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Extended Interval Arithmetic Involving Infinite Intervals

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In this paper we introduce an extended interval arithmetic structure $\mathcal{M}^*$ involving infinite intervals which is a further extension of the extended interval arithmetic $\mathcal{M}$ using finite intervals and an extended set of basic interval arithmetic operations. The structure $\mathcal{M}^*$ is compared to other known interval arithmetic structures. We discuss three possible directions of extensions of the conventional real interval arithmetic $\mathcal{S}$, which have been developed as algebraic structures during the last two decades and which have proved to be useful for applications in interval analysis: i) the extension $\mathcal{M}$ by supplementary nonstandard interval-arithmetic operations, ii) the extensions $I(R^*)$ and $\mathcal{M}^*$ by infinite intervals and iii) the extension $\mathcal{K}$ by generalized intervals. These three types of extended structures have been introduced in a logically consecutive order by using uniform notations and can be considered as substructures of a single extended structure. The paper contains new expressions for the interval-arithmetic operations by means of the end-points of the operands which are suitable for software implementation and new relations in the extended interval arithmetic structures involved. The space $\mathcal{M}^*$ involving infinite (but proper) intervals and additional (nonstandard) operations finds a natural place among the considered extensions. In this paper we also propose a field of applications for the interval spaces $\mathcal{K}$ and $\mathcal{K}^*$ using generalized intervals. The idea is based on a correspondence between the spaces $\mathcal{M}$ and $\mathcal{K}$, resp. between $\mathcal{M}^*$ and $\mathcal{K}^*$, demonstrated in the paper which can be used to transfer numerical applications in $\mathcal{M}$, resp. $\mathcal{M}^*$, into corresponding applications in $\mathcal{K}$, resp. $\mathcal{M}^*$.

1. Introduction

The algebraic incompleteness of the conventional interval arithmetic as introduced and developed in [1, 29, 33, 41, 42] etc. and its limited applications led to various proposals for possible extensions. In this paper we survey several extended interval arithmetic structures, that have proved to be useful for applications in interval analysis and related fields. Our survey uses uniform notations so that the reader can easily follow the various directions of the presented extended algebraic structures. In a logically consecutive order we introduce: the conventional interval arithmetic $\mathcal{S}$, the extended interval arithmetic $\mathcal{M}$ using two basic nonstandard operations and a basic nonstandard

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relation [5,6], [22]–[27], the extensions $\mathscr{S}^*$ and $\mathscr{M}^*$ of $\mathscr{S}$, resp. $\mathscr{M}$, by infinite intervals, the extended interval arithmetic $\mathscr{K}$ using generalized intervals and its extension $\mathscr{K}^*$ by the corresponding infinite elements [9], [14]–[19], [31]. We give below a short motivation for the use of the considered interval-arithmetic extensions.

Extended interval arithmetic operations. One of the main fields for applications of interval arithmetic is the computation of ranges of functions over intervals from the set $I(R)$ of all closed intervals on the real line R (see e. g. [35]). If a function f is defined in a domain D, then its range over an interval $X \subseteq D$ is the set $f(X) = \{f(x) \mid x \in X\}$. Let $y = f(x, t)$be a continuous function of two variables defined on $D \times T$. Fig. 1 presents a typical situation when the argument x is considered as parameter taking values from an interval $X = [x^-, x^+] \subseteq D$, and f is considered as a family of functions on t depending on a parameter x. Let us fix two values t_1, t_2 for the argument t and consider the functions $f_1(x) = f(x; t_1)$, $f_2(x) = f(x; t_2)$. Providing we know the ranges of f_1, f_2 over X, i. e.:

$$f_1(X) = f(X; t_1) = \{f(x; t_1) | x \in X\}, \quad f_2(X) = f(X; t_2) = \{f(x; t_2) | x \in X\},$$

we seek the range of the composite function $h(x) = f_1(x) * f_2(x)$, $* \in \{+, -, \times, /\}$ using the already known ranges $f_1(X)$, $f_2(X)$. This is a key problem, since its solution contributes to the solution of the problem of finding the range of an arbitrary rational function. To be more specific, consider the case $* = +$. We know that if f_1 and f_2 are continuous on X, then the ranges $f_1(X)$, $f_2(X)$ are intervals and for the range of $h(x) = f_1(x) + f_2(x)$ over X we have $h(X) = \{f_1(x) + f_2(x) | x \in X\} \subseteq f_1(X) + f_2(X)$, where $f_1(X) + f_2(X)$ is the sum of the intervals $f_1(X)$ and $f_2(X)$ defined by $[a^-, a^+] + [b^-, b^+] = [a^- + b^-, a^+ + b^+]$. Moreover, if f_1, f_2 are monotone in X and are both nondecreasing or are both nonincreasing on X, then the above inclusion becomes an equality relation, i. e. $h(X) = \{f_1(x) + f_2(x) | x \in X\} = f_1(X) + f_2(X)$. Since monotone and partially monotone functions are widely used in practical applications, equality relations of the above type can be very useful. However, such equality relations are only true in "half" of the situations, namely when both functions are "equally" monotone. In "the other half" of the cases when one of the functions is monotone increasing and the other is monotone decreasing (see Fig. 1) the familiar interval arithmetic $\mathscr{S}$ can not provide an exact expression for $h(X)$ and the corresponding inclusion may not be sufficiently sharp. An exact expression in this situation can be obtained by means of extended interval arithmetic. There are two different approaches, both of which provide exact expressions in such a situation. The interval arithmetic space $\mathscr{M}$ introduced in section 3 provides a supplementary nonstandard interval-arithmetic operation "$\mathscr{M}$-addition", defined by $[a^-, a^+] +^- [b^-, b^+] = [\min\{a^- + b^+, a^+ + b^-\}, \max\{a^- + b^+, a^+ + b^-\}]$. We can assert that $h(X) = f_1(X) +^- f_2(X)$ whenever f_1 and f_2 are differently monotone on X and the sum h is monotone, which covers "the other half" of the practical situations. Supplementary "$\mathscr{M}$-operations" are introduced for subtraction, multiplication and division as well in the following way. Let $A = [a^-, a^+]$, $B = [b^-, b^+]$,

$E=\{a^- * b^-, a^- * b^+, a^+ * b^-, a^+ * b^-\}, * \in\{+, -, \times, /\}$. Consider the situation when all four points of E are different. Then the set E_1 of the two outer points of E determins the usual interval-arithmetic operations by $A * B=[\min E, \max E]=\text{hull}\,(E_1)$. The set $E\backslash E_1$ consisting of the remaining two (inner) elements of E defines then the corresponding $\mathcal{M}$-operations by $A *^- B=\text{hull}\,(E\backslash E_1)$. Using these supplementary interval-arithmetic operations in combination with the standard ones we can efficiently compute ranges of monotone functions (see [27] and section 8). If monotonicity can not be assumed interval arithmetic operations can be emploid for obtaining inner and outer inclusions: $f_1(X) *^- f_2(X)\subseteq(f_1 * f_2)(X)\subseteq f_1(X) * f_2(X)$.

Generalised and directed intervals. Wide possibilities for practical applications provides the interval-arithmetic space $\mathcal{K}$ developed in [31], [14]—[19]. Let us give the main idea of this possibility, pointing out thereby a large field of application of the interval structure $\mathcal{K}$, which seems not to be noticed by now. The support of $\mathcal{K}$ is the set of all ordered couples of real numbers, called generalised intervals. This set is equivalent to the set of all couples of the form $(\tau, [a, b])$, $\tau\in\{+, -\}$, $[a, b]\in I(R)$. Here a binary variable τ provides additional information which can be interpreted as a "direction" of the interval in which some variable (possibly defined by a functional value) traces the interval. Because of this interpretation we may call such couples "directed intervals". Directed intervals can be conveniently stored in the form of two-dimensional vectors (called generalized intervals) by setting $[c, d]=\{[a, b]$, if $\tau=+$; $[b, a]$, if $\tau=-\}$. The set of directed intervals is isomorphic to the set of generalized intervals and obeys simple interval arithmetic rules: addition of directed intervals is defined as addition of two-dimensional vectors and multiplication is defined by an isometric extension of the corresponding definition for intervals from $I(R)$. Using such an arithmetic we may put in correspondence to the range $f(X)$ the generalized interval $f[X]=[f(x^-), f(x^+)]$. Note that $f[X]$ is a proper interval (or an interval directed from left to right) if f is nondecreasing on X and is an improper interval (or an interval directed from right to left) if f is nonincreasing on X (Fig. 1). So the generalized interval $f[X]$ is the range $f(X)$ together with a supplementary information w. r. t. the type of monotonicity of f, which is provided by the order of the end-points. Now if f_1, f_2 and $h=f_1+f_2$ are monotone, then the generalized interval $h[X]=f_1[X]+f_2[X]$ again corresponds to the range $h(X)$ in the sense that $h(X)$ and $h[X]$ have same endpoints, but $h[X]$ carries an additional information concerning the type of monotonicity of h on X (see section 8).

