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On the Best Approximation of Riemann Integrable Functions by Trigonometric Polynomials

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For spaces of (2π-periodic) Riemann integrable functions this paper suggests a concept for a best approximation by (trigonometric) polynomials which fits together with recent results for linear approximation processes. Indeed, in terms of the τ-modulus direct (Jackson-type) as well as inverse (Bernstein-type) theorems are established so that a constructive theory of functions may be developed in these spaces. As a consequence of previous quantitative extensions of the uniform boundedness principle it is shown that the notion of a best approximation, induced by the standard Lebesgue theory, would be too small for inverse theorems to hold true. A final section deals with the problem of a direct comparison of the present concept with the well-known one of best onesided approximation.

1. Introduction

Continuing our previous investigations on a sequential convergence in the space of Riemann integrable functions (see [2]), it is the aim of this paper to develop a notion of best approximation by polynomials which fits together with recent results (see [7, 15]) for linear approximation processes (e. g., for the delayed means of de La Vallée Poussin).

To this end, let $B_{2\pi}$, $R_{2\pi}$ or $C_{2\pi}$ be the spaces of (complex-valued) functions, everywhere defined and 2π-periodic on the real axis $\mathbb{R}$, which are bounded (with $\|f\|_B := \sup\{|f(u)| : u \in \mathbb{R}\}$), Riemann integrable over $[-\pi, \pi]$ or continuous, respectively. For $f \in B_{2\pi}$, $\delta > 0$ set

$$(1.1)\quad \|f\|_\delta := \overline{\int} M(f, x, \delta)\,dx, \qquad M(f, x, \delta) := \sup\{|f(y)| : y \in U_\delta(x)\},$$

where $U_\delta(x) := [x-\delta, x+\delta]$, and $\overline{\int} g := \overline{\int}_{-\pi}^{\pi} g(u)\,du$ denotes the upper Riemann integral of $g \in B_{2\pi}$. Obviously, $\|f\|_\delta$ defines a norm on $B_{2\pi}$ for each $\delta > 0$ satisfying

$$(1.2)\qquad \|f\|_{2\delta} \leqq 2\,\|f\|_\delta, \quad \|f\|_{\lambda\delta} \leqq 2(1+\lambda)\,\|f\|_\delta \quad (\lambda > 0).$$

Let $\mathbb{N}$ be the set of natural numbers and Π_n be the set of trigonometric polynomials of degree less than or equal to $n \in \mathbb{N} \cup \{0\}$. The best approximation by polynomials to be discussed here is then defined on $B_{2\pi}$ by

$$(1.3)\qquad E_n[f, R] := \inf\{\|f - p_n\|_{1/n} : p_n \in \Pi_n\} \quad (f \in B_{2\pi})$$

with the interpretation $\|f-p_n\|_{1/n}:=2\pi\|f-p_n\|_B$ for $n=0$. Note that this definition does not involve any additional constraints, in contrast to the notion of a best onesided approximation (cf. (4.1)). Among the elementary properties needed let us mention the following ones: For $f,\ g\in B_{2\pi}$ and scalar α

$$(1.4)\quad E_n[f+g,\ R]\leqq E_n[f,\ R]+E_n[g,\ R],\quad E_n[\alpha f,\ R]=|a|E_n[f,\ R],$$
$$E_{n+1}[f,\ R]\leqq E_n[f,\ R],\quad E_n[f+p,\ R]=E_n[f,\ R]\quad (p\in\Pi_n),$$

and the functional E_n is bounded on $B_{2\pi}$ in the sense that for each $n\in\mathrm{N}\cup\{0\}$ one has $E_n[f,\ R]\leqq\|f\|_{1/n}$ for all $f\in B_{2\pi}$.

