 1 - 1/j + 1|$.

By d.) each (S_k, d) is a complete metric space (on $S_k d$ can be thought of as the Euclidean metric on S_k imbedded into R^2 as is suggested by Fig. 5) and since the distance between points of different S_k's is at least 2 and is at most 4 (see a.), b.), c.)), we get that (X, d) is complete.

Our definitions give easily that T is continuous, T has no fixed point and $y_1^{(\infty)}$ is a cluster point of $T^n y_1^{(1)}$ (since $y_1^{(i)} \in \{T^n y_1^{(1)}\}_{n=1}^{\infty}$ for each $i \geqq 1$).

Finally, (ii) can be checked by straightforward verification.

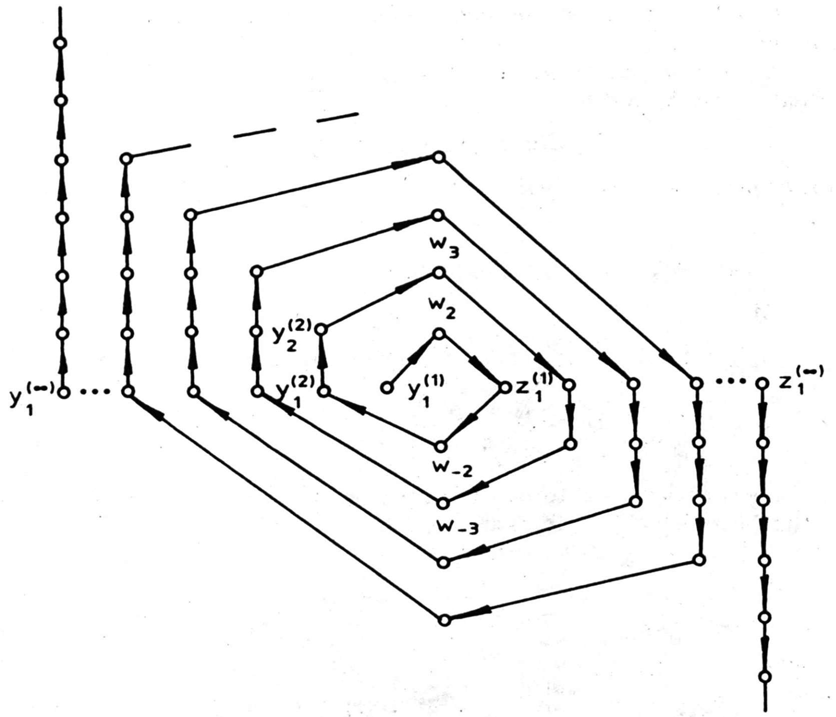

Fig. 5

Example 7. *There is a complete metric space X and a uniformly continuous mapping $T: X \to X$ such that*

(i) to every $x, y \in X$, $x \neq y$ there is a $p = p(x, y)$ with

$$d(T^p x, T^p y) < \frac{1}{2} d(x, y),$$

(ii) T does not have any fixed point.

Proof. Let $X = \{x_i\}_{i=1}^{\infty} \subseteq R$ be defined by $x_i = \Sigma_{j=1}^{i} 1/j$, and let T be the mapping $Tx_i = x_{i+1}$ $(i = 1, 2, \ldots)$. This X and T clearly satisfy all of our requirements.

References

1. B. E. Rhoades. A comparison of various definitions of contractive mappings. *Trans. of the Amer. Math. Soc.*, **226**, 1977, 257-290.

Attila József University
Bolyai Institute of Mathematics
Szeged
Aradi vértanúk tere 1.
H-6720 HUNGARY

Received 16.05.1989