Another useful extension is the one by infinite intervals. This extension allows to consider monotone functions possibly tending to infinity at certain points of their interval domain, having as ranges infinite intervals of the form $[-\infty, a]$, $[b, \infty]$, $[-\infty, \infty]$, $[a, \infty]\cup[-\infty, b]$ etc. (see Fig. 2). Using infinite intervals we can define division by intervals containing zero obtaining thereby an algebraically closed interval space. In sections 5, 6 we extend the $\mathcal{M}$-operations for infinite intervals and give explicit expressions for the interval-arithmetic operations involving infinite intervals.

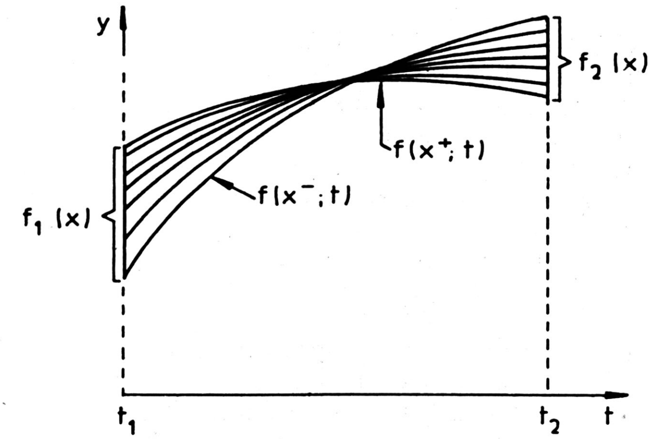

Fig. 1

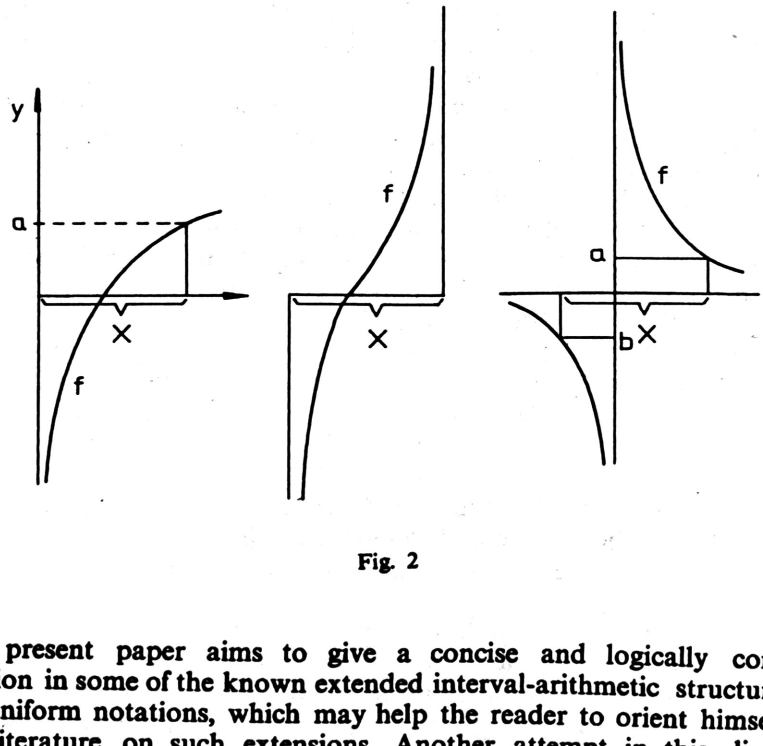

Fig. 2

The present paper aims to give a concise and logically consequent introduction in some of the known extended interval-arithmetic structures using thereby uniform notations, which may help the reader to orient himself in the existing literature on such extensions. Another attempt in this direction is undertaken in [8]; however infinite intervals are not considered there.

2. The conventional interval space $\mathscr{S}=(I(R), +, \times, /, \subseteq)$

An interval $[a, b]$, $a, b, \in R$, $a \leqq b$, is a compact set on the real line R defined by $[a, b]=\{x | a \leqq x \leqq b\}$. The set of all intervals will be denoted by

(1) $$I(R)=\{[a, b] \,|\, a, b \in R, a \leqq b\}.$$

The functionals $(\cdot)^-, (\cdot)^+ : I(R) \to R$ assign to each interval $A \in I(R)$ the left, resp. the right endpoint of A, that is $A=[(A)^-, (A)^+]$. For the endpoints $(A)^-$, $(A)^+$ we shall also use the notations A^- or a^-, resp. A^+ or a^+ (as in [22]). Hence, for $A \in I(R)$ the symbol A^s (or a^s) with $s \in \{+, -\}$ denotes certain end-point of A, which can be the left one or the right one depending on the value of s. We define the "product" st for $s, t \in \{+, -\}$ by $++=--=+$, $+-=-+=-$, so that $a^{++}=a^{--}=a^+$ etc.

Denote the set of intervals containing zero by

$$Z=\{A \in I(R) \,|\, 0 \in A\}=\{[a^-, a^+] \in I(R) \,|\, a^- \leqq 0 \leqq a^+\};$$

the set of intervals which do not contain zero (so-called zero free intervals) is

$$I(R) \backslash Z=\{A \in I(R) \,|\, 0 \notin A\}=\{A \in I(R) \,|\, a^+<0 \text{ or } a^->0\}.$$

Denote by $Z^*=\{A \in I(R) \,|\, a^-<0<a^+\}$ the set of all intervals containing a neighbourhood of zero. Then $I(R) \backslash Z^*=\{A \in I(R) \,|\, a^+ \leqq 0 \text{ or } a^- \geqq 0\}$ is the set of all intervals that may contain zero only as an endpoint. Define a "sign functional" $\sigma : I(R) \backslash Z^* \to \{+, -\}$, by means of

$$\sigma(A)=\begin{cases} +, & \text{if } 0 \leqq a^-; \\ -, & \text{if } a^+ \leqq 0,\ A \neq [0, 0]. \end{cases}$$

The interval arithmetic structure $\mathscr{S}=(I(R), +, \times, /, \subseteq)$ [29]–[41] consists of the supporting set $I(R)$ together with a relation for inclusion $\subseteq$ and three operations: addition $+ : I(R) \times I(R) \to I(R)$, multiplication $\times : I(R) \times I(R) \to I(R)$ and inversion (reciprocal value) $/ : I(R) \backslash Z \to I(R)$, defined for $A=[a^-, a^+]$, $B=[b^-, b^+]$ by (in the sequel "$\leftrightarrow$" means "iff", "$\wedge$" means "and"):

(2) $$A \subseteq B \leftrightarrow (b^- \leqq a^-) \wedge (a^+ \leqq b^+), \quad \text{for } A, B \in I(R),$$

(3) $$A+B=[a^-+b^-, a^++b^+], \quad \text{for } A, B \in I(R),$$

(4) $$A \times B=\begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)}, a^{\sigma(B)} b^{\sigma(A)}], & \text{for } A, B \in I(R) \backslash Z, \\ [a^\delta b^{-\delta}, a^\delta b^\delta],\ \delta=\sigma(A), & \text{for } A \in I(R) \backslash Z,\ B \in Z, \\ [a^{-\delta} b^\delta, a^\delta b^\delta],\ \delta=\sigma(B), & \text{for } A \in Z,\ B \in I(R) \backslash Z, \end{cases}$$

(5) $$A \times B=[\min\{a^- b^+, a^+ b^-\}, \max\{a^- b^-, a^+ b^+\}], \text{ for } A, B \in Z,$$

(6) $$1/B=[1/b^+, 1/b^-],\ B \in I(R) \backslash Z.$$

Remarks. The definition of "$\times$" is given by two separate formulae (4) and (5), for the sake of easier reference. We can formally replace Z in (4) and (5) by Z^*; then the definition of multiplication gains some computational advantages. The case when some of the intervals in (4), (5) is zero can be treated on a computer separately by setting $[0, 0]\times B=A\times[0, 0]=[0, 0]\times[0, 0]=[0, 0]=0$.