Since $B_{2\pi}$ under $\|\cdot\|_{1/n}$ is a normed linear space, for each $f\in B_{2\pi}$ there exist elements of best approximation $p_n^*\in\Pi_n$ thus

$$(1.5)\qquad E_n[f,\ R]=\|f-p_n^*\|_{1/n}\qquad (n\in\mathrm{N}\cup\{0\}).$$

A first important property of the concept (1.3) is the following theorem of Weierstraß-type: For $f\in B_{2\pi}$ there holds true

$$(1.6)\qquad f\in \mathrm{R}_{2\pi}\leftrightarrow E_n[f,\ R]=o(1)\qquad (n\to\infty),$$

analogously to the classical Weierstraß result which for $f\in B_{2\pi}$ states:

$$f\in C_{2\pi}\leftrightarrow E_n[f,\ C]=o(1)\qquad (n\to\infty),$$

where as usual one defines

$$(1.7)\qquad E_n[f,\ C]:=\inf\{\|f-p_n\|_B : p_n\in\Pi_n\}.$$

Indeed, if for $f\in B_{2\pi}$ one has $E_n[f,\ R]=o(1)$, then $f\in R_{2\pi}$ as a consequence of Riemann's criterion since $(p_n\in\Pi_n\subset R_{2\pi})$ with lower Riemann integral

$$\bar{\int} f-\underline{\int} f=\bar{\int}(f-p_n)+\bar{\int}(p_n-f)\leqq 2\bar{\int}|f-p_n|\leqq 2\|f-p_n\|_{1/n}.$$

On the other hand, if $f\in R_{2\pi}$, one may apply the Jackson-type Theorem 3.1 (together with (1.9)).

To develop a constructive theory of functions with regard to the quantity (1.3), it is essential that a suitable structural counterpart is already available in terms of the τ-modulus $(f\in B_{2\pi},\ \delta>0,\ r\in\mathrm{N})$

$$(1.8)\qquad \tau_r(f,\ \delta):=\bar{\int}\omega_r(f,\ x,\ \delta)\,dx,$$
$$\omega_r(f,\ x,\ \delta):=\sup\{|\Delta_h^r f(t)| : t,\ t+rh\in U_\delta(x)\},$$
$$\Delta_h^r f(t):=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f(t+jh),$$

introduced by the Bulgarian school of approximation. Indeed, to mention a first property in this connection, parallel to (1.6) one has that for $f\in B_{2\pi}$

$$(1.9)\qquad f\in R_{2\pi}\leftrightarrow \tau_1(f,\ \delta)=o(1)\qquad (\delta\to 0+).$$

For further details and elementary properties of $\tau_r(f, \delta)$ see [12]. There is a close relation of the τ-modulus to the δ-norm (1.1), underlying the definition of $E_n[f, R]$: If a K-functional is defined for $\delta>0$, $r\in\mathbf{N}$ by

$$(1.10)\qquad K_r(f, \delta):=\inf\{\|f-g\|_\delta+\delta^r\int_{-\pi}^{\pi}|g^{(r)}(u)|du : g\in ACR_{2\pi}^r\},$$

where $ACR_{2\pi}^r$ is the set of r-times absolutely continuous functions with rth derivative in $R_{2\pi}$, then there holds true the equivalence (cf. [6, 11])

$$(1.11)\qquad C_r'\tau_r(f, \delta)\leqq K_r(f, \delta)\leqq C_r''\tau_r(f, \delta).$$

With the aid of these concepts one may now work out a constructive theory of functions for the space $R_{2\pi}$, parallel to the classical theory of Jackson—Bernstein for $C_{2\pi}$. See Section 3 for details.

There are of course further candidates for the notion of a best approximation on $R_{2\pi}$. For example, one may consider the quantity $E_n[f, L^1]$ (cf. (2.4)), induced by the standard concept for the Lebesgue space $L_{2\pi}^1$. Among the many disadvantages with regard to $R_{2\pi}$, it is shown in Section 2 that for this candidate one cannot expect inverse theorems, reasonable for $R_{2\pi}$. Let us mention that Section 2 starts with a brief outline of the relation of (1.3) to error functionals, recently introduced in connection with a sequential convergence on $R_{2\pi}$. For classes of functions, determined by the asymptotic behaviour of τ-moduli, it is well known that they may be characterized by the rate of convergence of the best onesided approximation. Therefore Section 4 is devoted to the problem of a direct comparison of this notion with (1.3).