In the special case when A is a degenerated interval of the form $A=[a, a]=a$, we have $A\times B=a\times B=[ab^{-\sigma(a)}, ab^{\sigma(a)}]=\{[ab^-, ab^+]$, if $a\geqq 0$; $[ab^+, ab^-]$, if $a<0\}$. The operation $a\times B$ will be further referred as "scalar multiplication" and will be also denoted $a.B$ or just aB. Scalar multiplication with $a=-1$ is called "negation"; we have $(-1)\times B=-B=-[b^-, b^+]=[-b^+, -b^-]$. The operation $A-B$ is defined in $\mathcal{S}$ as a compound operation composed by the operations addition and negation by

$$(7)\ A-B=A+(-1)\times B=A+(-B)=[a^- - b^+,\ a^+ - b^-],\quad A,\ B\in I(R).$$

Therefore we exclude the operation "$-$" from the set of basic operations of $\mathcal{S}$. On the other hand, inversion $1/B$ in $\mathcal{S}$ can not be derived from the basic operations "$+$" and "$\times$" (e. g. the element "$1/B$" is not inverse w. r. t. "$\times$") and thus has to be included in the notation of the algebraic structure $\mathcal{S}=(I(R), +, \times, /, \subseteq)$. After defining $1/B$ as an independent monadic interval operation we then define the operation division A/B as a compound operation composed by multiplication (4) and the operation $1/B$ given by (6):

$$(8)\qquad A/B=A\times(1/B),\quad A\in I(R),\quad B\in I(R)\backslash Z.$$

Calculation of the end-points of the expression $A\times(1/B)$ by substituting (6) in (8) produces

$$A/B=\begin{cases}[a^{-\sigma(B)}/b^{\sigma(A)},\ a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A,\ B\in I(R)\backslash Z,\\ [a^{-\delta}/b^{-\delta},\ a^{\delta}/b^{-\delta}],\ \delta=\sigma(B), & \text{for } A\in Z,\ B\in I(R)\backslash Z.\end{cases}$$

We next recall the basic laws of the interval space $\mathcal{S}=(i\ (R),+, \times, /, \subseteq)$ (see e. g. [1]). If not specified, $A, B, C,\ldots$ denote elements of $I(R)$, resp. of $I(R)\backslash Z$ when used as divisors.

S1. $A+B=B+A,\quad A\times B=B\times A.$

S2. $(A+B)+C=A+(B+C),\quad (A\times B)\times C=A\times(B\times C).$

S3. $X=[0,0]=0$ and $Y=[1, 1]=1$ are the unique neutral elements with respect to addition and multiplication; that is,

$$A=X+A\leftrightarrow X=[0,0];\ A=Y\times A\leftrightarrow Y=[1,\ 1].$$

S4. No element $A\in I(R)$ with $A^-\neq A^+$ has an inverse with respect to $+$ and no element $A\in I(R)\backslash Z$ has an inverse w. r. t. $\times$. The elements $-A$ and $1/A$ (which might be suspected for such inverse elements, but are not) satisfy

$$0\in A+(-A)=A-A,\quad \text{resp. } 1\in A\times(1/A)=A/A.$$

S5. For arbitrary A, B, $C \in I(R)$ we have $A(B+C) \subseteq AB+AC$. An equality relation holds true in the following special cases [33]

$$a(B+C)=aB+aC, \qquad a \in R,$$

$$A(B+C)=AB+AC, \quad \text{if } B, C \in I(R)\backslash Z, \quad \sigma(B)=\sigma(C).$$

S6. Let $* \in \{+, -, \times, /\}$. Then $X \subseteq X_1 \Rightarrow X * C \subseteq X_1 * C$. As a corollary, $X \subseteq X_1$, $Y \subseteq Y_1 \Rightarrow X * Y \subseteq X_1 * Y_1$.

S7. $I(R)$ is a lattice w. r. t. $\subseteq$. The lattice operations w. r. t. $\subseteq$ are the intersection (the meet) and the connected union (the joint) of two intervals:

$$\inf_{\subseteq}(A, B)=\begin{cases}[\max\{A^-, B^-\}, \min\{A^+, B^+\}], & \text{if } \max\{A^-, B^-\} \leqq \min\{A^+, B^+\},\\ \emptyset, & \text{otherwise}\end{cases}$$
$$=[A \wedge B],$$

$$\sup_{\subseteq}(A, B)=[\min\{A^-, B^-\}, \max\{A^+, B^+\}]=[A \vee B].$$

In the special case when the intervals A, B are degenerated (point) intervals, $A=\alpha$, $B=\beta$, α, $\beta \in R$, then the joint $[A \vee B]=[\alpha \vee \beta]$ provides another form of representation of an interval with known endpoints α, β (but possibly with unknown order of these endpoints).

A detailed study of the properties of the relations meet and joint (also in combination with the arithmetic operations) is given in [41]).

We end this section by noting that the operations $+, -, \times, /$ in $\mathcal{S}$ defined by (3)–(8) satisfy the relations:

$$A * B=\{a * b \mid a \in A, b \in B\}, \; * \in \{+, -, \times, /\}, \tag{9}$$
$$A, B \in I(R) \; (B \in I(R)\backslash Z \text{ for } * = /),$$

which are the basis for the practical application of interval arithmetic. In most textbooks on applied interval analysis relations (9) are used as definitions. However, such definitions are of little use for an abstract presentation of the interval arithmetic as algebraic structure; they are not suitable for software implementations as well. More details on the algebraical aspects of the conventional interval-arithmetic structure $\mathcal{S}$ can be found in [1, 14, 28, 33, 36].

We next consider an extension of $\mathcal{S}$ by introducing two new operations and a new relation in $\mathcal{S}$.

3. The extended interval space $\mathcal{M}=(I(R), +, +^-, \times, \times^-, \subseteq, \leqq)$

For $A=[a^-, a^+] \in I(R)$ define $\omega(A)=a^+-a^-$ and $\mu(A)=(a^-+a^+)/2$. Define $\chi(A)=\{a^-/a^+ \text{ if } 0 \leqq \mu(A);\; a^+/a^- \text{ if } 0>\mu(A)\}$, for $A \neq [0, 0]$ [34, 35]. In addition to the sign functional σ we shall use the functionals φ, ψ: $I(R) \times I(R) \rightarrow \{+, -\}$ defined by:

$$\varphi(A, B) = \begin{cases} +, & \text{if } \omega(A) \geqq \omega(B); \\ -, & \text{otherwise,} \end{cases} \qquad \psi(A, B) = \begin{cases} +, & \text{if } \chi(A) \geqq \chi(B); \\ -, & \text{otherwise.} \end{cases}$$

The interval-arithmetic structure $\mathscr{M} = (I(R), +, +^-, \times, \times^-, \subseteq, \leqq)$ is defined [22]−[26] as an extension of $\mathscr{S} = (I(R), +, \times, /, \subseteq)$. In addition to the basic concepts (2)−(5) $\mathscr{M}$ involves two independent operations $+^-$, $\times^-$ and a relation $\leqq$ defined by:

$$(10) \qquad A +^- B = [a^{-\alpha} + b^{\alpha}, \alpha^{\alpha} + b^{-\alpha}], \ \alpha = \varphi(A, B), \quad \text{for } A, B \in I(R),$$

$$(11) \ A \times^- B = \begin{cases} [a^{\sigma(B)\varepsilon} b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon} b^{\sigma(A)\varepsilon}], \ \varepsilon = \psi(A, B), \ \text{for } A, B \in I(R) \backslash Z, \\ [a^{-\delta} b^{-\delta}, a^{-\delta} b^{\delta}], \ \delta = \sigma(A), \ \text{for } A \in I(R) \backslash Z, \ B \in Z, \\ [a^{-\delta} b^{-\delta}, a^{\delta} b^{-\delta}], \ \delta = \sigma(B), \ \text{for } A \in Z, \ B \in I(R) \backslash Z, \\ \max\{a^- b^+, a^+ b^-\}, \min\{a^- b^-, a^+ b^+\}] \ \text{for } A, B \in Z, \end{cases}$$

$$(12) \qquad A \leqq B \leftrightarrow (a^- \leqq b^-) \wedge (a^+ \leqq b^+), \ \text{for } A, B \in I(R).$$

The variables $\alpha = \varphi(A, B)$, $\varepsilon = \psi(A, B)$ in (10), (11) are responsible for the order of the endpoints in the results of the operations and provide an elegant software implementation of the above operations. If we do not need to know this order, then we can use a representation by means of a joint. Then the formulae involving α, ε obtain the following form

$$A +^- B = [(a^- + b^+) \vee (a^+ + b^-)], \qquad \text{for } A, B \in I(R),$$

$$A \times^- B = [(a^{\sigma(B)} b^{-\sigma(A)}) \vee (a^{-\sigma(B)} b^{\sigma(A)})], \quad \text{for } A, B \in I(R) \backslash Z.$$

Note that $A +^- (-A) = 0$, $A \times^- (1/A) = 1$, which means that $-A = [-a^+, -a^-]$ and $1/A = [1/a^+, 1/a^-]$ are inverse elements with respect to the basic $\mathscr{M}$-operations $+^-$ and $\times^-$. Recall that the operator $1/A$ can not be related to (or composed by means of) the basic interval-arithmetic operations "+" and "×" and therefore should be considered as independent operator in $\mathscr{S}$, but the same operator $1/A$ should not be considered as an independent operator in $\mathscr{M}$, since it is an inverse element of the basic $\mathscr{M}$-operation $\times^-$. The algebraic structure $\mathscr{M}$ defined by (1)−(5), (10)−(12) involves the following four compound operations composed by means of the basic interval arithmetic operations $+$, $+^-$, $\times$, $\times^-$ and the inverse elements $-A$ and $1/A$ with respect to the operations $+^-$ and $\times^-$, resp.: the compound operations $A - B = A + (-B)$ and $A/B = A \times (1/B)$ defined by (7), resp. (8) and the operations:

$$(13) \qquad A -^- B = A +^- (-B) = [a^{-\alpha} - b^{-\alpha}, a^{\alpha} - b^{\alpha}], \quad \alpha = \varphi(A, B),$$

(14) $A/^{-}B = A \times^{-} (1/B)$

$$= \begin{cases} [a^{\sigma(B)\varepsilon}/b^{\sigma(A)\varepsilon},\ a^{-\sigma(B)\varepsilon}/b^{-\sigma(A)\varepsilon}], & \varepsilon=\psi(A,\ B), \text{ for } A,\ B\in I(R)\backslash Z, \\ [a^{-\delta}/b^{\delta},\ a^{\delta}/b^{\delta}], & \delta=\sigma(B) \text{ for } A\in Z,\ B\in I(R)\backslash Z. \end{cases}$$

Using the operation "joint" we can again reformulate the above expressions involving the variables α and ε in the following more comprehensive form (but not so suitable for programming):

$$A -^{-} B = [(a^{-} - b^{-}) \vee (a^{+} - b^{+})],$$

$$A/^{-}B = [(a^{\sigma(B)}/b^{\sigma(A)}) \vee (a^{-\sigma(B)}/b^{-\sigma(A)})], \text{ for } A,\ B\in I(R)\backslash Z.$$

The four basic operations defined by (3)−(5), (10), (11) together with the four compound operations (7), (8), (13), (14) summarize eight interval arithmetic operations in $\mathscr{M}$. Four of them coincide with the conventional interval arithmetic operations in $\mathscr{S}$, namely the operations $+, -, \times, /$, defined by (3), (7), (4)−(5), (8), resp., which will be further referred as $\mathscr{S}$-operations: $\mathscr{S}$-addition, $\mathscr{S}$-subtraction etc. The supplementary interval-arithmetic operations $+^{-}, \times^{-}, -^{-}, /^{-}$ will be further referred as nonstandard operations or $\mathscr{M}$-operations.

Recall that the interval-arithmetic operation for division "/" can be composed by means of the operation $\times$ and the inverse element $1/A$ w. r. t. $\times^{-}$ (analogously to the situation with the subtraction "−" which is also a compound operation). This is the reason to define the algebraic structure $\mathscr{M}$ as the set $(I(R), +, +^{-}, \times, \times^{-}, \subseteq, \leqq)$, excluding thereby division (or inversion) from the set of independent (basic) operations of $\mathscr{M}$.

Remark. In our previous publications [5,6], [22]−[27] we assume as basic the operations $+, -^{-}, \times$ and $/^{-}$. Under such an assumption the $\mathscr{M}$-operations $+^{-}, \times^{-}$ (which have been assumed as basic in this presentation) are composed from the chosen basic operations by means of: $A +^{-} B = A -^{-} (-B)$, $A \times^{-} B = A/^{-}(1/B)$. The operations $-^{-}$ and $/^{-}$ (which have been previously assumed as basic) have been denoted by − and /, respectively, and the operations $+^{-}, -, \times^{-}$ and / have been denoted by $\oplus, \ominus, \otimes$, and $\oslash$, respectively. Our earlier notations for $\mathscr{M}$-subtraction and $\mathscr{M}$-division (− and /, resp.) are in confusion with the notations for the $\mathscr{S}$-subtraction and for the $\mathscr{S}$-division as adopted in the literature on interval analysis [1], [29]−[35], [41].

In what follows we use the notations $+^{++} = +^{--} = +^{+} = +$, $+^{+-} = +^{-+} = +^{-}$, $\times^{++} = \times^{--} = \times^{+} = \times$, $\times^{+-} = \times^{-+} = \times^{-}$. For $A\in I(R)\backslash Z$ we shall denote $|A| = \sigma(A)A = \{A, \text{ if } \sigma(A)=+\ ;\ -A, \text{ if } \sigma(A)=-\}$.

In addition to relations S1−S7 the extended interval space $\mathscr{M}$ obeys the following laws:

M1. For $A,\ B\in I(R)$ we have $A +^{-} B = B +^{-} A$, $A \times^{-} B = B \times^{-} A$.

M2. For $A,\ B,\ C\in I(R)$ we have

$$(A+B)+^-C=A+^{\varphi(B,C)}(B+^-C);$$

$$(A+^-B)+C=\begin{cases}A+^{-\varphi(B,C)}(B+^-C) & \text{if } \omega(A)\geqq\omega(B),\\ A+^-(B+C) & \text{if } \omega(A)<\omega(B);\end{cases}$$

$$(A+^-B)+^-C=\begin{cases}A+^{-\varphi(B,C)}(B+^-C) & \text{if } \omega(A)<\omega(B),\\ A+^-(B+C) & \text{if } \omega(A)\geqq\omega(B).\end{cases}$$

For $A, B, C\in I(R)\backslash Z$ we have:

$$(A\times B)\times{}^-C=A\times^{-\psi(B,C)}(B\times{}^-C);$$

$$(A\times{}^-B)\times C=\begin{cases}A\times^{\psi(B,C)}(B\times{}^-C) & \text{if } \chi(A)\leqq\chi(B),\\ A\times{}^-(B\times C) & \text{if } \chi(A)>\chi(B);\end{cases}$$

$$(A\times{}^-B)\times{}^-C=\begin{cases}A\times^{\psi(B,C)}(B\times{}^-C) & \text{if } \chi(A)\geqq\chi(B),\\ A\times{}^-(B\times C) & \text{if } \chi(A)<\chi(B).\end{cases}$$

M3. $X=[0, 0]=0$ and $Y=[1, 1]=1$ are the unique neutral elements with respect to $\mathcal{M}$-addition and $\mathcal{M}$-multiplication; that is,

$$A=X+^-A=A+^-X \quad \text{for all } A\in I(R)\leftrightarrow X=[0, 0],$$

$$A=Y\times{}^-A=A\times{}^-Y \quad \text{for all } A\in I(R)\leftrightarrow Y=[1, 1].$$

M4. Every element $A\in I(R)$ has an unique inverse element with respect to $+^-$ and every element $A\in I(R)\backslash Z$ possesses an unique inverse element with respect to $\times^-$. These are the elements $-A$, resp. $1/A$, i. e.: $0=A+^-(-A)=A-^-A$ and $1=A\times{}^-(1/A)=A/^-A$.