2. Riemann convergence

Based upon work of G. Pólya [10], the following notion of a sequential convergence was introduced in [2]: A sequence $\{f_n\}\subset B_{2\pi}$ is called (Riemann) R-convergent to $f\in B_{2\pi}$ if for $n\to\infty$

$$(2.1)\qquad \|f_n\|_B=\mathcal{O}(1),\quad \bar{\int}\sup_{k\geqq n}|f_k-f|=o(1).$$

It follows that the (sequential) completion of the set of trigonometric polynomials then yields $R_{2\pi}$. To discuss quantitative aspects of the R-convergence, we started with estimates of the R-error

$$(2.2)\qquad \bar{\int}\sup_{k\geqq n}|T_kf-f|$$

of some (linear) process $\{T_n\}$ versus the τ-modulus (1.8) (cf. [8, 9]). In this setting, however, there cannot hold true inverse results (see [7, 14]). Therefore we strengthened (2.2) to the R-δ-error (cf. (1.1))

$$(2.3)\qquad \|\sup_{k\geqq n}|T_kf-f|\|_{\delta_n}$$

for an appropriate sequence of positive numbers $\{\delta_n\}$, monotonely decreasing to zero. Now direct theorems in terms of the τ-modulus can indeed be supplemented by corresponding inverse results, delivering complete equivalence assertions (see [7, 15]). Note that a concept of sequential convergence based on (2.3) (instead of (2.2)) is consistent with the R-convergence as introduced in (2.1).

Continuing this procedure, one may ask to use the error functional (2.3) in order to introduce the notion of a best approximation in $R_{2\pi}$. To this end, let us consider

$$E_n^{\cdot}[f,\ R]:=\inf\{\|\sup_{k\geq n}|f-p_k|\,\|_{1/n}:p_k\in\Pi_k\}\qquad (f\in R_{2\pi}).$$

Obviously, one has $E_n[f,\ R]\leq E_n^{\cdot}[f,\ R]$. On the other hand there holds true

$$\begin{aligned}E_n^{\cdot}[f,\ R]&\leq\inf\{\|\sup_{k\geq n}|f-p_k|\,\|_{1/n}:p_k=p_n \text{ for all } k\geq n \text{ with } p_n\in\Pi_n\subset\Pi_k\}\\&=\inf\{\|f-p_n\|_{1/n}:p_n\in\Pi_n\}=:E_n[f,\ R].\end{aligned}$$

Thus one in fact arrives at the definition (1.3).

In this connection one may mention that if one starts with the (smaller) error functional (2.2), then by the same arguments

$$\inf\{\overline{\int}\sup_{k\geq n}|f-p_k|:p_k\in\Pi_k\}=E_n[f,\ L^1]\qquad (f\in R_{2\pi})$$

with the standard integral (L^1-) error

$$(2.4)\qquad E_n[f,\ L^1]:=\inf\{\|f-p_n\|_1:=\int_{-\pi}^{\pi}|f(u)-p_n(u)|du:p_n\in\Pi_n\}.$$

Note the abuse in the notation of the δ-norm $\|f\|_\delta$ for $\delta=1$ (cf. (1.1)) and of the integral norm $\|f\|_1$ (cf. (2.4)) of $f\in R_{2\pi}$, but in the following there will be no danger of confusion.

As for (2.2), however, the integral error (2.4), well-defined on $R_{2\pi}$, would be too small for inverse theorems to hold in connection with τ-moduli. Indeed, whereas one has the direct (Jackson-type) estimate

$$E_n[f,\ L^1]\leq E_n[f,\ R]\leq C\,\tau_1(f,\ 1/n)\qquad (f\in R_{2\pi})$$

(the first inequality being obvious, for the second one see Theorem 3.1), there cannot hold true a corresponding inverse (Bernstein-type) assertion. This (cf. Theorem 2.2) will be a consequence of the following quantitative extension of the classical uniform boundedness principle: For a Banach space X with norm $\|\cdot\|_X$ let X^* be the space of non-negative, sublinear, bounded functionals T on X, i.e., for $f,\ g\in X$ and scalar α

$$0\leq T(f+g)\leq Tf+Tg,\quad T(\alpha f)=|\alpha|\,Tf,\quad \sup\{Tf:f\in X,\ \|f\|_X\leq 1\}<\infty.$$

Let $\omega(\delta)$ be an abstract modulus of continuity, thus a function, strictly positive, continuous, monotonely increasing, and subadditive on $(0,\ \infty)$, satisfying

$$(2.5)\qquad \omega(\delta)=o(1),\quad \delta=o(\omega(\delta)).$$

Then there holds true

Theorem 2.1: *Let $\{\varphi_n\}$ be a strictly decreasing nullsequence, $\sigma(\delta)>0$ for $\delta>0$, and ω a modulus satisfying* (2.5). *If for U_δ, $R_n \in X^*$ there are testelements $h_n \in X$ with* ($\delta \to 0+$, $n \to \infty$):

$$\|h_n\|_X = \mathcal{O}(1), \tag{2.6}$$

$$U_\delta h_n \leqq M \min\{1, \sigma(\delta)/\varphi_n\}, \quad (\delta>0,\ n\in \mathsf{N}), \tag{2.7}$$

$$R_n h_n \neq o(1), \tag{2.8}$$

then there exist counterexamples $f_\omega \in X$ with

$$U_\delta f_\omega = \mathcal{O}(\omega(\sigma(\delta))), \tag{2.9}$$

$$R_n f_\omega \neq o(\omega(\varphi_n)). \tag{2.10}$$

For a proof see [3] and the literature cited there.

Theorem 2.2: *For each modulus ω with* (2.5) *there exists $f_\omega \in R_{2\pi}$ with*

$$\tau_1(f_\omega, \delta) = \mathcal{O}(\omega(\delta)),$$

$$E_n[f_\omega, R] \neq o(\omega(\frac{1}{n})),$$

$$E_n[f_\omega, L^1] = 0 \qquad (n \in \mathsf{N}).$$

Proof: Consider the linear space $X := \{f \in R_{2\pi} : \int |f| = 0\}$ which is a Banach space under the norm $\|\cdot\|_B$. Indeed, $R_{2\pi}$ being a Banach space under $\|\cdot\|_B$, let $f \in R_{2\pi}$, $\{f_n\} \subset X$ be such that $\|f_n - f\|_B = o(1)$. Then

$$\int |f| \leqq \int |f_n| + \int |f_n - f| \leqq 2\pi \|f_n - f\|_B = o(1),$$

thus $f \in X$ so that X is complete. Therefore Theorem 2.1 may be applied to X, $\varphi_n = 1/n$, $\sigma(\delta) = \delta$, $U_\delta f = \tau_1(f, \delta)$, $R_n f = E_n[f, R]$, and the testelements $h_n(u) = 1$ for $u = 2\pi j/7n$, $j \in \mathsf{Z}$ (set of integers), and $=0$ elsewhere. Obviously, $h_n \in X$ with $\|h_n\|_B = 1$ and (cf. [12, p. 10])

$$\tau_1(h_n, \delta) \leqq \begin{cases} 2\|h_n\|_\delta \leqq 4\pi \\ 2\delta[\operatorname{Var} h_n]_0^{2\pi} \leqq 28\delta n \end{cases},$$

which gives (2.6, 7). In order to show $E_n[h_n, R] \neq o(1)$, let $p_n \in \Pi_n$ be arbitrary. For each $x \in [0, 2\pi)$ there exists $0 \leqq j \leqq 7n-1$ such that $x_j := 2\pi j/7n \in (x-1/n, x+1/n)$ since $x_j - x_{j-1} = 2\pi/7n < 1/n$. This implies ($\operatorname{Re} p_n$ denoting the real part of p_n)