M5. In addition to the distributivity law S5 we have in $\mathcal{M}$ distributive laws involving equality relations

$$a(B+^-C)=aB+^-aC,\ a\in R,$$

$$A(B+C)=AB+^-AC, \text{ if } B, C\in I(R)\backslash Z,\ \sigma(B)=-\sigma(C).$$

Moreover, for $A, B, C, A+B\in I(R)\backslash Z$ we have

$$(A+B)\times C=\begin{cases}(A\times C)+(B\times C) & \text{if } \sigma(A)=\sigma(B);\\ (A\times C)+^{\psi(C,B)}(B\times{}^-C) & \text{if } \sigma(A)=-\sigma(B)=\sigma(A+B),\\ (A\times{}^-C)+^{\psi(C,A)}(B\times C) & \text{if } \sigma(A)=-\sigma(B)=-\sigma(A+B);\end{cases}$$

$$(A+B)\times{}^-C=\begin{cases}(A\times{}^-C)+^{\psi(A,C)\psi(B,C)}(B\times{}^-C) & \text{if } \sigma(A)=\sigma(B);\\ (A\times{}^-C)+^{\psi(C,A)}(B\times C) & \text{if } \sigma(A)=-\sigma(B)=\sigma(A+B),\\ (A\times C)+^{\psi(C,B)}(B\times{}^-C) & \text{if } \sigma(A)=-\delta(B)=-\sigma(A+B);\end{cases}$$

$$(A+^{-}B)\times C=\begin{cases}(A\times C)+^{-\psi(C,B)}(B\times {}^{-}C) & \text{if } \sigma(A)=\sigma(B),\ \omega(A)\geqq\omega(B),\\ (A\times {}^{-}C)+^{\psi(C,A)}(B\times C) & \text{if } \sigma(A)=\sigma(B),\ \omega(A)<\omega(B),\\ (A\times C)+^{-}(B\times C) & \text{if } \sigma(A)=-\sigma(B),\ \xi(A,\ B)\geqq 0,\\ (A\times {}^{-}C)+^{\psi(C,A)\psi(C,B)}(B\times {}^{-}C) & \text{if } \sigma(A)=-\sigma(B),\ \xi(A,\ B)<0;\end{cases}$$

$$(A+^{-}B)\times {}^{-}C=\begin{cases}(A\times {}^{-}C)+^{-\psi(C,A)}(B\times C) & \text{if } \sigma(A)=\sigma(B),\ \omega(A)\geqq\omega(B),\\ (A\times C)+^{\psi(C,B)}(B\times {}^{-}C) & \text{if } \sigma(A)=\sigma(B),\ \omega(A)<\omega(B),\\ (A\times {}^{-}C)+^{-\psi(C,A)\psi(C,B)}(B\times {}^{-}C) & \text{if } \sigma(A)=-\sigma(B),\ \xi(A,\ B)\geqq 0,\\ (A\times C)+^{-}(B\times {}^{-}C) & \text{if } \sigma(A)=-\sigma(B),\ \xi(A,\ B)<0,\end{cases}$$

wherein $\xi(A,\ B)=\sigma(A\times(A+^{-}B))\ \varphi(A,\ B)$.

M6. Let $*\in\{+^{-},\ -^{-}\}$ and $X, X_1, Y, Y_1\in I(R)$. Under the assumption $X\supseteq X_1$, $Y\subseteq Y_1$ we have

$$\text{If } \omega(X)\leqq\omega(Y),\quad \text{then } X*Y\subseteq X_1*Y_1,$$

$$\text{if } \omega(X_1)\geqq\omega(Y_1),\quad \text{then } X*Y\supseteq X_1*Y_1.$$

Let $*\in\{\times^{-},\ /^{-}\}$, and $X, X_1, Y, Y_1\in I(R)\backslash Z$, are such that $X\supseteq X_1$, $Y\subseteq Y_1$. Under this assumption we have:

$$\text{If } \min\{\chi(X),\ \chi(X_1)\}\geqq\max\{\chi(Y),\ \chi(Y_1)\},\quad \text{then } X*Y\subseteq X_1*Y_1,$$

$$\text{If } \max\{\chi(X),\ \chi(X_1)\}\leqq\min\{\chi(Y_1)\},\ \chi(Y_1)\},\quad \text{then } X*Y\supseteq X_1*Y_1.$$

Let $*\in\{+,\ +^{-}\}$ and $X, X_1, Y, Y_1\in I(R)$. Then $X\leqq X_1,\ Y\leqq Y_1 \Rightarrow X*Y\leqq X_1*Y_1$.

Let $*\in\{-,\ -^{-}\}$ and $X, X_1, Y, Y_1\in I(R)$. Then $X\leqq X_1,\ Y\geqq Y_1\Rightarrow X*Y\leqq X_1*Y_1$.

Let $*\in\{\times,\ \times^{-}\}$ and $X, X_1, Y, Y_1\in I(R)\backslash Z$. Then $|X|\leqq|X_1|,\ |Y|\leqq|Y_1|\Rightarrow X*Y\leqq X_1*Y_1$.

Let $*\in\{/,\ /^{-}\}$ and $X, X_1, Y, Y_1\in I(R)\backslash Z$. Then $|X|\leqq|X_1|,\ |Y|\geqq|Y_1|\Rightarrow X*Y\leqq X_1*Y_1$.

M7. $I(R)$ is a lattice w. r. t. $\leqq$. The lattice operations w. r. t. $\leqq$ are:

$\inf_{\leqq}(A,\ B)=[\min\{A^{-},\ B^{-}\},\ \min\{A^{+},\ B^{+}\}]$,

$\sup_{\leqq}(A,\ B)=[\max\{A^{-},\ B^{-}\},\ \max\{A^{+},\ B^{+}\}]$.

If $A\leqq B$, then we shall write the joint $C=[AvB]$ in the form $C=[A,\ B]$, and say that A, B are the left, resp. the right, (interval) endpoints of C.

The $\mathcal{M}$-operations satisfy the relations $A*^{-}B\subseteq A*B$ and are useful for obtaining presentations for the ranges of functions and inner inclusions. This is illustrated by the following example.

Example. Consider the function $h(x)=1-x+x^2$, $x\in X=[0, 1]$. Denoting $f(x)=1-x$, $g(x)=x^2$, we have $F(X)=1-X$, $G(X)=X\times X=X^2$. We divide the interval [0, 1] into two subintervals [0, 1/2], [1/2, 1] such that f, g and h are monotone in each subintervals. According to a theorem [27] we can compute exactly the range of h in each subinterval using $\mathscr{M}$-addition: $H([0, 1/2])=(1-X)+^-X^2=[3/4, 1]$, $H([1/2, 1])=(1-X)+^-X^2=[3/4, 1]$, obtaining finally $H(X)=H([0, 1/2])\cup H([1/2, 1])=[3/4, 1]$. If we do not use monotonocity arguments (see section 8) then we can only get inclusions; standard addition produces an outward inclusion whereas $\mathscr{M}$-addition produces inner inclusion. Indeed we have $F(X)+G(X)=[0, 1]+[0, 1]=[0,2]$ and $F(X)+^-G(X)=[0, 1]+^-[0, 1]=[1, 1]$.

4. Extensions by infinite (inner) intervals

Extensions by infinite intervals are considered in [2], [4], [9], [10]–[21], [30], [37]–[40]. Such extensions are useful for the developement of: i) the theoretical backgrounds of differentiation and integration of functions (in particular, certain classes of discontinuos functions and so-called segment functions); ii) algebraically closed interval and computer arithmetic structures; iii) various interval numerical methods, in particular methods involving division by intervals containing zero. In this section we extend the set $I(R)$ into a set $\mathfrak{I}$ involving infinite intervals and extend the definition domains of the corresponding relations and operations in $\mathscr{S}$, resp. $\mathscr{M}$, obtaining thereby the extended interval structures $\mathscr{S}_3=(\mathfrak{I}, +, \times, /, \subseteq)$ and $\mathscr{M}_3=(\mathfrak{I}, +, +^-, \times, \times^-, \subseteq, \leqq)$. We first extend the real line R by two supplementary elements, denoted by $-\infty$, ∞, satisfying by definition the relation $-\infty<a<\infty$ for all $a\in R$. The extended real line will be denoted $R^*=R\cup\{-\infty, \infty\}=[-\infty, \infty]$. We next extend the set $I(R)$ by a set of intervals of the form:

$$[-\infty, \alpha]=\{x|x\leqq\alpha\}\ [\beta, \infty]=\{x|x\geqq\beta\}, \alpha, \beta\in R,$$

called (nondegenerate) infinite inner intervals, and intervals of the form

$$[-\infty, -\infty]=-\infty, [\infty, \infty]=\infty, [-\infty, \infty]=R^*,$$

further referred as degenerate infinite inner intervals. The whole set of infinite inner intervals (briefly : ii-intervals) will be denoted by

$$\mathscr{I}=\{[-\infty, \alpha], [\beta, \infty], [-\infty, -\infty], [-\infty, \infty], [\infty, \infty] | \alpha, \beta\in R\}$$

$$=\{[-\infty, \alpha], [\beta, \infty] | \alpha, \beta\in R\}\cup\{-\infty, R^*, \infty\}$$

$$=\{[-\infty, \alpha], [\beta, \infty] | \alpha, \beta\in R^*\}.$$

Denote further $\mathfrak{I}=I(R)\cup\mathscr{I}$. We can briefly write

$$\mathfrak{I}=\{[\alpha, \beta] | \alpha, \beta\in R^*, \alpha\leqq\beta\}.$$

We shall classify the familiar intervals from $I(R)$ as finite inner intervals or briefly fi-intervals; and the intervals from the set $\mathfrak{I}=I(R)\cup\mathcal{J}$ (that is the fi-intervals together with the ii-intervals) will be classified as inner intervals or briefly i-intervals.