$$\sup_{y \in U_{1/n}(x)} |h_n(y) - p_n(y)| \geqq \lim_{\delta \to 0+} M(h_n - p_n, x_j, \delta) \geqq \lim_{\delta \to 0+} M(h_n - \operatorname{Re} p_n, x_j, \delta)$$

since $|h_n(u) - p_n(u)| \geqq |h_n(u) - \operatorname{Re} p_n(u)|$. If $\operatorname{Re} p_n(x_j) \leqq 1/2$, one may continue to

$$\lim_{\delta \to 0+} M(h_n - \operatorname{Re} p_n, x_j, \delta) \geqq |h_n(x_j) - \operatorname{Re} p_n(x_j)| \geqq 1/2.$$

On the other hand, if Re $p_n(x_j) > 1/2$, then $|h_n(y) - \operatorname{Re} p_n(y)| = \operatorname{Re} p_n(y)$ for $y \neq x_j$ which implies

$$\lim_{\delta \to 0+} M(h_n - \operatorname{Re} p_n, x_j, \delta) \geqq \operatorname{Re} p_n(x_j) \geqq 1/2$$

in view of the continuity of p_n. Altogether it follows that

$$\sup_{y \in U_{1/n}(x)} |h_n(y) - p_n(y)| \geqq 1/2,$$

thus $\| h_n - p_n \|_{1/n} \geqq \pi$ for $p_n \in \Pi_n$ which establishes (2.8). The assertion of Theorem 2.2 now follows by (2.9, 10) since $f_\omega \in X$ and (cf. (2.4))

$$E_n[f_\omega, L^1] \leqq \int |f_\omega| = 0 \qquad (n \in \mathrm{N}).$$

■

3. Constructive theory of functions

In connection with the concept (1.3) of a best approximation in $R_{2\pi}$ one may now develop direct and inverse theorems, quite parallel to the classical Jackson-Bernstein theory for (1.7) in $C_{2\pi}$. Let us exemplify this by the following results.

Theorem 3.1: *Given $r \in \mathrm{N}$, there holds true the Jackson-type theorem*

$$E_n[f, R] \leqq C_r \tau_r(f, 1/n) \qquad (f \in R_{2\pi}). \tag{3.1}$$

Proof: Obviously, one has that for each $g \in ACR^r_{2\pi}$

$$E_n[f-g, R] \leqq \|f-g\|_{1/n}.$$

Therefore, if one can show the Jackson-type inequality

$$E_n[g, R] \leqq C_r n^{-r} \|g^{(r)}\|_1 \qquad (g \in ACR^r_{2\pi}), \tag{3.2}$$

then the assertion follows in view of (1.10, 11). To prove (3.2), as usual (cf. [1, p. 97]) define operators $U_{r,n}$ via ($K^j_n := K_n \cdot K^{j-1}_n$)

$$U_{r,n} f := \sum_{j=1}^{r} (-1)^{j+1} \binom{r}{j} K^j_n f \qquad (f \in R_{2\pi}),$$

the Fejér—Korovkin process $\{K_n\}$ being given by

$$K_n f(x) := \frac{1}{\pi(n+2)} \sin^2\left(\frac{\pi}{n+2}\right) \int_{-\pi}^{\pi} f(x-u) \frac{\cos^2 \frac{(n+2)u}{2}}{\left(\cos u - \cos \frac{\pi}{n+2}\right)} du.$$

Obviously, $U_{r,n} f \in \Pi_n$ and $U_{r,n} - I = (-1)^{r-1} (K_n - I)^r$ where I is the identity. Moreover, in [7] it was shown that

$$\|\sup_{j\geqq n}|K_j g-g|\|_{1/n}\leqq\frac{C}{n}\|g'\|_1 \qquad (g\in ACR^1_{2\pi}).$$