Using formula (2) and that $-\infty\leqq a\leqq\infty$ for all $a\in R^*$ we extend the definition domain of the relation $\subseteq$ from $I(R)\times I(R)$ into $\mathfrak{I}\times\mathfrak{I}$ by (in what follows we denote $A=[a^-, a^+]\in\mathfrak{I}$ and $B=[b^-, b^+]\in\mathfrak{I}$):

$$A\subseteq B \leftrightarrow b^-\leqq a^- \text{ and } a^+\leqq b^+.$$

Examples. We have $[-\infty, 0]\subseteq[-\infty, \infty]$, $[0, \infty]\subseteq R^*$.

We consider below two methods to define arithmetic operations in $\mathfrak{I}$.

Method 1. This method is used in [4], [14], [21]. We first extend the domain of the real arithmetic operations from R into R^*. The results of the operations in R^* will not necessarily belong to R^*; there will be situations when the result is equal to R^* itself. We define the arithmetic $(R^*, +, \times, /, \leqq)$ as follows ($*$ stands for a well defined result in R):

Addition $a+b=b+a$

b \ a	$-\infty$	$a\in R$	∞
$-\infty$	$-\infty$	$-\infty$	R^*
$b\in R$		$*$	∞
∞			∞

Multiplication $a.b=b.a$

b \ a	$-\infty$	$-\infty<a<0$	0	$0<a<\infty$	∞
$-\infty$	∞	∞	0	$-\infty$	$-\infty$
$-\infty<b<0$		$*$	0	$*$	$-\infty$
0			0	0	0
$0<b<\infty$				$*$	∞
∞					∞

Inversion $1/b$, $b \neq 0$

b	$1/b$
$-\infty$	0
$b \in R$, $b \neq 0$	*
∞	0

Negation $-b=(-1).\, b$ is a special case of multiplication:

b	$-b$
$-\infty$	∞
$b \in R$	*
∞	$-\infty$

In $(R^*, +, \times, /, \leqq)$ we define the composite operations $a-b$, a/b by $a-b=a+(-b)$ and $a/b=a.(1/b)$, $b \neq 0$; from these definitions we obtain the following tables for the results of the corresponding operation

Subtraction $a-b=a+(-b)$

b \ a	$-\infty$	$a \in R$	∞
$-\infty$	R^*	∞	∞
$b \in R$	$-\infty$	*	∞
∞	$-\infty$	$-\infty$	R^*

Division $a/b=a(1/b)$, $b \neq 0$

b \ a	$-\infty$	$-\infty<a<0$	0	$0<a<\infty$	∞
$-\infty$	0	0	0	0	0
$-\infty<b<0$	∞	*	0	*	$-\infty$
$0<b<\infty$	$-\infty$	*	0	*	∞
∞	0	0	0	0	0

In order to close the arithmetic over R^* we can consider it as algebraic structure with supporting set the set $\mathscr{R}$ of all elements of R^* together with R^* itself, $\mathscr{R}=\{x|x\in R^*\}\cup\{R^*\}$. We then define $a+R^*=R^*$ for $a\in\mathscr{R}$, $a\times R^*=R^*$ for $a\in\mathscr{R}$, $a\neq 0$, $0\times R^*=0$ etc.

Remark. Similar definitions of arithmetic operations on the extended real line R^* can be found in [4], [14], [21], [40] etc. Our definitions differ from the one's known to us in the situations when the result defined by the above tables coincides with the interval R^*. Most of the familiar definitions consider these situations as "undefined". The settings $(\pm\infty)\times 0=0$, resp. $(\pm\infty)/(\pm\infty)=0$ are controversial.

Denote by $Z_{\mathfrak{I}}$ the set of all i-intervals, which contain zero, that is

$$Z_{\mathfrak{I}}=\{[a^-, a^+]\,|\,a^-\leqq 0\leqq a^+,\ a^-, a^+\in R^*\}.$$

We now extend the domains of the arithmetic operations $+, -, \times, /$ from $I(R)$ into $\mathfrak{I}$ by means of the following definition.

We define $A*B$, $*\in\{+, -, \times, /\}$, for i-intervals (except for division by intervals containing zero) by means of formulae (2)–(8) modified by formally replacing in these formulae $I(R)$ by $\mathfrak{I}$ and Z by $Z_{\mathfrak{I}}$, i. e.

(15) $$A+B=[a^-+b^-,\ a^++b^+],\ \text{for } A, B\in\mathfrak{I},$$

(16) $$A\times B=\begin{cases}[a^{-\sigma(B)}b^{-\sigma(A)},\ a^{\sigma(B)}b^{\sigma(A)}], & \text{for } A, B\in\mathfrak{I}\setminus Z_{\mathfrak{I}},\\ [a^{\delta}b^{-\delta},\ a^{\delta}b^{\delta}],\ \delta=\sigma(A), & \text{for } A\in\mathfrak{I}\setminus Z_{\mathfrak{I}},\ B\in Z,\\ [a^{-\delta}b^{\delta},\ a^{\delta}b^{\delta}],\ \delta=\sigma(B), & \text{for } A\in Z_{\mathfrak{I}},\ B\in\mathfrak{I}\setminus Z_{\mathfrak{I}},\end{cases}$$

(17) $$A\times B=[\min\{a^-b^+, a^+b^-\},\ \max\{a^-b^-, a^+b^+\}],\ \text{for } A, B\in Z_{\mathfrak{I}},$$

(18) $$1/B=[1/b^+,\ 1/b^-],\ B\in\mathfrak{I}\setminus Z_{\mathfrak{I}},$$

(19) $$A-B=A+(-1)\,.\,B=A+(-B)=[a^--b^+,\ a^+-b^-],\ A, B\in\mathfrak{I},$$

(20) $$A/B=A\times(1/B)=\begin{cases}[a^{-\sigma(B)}/b^{\sigma(A)},\ a^{\sigma(B)}/b^{-\sigma(A)}], & \text{for } A, B\in\mathfrak{I}\setminus Z_{\mathfrak{I}},\\ [a^{-\delta}/b^{-\delta},\ a^{\delta}/b^{-\delta}],\ \delta=\sigma(B), & \text{for } A\in Z_{\mathfrak{I}},\ B\in\mathfrak{I}\setminus Z_{\mathfrak{I}}.\end{cases}$$

We note that the endpoints of the resulting intervals, computed by (15)–(20) are elements of $\mathscr{R}$. If both endpoints are elements of R^*, then the result belongs to $\mathfrak{I}$. If some of the computed endpoints is equal to R^*, then the resulting interval (with possibly interval endpoints in the sense of S7 or M7) is equal to R^*. By all means the resulting interval is an element of $\mathfrak{I}$. It can be verified that the basic relations (9) hold true with a formal substitution of $I(R)$ by $\mathfrak{I}$ and of Z by $Z_{\mathfrak{I}}$ in (9). The application of the above definition is illustrated below.

Examples. For $a, b\in R^*$ we have

$$[a, \infty]+[-\infty, -\infty]=[a-\infty, \infty-\infty]=[-\infty, R^*]=[(-\infty)\vee R^*]=R^*,$$

$$[-\infty, a]+[\infty, \infty]=[-\infty+\infty, a+\infty]=[R^*, +\infty]=[R^*\vee(+\infty)]=R^*,$$

$$[-\infty, a]+[b, \infty]=[-\infty, \infty]=R^*$$

$$[-1, 1]\times[\infty, \infty]=[-\infty, \infty]=R^*,$$

$$[-1, 1]\times[-\infty, \infty]=[-\infty, \infty]=R^*,$$

$$1/[a, \infty]=[0, 1/a], \text{ for } a>0,$$

$$1/[-\infty, a]=[1/a, 0], \text{ for } a<0,$$

$$1/[\infty, \infty]=1/[-\infty, -\infty]=[0, 0],$$

$$[1, \infty]/[1, \infty]=[1, \infty]\times(1/[1, \infty])=[1, \infty]\times[0, 1]=[0, \infty],$$

$$[-\infty, \infty]/[1, \infty]=[-\infty, \infty]\times[0, 1]=[-\infty, \infty]=R^*.$$

It is possible to extend the definitions of the sign functionals involved in the definition of the $\mathscr{M}$-operations and to use the above method to obtain extensions of these operations in $\mathfrak{I}$. In what follows we shall do this in a direct way (Method 2 below).

Method 2. The definitions of the interval-arithmetic operations in $\mathfrak{I}$ by Method 1 are suitable for direct implementation on computer. However, formulae (15)–(20) does not give an easy overview on the type of the computed results for the various classes of operands, like $A, B\in\mathfrak{I}$; $A\in\mathfrak{I}$, $B\in I(R)$ etc. This can be achieved by deriving special expressions for computing with nondegenerate i-intervals involving their finite endpoints. These formulae avoid the operations between elements in R^* as prescribed in formulae (15)–(20) (except when degenerated ii-intervals are involved) and are also suitable for software/hardware applications.

We first introduce special notations for intervals from $\mathscr{I}$.

Denote $\mathscr{I}^-=\{[a, \infty]\,|\,a\in R\}$, $\mathscr{I}^+=\{[-\infty, a]\,|\,a\in R\}$. Denoting $A=[a, \infty]=(a, -)$ for $A\in\mathscr{I}^-$ and $A=[-\infty, a]=(a, +)$ for $A\in\mathscr{I}^+$ we obtain the uniform notation

$$A=(a, i(A)),\ A\in\mathscr{I}/\{R^*\},$$

wherein for $A\neq R^*$

$$i(A)=\begin{cases}-, & \text{if } A=[a, \infty]\in\mathscr{I}^-;\\ +, & \text{if } A=[-\infty, a]\in\mathscr{I}^+.\end{cases}$$

Using these notations we can formulate the standard arithmetic in $\mathfrak{I}$ in the following equivalent to Method 1 way.

Inclusion. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}$,

$$A\subseteq B\leftrightarrow\begin{cases} i(A)=i(B)=+\text{and}\quad a\leqq b; \\ i(A)=i(B)=-\text{and}\quad a\geqq b; \\ i(A)=-i(B)\quad\text{and}\quad B=R^*. \end{cases}$$

Addition. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}$ the sum $A+B\in\mathscr{I}$ and is given by

$$A+B=\begin{cases} (a+b,\ i), & \text{if } i(A)=i(B)=i; \\ R^*, & \text{if } i(A)=-i(B). \end{cases}$$

For $A=(a,\ i(A))\in\mathscr{I}$, $B=[b^-,b^+]\in I(R)$ the sum $A+B\in\mathscr{I}$ and is given by

$$A+B=(a+b^{i(A)},\ i(A)).$$

Multiplication. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}$ the product $A\times B\in\mathscr{I}$ and is given by

$$A\times B=\begin{cases} (ab,\ i(A)i(B)), & \text{if } A,\ B\in\mathscr{I}\backslash Z^*_{\mathscr{I}}; \\ R^*, & \text{otherwise,} \end{cases}$$

wherein $Z^*_{\mathscr{I}}=\{A\in\mathscr{I}\mid a^-<0<a^+\}$, so that $\mathscr{I}\backslash Z^*_{\mathscr{I}}$ is the set of all ii-intervals which do not contain zero as inner point (that is do not contain a two-sided neighbourhood of zero).

For $A=(a,\ i(A))\in\mathscr{I}$, $B=[b^-,b^+]\in I(R)$ the product $A\times B\in\mathscr{I}$ and is given by

$$A\times B=\begin{cases} (ab^{\zeta(A)\sigma(B)},\ i(A)\sigma(B)), & \text{if } B\in I(R)\backslash Z; \\ R^*, & \text{if } B\in Z, \end{cases}$$

wherein

$$\zeta(A)=\begin{cases} -, & \text{if } A\in\mathscr{I}\backslash Z^*_{\mathscr{I}} \\ +, & \text{if } A\in Z^*_{\mathscr{I}}. \end{cases}$$

Let $Z_{\mathscr{I}}=\{A\in\mathscr{I}\mid a^-\leqq 0\leqq a^+\}$, so that $\mathscr{I}\backslash Z_{\mathscr{I}}$ is the set of all ii-intervals which do not contain zero.

Inversion. For $B=(b,\ i(B))\in\mathscr{I}\backslash Z_{\mathscr{I}}$ the reciprocal interval $1/B\in I(R)\backslash Z$ is given by

$$1/(b,\ -)=1/[b,\ \infty]=[0,\ 1/b],\ b>0;\quad 1/(b,\ +)=1/[-\infty,\ b]=[1/b,\ 0],\ b<0.$$

We next give the expressions for the composite operations negation, subtraction and division.

Negation. For $A=(a,\ i(A))\in\mathscr{I}$ the interval $-A\in\mathscr{I}$ is given by

$$-A=-(a,\ i(A))=(-a,\ -i(A)).$$

Subtraction. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}$ the difference $A-B=A+(-B)\in\mathscr{I}$ and is given by

$$A-B=\begin{cases}(a-b,\ i(A)), & \text{if } i(A)=-i(B);\\ R^*, & \text{if } i(A)=i(B).\end{cases}$$

For $A=(a,\ i(A))\in\mathscr{I}$, $B=[b^-,b^+]\in I(R)$ the interval $A-B\in\mathscr{I}$ and is given by

$$A-B=(a-b^{-i(A)},\ i(A)).$$

For $A=[a^-,\ a^+]\in I(R)$, $B=(b,\ i(B))\in\mathscr{I}$ the interval $A-B\in\mathscr{I}$ and is given by

$$A-B=(a^{-i(B)}-b,\ -i(B)).$$

Division. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}\backslash Z_{\mathscr{I}}$ the interval ratio $A/B\in\mathscr{I}$ and is given by

$$A/B=\begin{cases}(a/b,\ i(A)i(B)), & \text{if } A\in Z^*_{\mathscr{I}}\\ (0,\ i(A)i(B)), & \text{if } A\in\mathscr{I}\backslash Z^*_{\mathscr{I}}.\end{cases}$$

For $A=(a,\ i(A))\in\mathscr{I}$, $B=[b^-,b^+]\in I(R)\backslash Z$ the interval $A/B\in\mathscr{I}$ and is given by

$$A/B=(a/b^{-i(A)\sigma(B)},\ i(A)\sigma(B)).$$

For $A=[a^-,\ a^+]\in I(R)$, $B=(b,\ i(B))\in\mathscr{I}\backslash Z_{\mathscr{I}}$ the interval $A/B\in I(R)$ and is given by

$$A/B=\begin{cases}[0,\ a^{\sigma(A)}/b]=[0,\ a^{-i(B)}/b], & \text{if } A\in I(R)\backslash Z \text{ and } \sigma(A)=-i(B),\\ [a^{\sigma(A)}/b,\ 0]=[a^{i(B)}/b,\ 0], & \text{if } A\in I(R)\backslash Z \text{ and } \sigma(A)=i(B),\\ [a^{i(B)}/b,\ a^{-i(B)}/b], & \text{if } A\in Z.\end{cases}$$

The interval structure thus obtained will be denoted by $\mathscr{S}_3=(\mathfrak{I},\ +,\ \times,/,\ \subseteq)$. We shall next extend the $\mathscr{M}$-operations in $\mathscr{S}_3$ obtaining thereby the structure $\mathscr{M}_3=(\mathfrak{I},\ +,\ +^-,\ \times,\ \times^-,\ \subseteq,\ \leqq)$.

Relation $\leqq$. For $A=(a,\ i(A))\in\mathscr{I}$, $B=(b,\ i(B))\in\mathscr{I}$,

$$A\leqq B\leftrightarrow\begin{cases}i(A)i(B)=+ & \text{and } a\leqq b\\ i(A)=+,\ i(B)=- & \text{for arbitrary endpoints } a,\ b,\\ i(A)=-,\ i(B)=+ & \text{and } A=B=R^*.\end{cases}$$

Examples. $-\infty \leqq R^*$, $R^* \leqq \infty$, $[-\infty, 0] \leqq [0, \infty]$, $[-\infty, 0] \leqq [-\infty, \infty]$, $[-\infty, \infty] \leqq [0, \infty]$.