Since in view of the convolution structure

$$(U_{r,n}g-g)'=(-1)^{r-1}(K_n-I)^r(g') \qquad (g\in ACR^1_{2\pi}),$$

one therefore has that for $g\in ACR^r_{2\pi}$

$$E_n[g,\ R]\leqq\|U_{r,n}g-g\|_{1/n}=\|(K_n-I)(K_n-I)^{r-1}g\|_{1/n}$$
$$\leqq Cn^{-1}\|((K_n-I)^{r-1}g)'\|_1=Cn^{-1}\|(K_n-I)^{r-1}(g')\|_1$$

so that iteration of the argument implies (3.2). ∎

Let us remark that the estimate (3.1) is sharp (cf. Theorem 2.2 for $r=1$). Indeed, there holds true

Corollary 3.2: *For each modulus ω with (2.5) there exists $f_\omega\in R_{2\pi}$ with $(r\in\mathbb{N})$*

$$\tau_r(f_\omega,\ \delta)=O(\omega(\delta^r)),$$
$$E_n[f_\omega,\ R]\neq o(\omega(n^{-r})).$$

Proof: We apply Theorem 2.1 to $X=R_{2\pi}$ with norm $\|\cdot\|_B$, $\varphi_n=n^{-r}$, $\sigma(\delta)=\delta^r$, $U_\delta f=\tau_r(f,\ \delta)$, $R_n f=E_n[f,\ R]$, and the testelements $h_n(u)=e^{i(n+1)u}$. Obviously, one has $h_n\in X$ with $\|h_n\|_B=1$ and (cf. [12, p. 10])

$$\tau_r(h_n,\ \delta)\leqq\begin{cases}2^r\|h_n\|_\delta\leqq 2^{r+1}\pi\\ 2\delta^r\|h_n^{(r)}\|_1=c_r\delta^r n^r\end{cases},$$

thus (2.6, 7). Moreover, for $f\in R_{2\pi}$

$$2\pi|\hat{f}(n+1)|:=\left|\int_{-\pi}^{\pi}f(u)e^{-i(n+1)u}du\right|\leqq E_n[f,\ L^1]\leqq E_n[f,\ R]$$

which implies $R_n h_n\geqq 2\pi$, thus (2.8). Now (2.9, 10) deliver the assertion. ∎

Theorem 3.3: *Given* $r\in\mathbb{N}$, there holds true the Steckin-type inequality

$$\tau_r(f,\ 1/n)\leqq M_r n^{-r}\sum_{j=0}^{n}(j+1)^{r-1}E_j[f,\ R] \qquad (f\in R_{2\pi}). \tag{3.3}$$

Proof: For $f\in R_{2\pi}$ let $p_j^*\in\Pi_j$ be an element of best approximation, thus $\|f-p_j^*\|_{1/j}=E_j[f,\ R]$. By (1.10, 11) one has that

$$\tau_r(f,\ 1/n)\leqq C_r[\|f-p_n^*\|_{1/n}+n^{-r}\|p_n^{*(r)}\|_1].$$

Since $E_j[f,\ R]$ is decreasing with regard to j, for the first summand there holds true

$$\|f-p_n^*\|_{1/n}=E_n[f,\ R]\leqq C_r n^{-r}\sum_{j=0}^{n}(j+1)^{r-1}E_j[f,\ R].$$

To estimate the second summand, consider first the case $n=2^s$, $s\geqq 0$. Since

$$\|p_{2^s}^{*(r)}\|_1=\|p_{2^s}^{*(r)}-p_0^{*(r)}\|_1\leqq\|(p_1^*-p_0^*)^{(r)}\|_1+\sum_{j=1}^{s}\|(p_{2^j}^*-p_{2^{j-1}}^*)^{(r)}\|_1,$$

the classical inequality of Bernstein (cf. [1, p. 99]) delivers

$$\|(p_{2^j}^*-p_{2^{j-1}}^*)^{(r)}\|_1\leqq 2^{jr}\|p_{2^j}^*-p_{2^{j-1}}^*\|_1\leqq 2^{jr}(\|p_{2^j}^*-f\|_1+\|p_{2^{j-1}}^*-f\|_1)$$

which leads to (cf. [13, p. 331])

$$\|p_{2^s}^{*(r)}\|_1\leqq C_r'\sum_{j=0}^{2^s}(j+1)^{r-1}E_j[f,R],$$

thus (3.3) for $n=2^s$. For arbitrary n the proof is now completed as usual. ■

As a consequence of (3.1, 3) note that there holds true a complete analogue of the classical Jackson-Bernstein equivalence theorem, namely: Given $r\in\mathbb{N}$ and $f\in R_{2\pi}$, one has

$$E_n[f,R]=\mathcal{O}(n^{-\alpha})\leftrightarrow\tau_r(f,\delta)=\mathcal{O}(\delta^\alpha), \tag{3.4}$$

provided $0<\alpha<r$.

4. Onesided approximation

Apart from (2.4) there is a further well-known candidate to measure the best approximation of Riemann integrable functions by (trigonometric) polynomials. The concept in question relies to the theory of onesided approximation and is given by

$$\tilde{E}_n[f,R]:=\inf\{\|p-q\|_1: p,q\in\Pi_n,\ p\leqq f\leqq q\}\qquad(f\in R_{2\pi}). \tag{4.1}$$

Indeed, there hold true results, completely parallel to those of Section 3, namely

$$\tilde{E}_n[f,R]\leqq C_r\tau_r(f,1/n)\leqq M_rn^{-r}\sum_{j=0}^{n}(j+1)^{r-1}\tilde{E}_j[f,R]\qquad(f\in R_{2\pi}). \tag{4.2}$$

For this and related material see [12, Chapter 8] and the literature cited there.

Obviously, one may connect the quantities $E_n[f,R]$ and $\tilde{E}_n[f,R]$ via (3.1, 3) and (4.2), delivering

$$\tilde{E}_n[f,R]\leqq C_rn^{-r}\sum_{j=0}^{n}(j+1)^{r-1}E_j[f,R]\qquad(f\in R_{2\pi}), \tag{4.3}$$

and vice versa. This raises the question whether one may improve these estimates to the direct comparison result

$$\tilde{E}_n[f,R]\leqq CE_n[f,R]\qquad(f\in R_{2\pi}), \tag{4.4}$$

and vice versa. In the following we will establish (4.4), but the converse situation remains open (apart from (4.6)).

To this end we follow the construction of a concrete onesided polynomial approximation process as given in [5]. Consider the Fejér kernel

$$F_n(x) := \begin{cases} \sin^2(\pi/2n)\dfrac{\sin^2(nx/2)}{\sin^2(x/2)}, & x\in(0,\ 2\pi) \\ n^2\sin^2(\pi/2n), & x=0 \end{cases}$$

Lemma 1.1: *For the kernel* $\{F_n\}$ *one has*

i) $F_n(x) \geqq 0$,

ii) $F_n(x) \geqq 1$ $\qquad (|x| \leqq \frac{\pi}{n})$,

iii) $\int\limits_0^{2\pi} F_n(x)\,dx = 2\pi\ n\ \sin^2\frac{\pi}{2n}$.

Setting $x_j := (2j-1)\pi/n$, onesided polynomials are now defined for $p_n \in \Pi_n$ and $f \in R_{2\pi}$ by (cf. (1.1))

$$P_n^{\pm} f(x) := p_n(x) \pm \sum_{j=1}^{n} F_n(x-x_j) M(f-p_n, x_j, \frac{\pi}{n}). \tag{4.5}$$

Lemma 4.2: *For* $p_n \in \Pi_n$ *and* $f \in R_{2\pi}$ *one has*

i) $P_n^{\pm} f \in \Pi_n$,

ii) $P_n^- f(x) \leqq f(x) \leqq P_n^+ f(x)$ $\qquad (x \in \mathbf{R})$.

Indeed, i) is obvious since $p_n \in \Pi_n$ and $F_n \in \Pi_{n-1}$. To check ii), let $x \in [0,\ 2\pi)$ be arbitrary. Then there exists $j \in \{1, \ldots, n\}$ such that $|x - x_j| \leqq \pi/n$. Together with Lemma 4.1 i), ii) this already implies

$$P_n^+ f(x) \geqq p_n(x) + M(f-p_n, x_j, \frac{\pi}{n}) \geqq f(x).$$

The assertion concerning P_n^- follows analogously (see [5]).

Theorem 4.3: *For* $p_n \in \Pi_n$ *and* $f \in R_{2\pi}$ *one has*

$$\| P_n^+ f - P_n^- f \|_1 \leqq c \, \| f - p_n \|_{1/n}.$$

Proof: By Lemma 4.1 iii) and the 2π-periodicity of F_n ·

$$\| P_n^+ f - P_n^- f \|_1 = 2 \| \sum_{j=1}^{n} F_n(\cdot - x_j) M(f-p_n, x_j, \frac{\pi}{n}) \|_1$$

$$\leqq 4\pi n \sin^2 \frac{\pi}{2n} \sum_{j=1}^{n} M(f-p_n, x_j, \frac{\pi}{n}).$$

Considering the upper Riemann sum Ω of $|f-p_n|$ for the partition $Z' := \{2j\pi/n : j = 0, \ldots, n\}$ of the interval $[0,\ 2\pi]$, there holds true

$$\Omega(Z', |f-p_n|) = (2\pi/n) \sum_{j=1}^{n} M(f-p_n, x_j, \frac{\pi}{n})$$

which implies

$$\| P_n^+ f - P_n^- f \|_1 \leqq \frac{\pi^2}{2} \Omega(Z', |f-p_n|) \leqq \frac{\pi^2}{2} \sup_{\|Z\| \leqq 2\pi/n} \Omega(Z, |f-p_n|)$$

with the usual maximum norm $\| \cdot \|$ for partitions. Setting $\delta = 2\pi/n$ and $g = |f-p_n|$, the proof is complete in view of (1.2) if one has shown

$$\sup_{\|Z\| \leqq \delta} \Omega(Z, g) \leqq \| g \|_\delta \qquad (\delta > 0).$$

To this end, let $Z = \{x_i\}$ be an arbitrary partition of $[0, 2\pi]$ with $\| Z \| \leqq \delta$. Setting $y_i := (x_{i-1} + x_i)/2$, one has

$$\sup_{x \in [x_{i-1}, x_i]} g(x) \leqq M(g, y_i, \delta/2).$$

For a further partition Z'' let $Z^* := Z \cup Z''$ be the common refinement. Obviously, if a subinterval S belonging to Z^* is a subset of $[x_{i-1}, x_i]$, then $M(g, y_i, \delta/2) \leqq \sup_{x \in S} M(g, x, \delta)$, and therefore

$$\Omega(Z, g) \leqq \Omega(Z^*, M(g, \cdot, \delta)) \leqq \Omega(Z'', M(g, \cdot, \delta)).$$

Since Z'' is arbitrary, this implies $\Omega(Z, g) \leqq \overline{\int} M(g, x, \delta)\, dx$ which completes the proof. ■

Corollary 4.4: *There holds true the (direct) comparison result*

$$\tilde{E}_n[f, R] \leqq c E_n[f, R] \qquad (f \in R_{2\pi}).$$

Proof: By Lemma 4.2, Theorem 4.3 for each $p_n \in \Pi_n$

$$\tilde{E}_n[f, R] \leqq \| P_n^+ f - P_n^- f \|_1 \leqq c \| f - p_n \|_{1/n},$$

which gives the assertion in view of the definition (1.3). ■

Corollary 4.4 in particular shows that the inverse assertion of Theorem 3.3 can also be derived from the corresponding (known, cf. (4.2)) result for the best onesided approximation. As already mentioned, the question whether there is a direct comparison result, opposite to (4.4), is still unsettled. There is a partial result within the frame of the construction (4.5), namely: For each $p_n \in \Pi_n$ there holds true (cf. [4], also [7a])

$$(4.6) \qquad E_n[f, R] \leqq C \inf \{ \| P_n^+ f - P_n^- f \|_1 : p_n \in \Pi_n \} \leqq C' E_n[f, R],$$

i.e., the quantity $E_n[f, R]$ can be characterized by the best approximation from a special class of onesided polynomials.

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