$\mathcal{M}$-addition. For $A=(a, i(A)) \in \mathcal{I}$, $B=(b, i(B)) \in \mathcal{I}$ the interval $A +^{-} B \in \mathcal{I}$ is defined by

$$A +^{-} B = \begin{cases} (a+b, \ -\sigma(a+b)), & \text{iff } i(A) = -i(B); \\ (-i(A))\infty, & \text{if } i(A) = i(B). \end{cases}$$

For $A=(a, i(A)) \in \mathcal{I}$, $B=[b^-, b^+] \in I(R)$ the interval $A +^{-} B \in \mathcal{I}$ and is given by

$$A +^{-} B = (a + b^{-i(A)}, \ i(A)).$$

$\mathcal{M}$-multiplication. For $A=(a, i(A)) \in \mathcal{I}$, $B=(b, i(B)) \in \mathcal{I}$ the interval $A \times^{-} B \in \mathcal{I}$ is defined by

$$A \times^{-} B = \begin{cases} (-i(A)i(B))\infty, & \text{if } A, B \in \mathcal{I} \backslash Z^*_{\mathcal{I}}; \\ (ab, \ -\sigma(ab)), & \text{otherwise.} \end{cases}$$

For $A=(a, i(A)) \in \mathcal{I}$, $B=[b^-, b^+] \in I(R)$ the interval $A \times^{-} B \in \mathcal{I}$ is defined by

$$A \times^{-} B = \begin{cases} (ab^{-\zeta(A)\sigma(B)} \ i(A)\sigma(B)), & \text{if } B \in I(R) \backslash Z; \\ (\mathrm{i}(A))\infty, & \text{if } B \in Z. \end{cases}$$

$\mathcal{M}$-subtraction. For $A=(a, i(A)) \in \mathcal{I}$, $B=(b, i(B)) \in \mathcal{I}$ the interval $A -^{-} B = A +^{-} (-B) \in \mathcal{I}$ and is given by

$$A -^{-} B = \begin{cases} (a-b, \ -\sigma(a-b)), & \text{if } i(A) = i(B); \\ (-i(A))\infty, & \text{if } i(A) = -i(B). \end{cases}$$

For $A=(a, i(A)) \in \mathcal{I}$, $B=[b^-, b^+] \in I(R)$ the interval $A -^{-} B \in \mathcal{I}$ and is given by

$$A -^{-} B = (a - b^{i(A)}, \ i(A)).$$

For $A=[a^-, a^+] \in I(R)$, $B=(b, i(B)) \in \mathcal{I}$ the interval $A -^{-} B \in \mathcal{I}$ and is given by

$$A -^{-} B = (a^{i(B)} - b, \ -i(B)).$$

$\mathcal{M}$-division. For $A=(a, i(A)) \in \mathcal{I}$, $B=(b, i(B)) \in \mathcal{I} \backslash Z_{\mathcal{I}}$ the interval $A /^{-} B = A \times^{-} (1/B) \in \mathcal{I}$ and is given by

$$A /^{-} B = \begin{cases} (a/b, \ -i(A)i(B)), & \text{if } A \in \mathcal{I} \backslash Z^*_{\mathcal{I}} \\ (0, \ -i(A)i(B)), & \text{if } A \in Z^*_{\mathcal{I}}. \end{cases}$$

For $A=(a, i(A)) \in \mathcal{I}$, $B=[b^-, b^+] \in I(R) \backslash Z$ the interval $A /^{-} B \in \mathcal{I}$ and is given by

$$A/^- B = (a/b^{\zeta(A)\sigma(B)},\ i(A)\sigma(B)).$$

For $A=[a^-,\ a^+]\in I(R)$, $B=(b,\ i(B))\in\mathcal{I}\backslash Z_{\mathcal{I}}$ the interval $A/^-B\in I(R)$ and is given by

$$A/^- B = \begin{cases} [0,\ a^{-\sigma(A)}/b]=[0,\ a^{i(B)}/b], & \text{if } A\in I(R)\backslash Z \text{ and } \sigma(A)=-i(B),\\ [a^{-\sigma(A)}/b,\ 0]=[a^{-i(B)}/b,\ 0], & \text{if } A\in I(R)\backslash Z \text{ and } \sigma(A)=i(B),\\ [0,\ 0], & \text{if } A\in Z. \end{cases}$$

The above expressions are also valid for the situation when the operands are degenerate ii-intervals of the form $-\infty$, R^*, ∞. In this case the extended real arithmetic $(R^*,\ +,\ \times,\ /,\ \leqq)$ from Method 1 should be used.

5. Extensions by infinite outer intervals (Kahan-intervals)

In this section we extend the set $\mathfrak{I}$ into a set $I(R^*)=\mathfrak{I}\cup\mathcal{O}$ involving infinite outer intervals (that is intervals which contain infinities in their interior). We then extend the definition domains of the corresponding relations and operations obtaining thus the extended interval structures $\mathcal{S}^*$, resp. $\mathcal{M}^*$, which admit division by intervals containing zero (intervals from $Z_{\mathfrak{I}}$). The space $\mathcal{S}^*$ is a substructure of $\mathcal{M}^*$; it is also a substructure of the space $\mathcal{K}^*$ introduced by E. Kaucher [14] which we shall mention in the end of section 6 (see also [9] and [21]).

Infinite outer intervals or briefly o-intervals, also known as Kahan-intervals (see [21]), are obtained as result of the operation A/B for $A\in I(R)$, $B\in Z^*$. To clarify this consider relation (9) for $A=1$, $B=[b^-,\ b^+]$ with $b^-<0<b^+$, i. e. $1/[b^-,\ b^+]=\{1/b\,|\,b\in[b^-,b^+]\}$. The right-hand side is not defined since $1/b$ is not defined for $b=0$. However, for sufficiently small $\varepsilon>0$

$$\begin{aligned} 1/[b^-,\ b^+] &= \lim_{\varepsilon\to 0}\{1/b\,|\,b\in[b^-,\ b^+]\backslash(-\varepsilon,\ \varepsilon)\}\\ &= \lim_{\varepsilon\to 0}\{1/b\,|\,b\in[b^-,\ -\varepsilon]\}\cup\lim_{\varepsilon\to 0}\{1/b\,|\,b\in[\varepsilon,\ b^+]\}\\ &= [1/b^+,\ \infty]\cup[-\infty,\ 1/b^-]. \end{aligned}$$

This is a reasonable motivation for considering sets of the form

$$\{x\,|\,x\geqq\alpha \text{ or } x\leqq\beta\}=[\alpha,\ \infty]\cup[-\infty,\ \beta],\ \alpha,\ \beta\in R,$$

which will be called o-intervals and will be denoted by $\langle\alpha,\ \beta\rangle$. To the set of o-intervals we shall add the limit cases $\langle\infty,\ \beta\rangle$, $\langle\alpha,\ -\infty\rangle$, $\langle\infty,\ -\infty\rangle$, defined by $\langle\infty,\ \beta\rangle=[-\infty,\ \beta]\cup\{\infty\}$, $\langle\alpha,\ -\infty\rangle=[\alpha,\ \infty]\cup\{-\infty\}$,
$\langle\infty,\ -\infty\rangle=\{\infty\}\cup\{-\infty\}=\{\infty,\ -\infty\}=\pm\infty$,
which will be referred as degenerated o-intervals. If $\alpha\leqq\beta$ then $\langle\alpha,\ \beta\rangle=[\alpha,\ \infty]\cup[-\infty,\ \beta]=R^*$; if $\alpha>\beta$ we shall say that the o-interval $\langle\alpha,\ \beta\rangle$ is normal. The definition of an o-interval obtains the form

(21) $\langle\alpha, \beta\rangle=\{x \mid x \geqq \alpha \text{ or } x \leqq \beta\}=[\alpha, \infty] \cup[-\infty, \beta], \quad \alpha, \beta \in R^*.$

The set of all o-intervals will be denoted by $\mathcal{O}^*=\{\langle\alpha, \beta\rangle \mid \alpha, \beta \in R^*\}$ and the set of all normal o-intervals will be denoted by $\mathcal{O}=\{\langle\alpha, \beta\rangle \mid \alpha, \beta \in R^*, a>\beta\}$ so that: $\mathcal{O}^*=\{\langle\alpha, \beta\rangle \mid \alpha, \beta \in R^*, \alpha>\beta\} \cup\{R^*\}=\mathcal{O} \cup\{R^*\}$.

We shall further say that the inner intervals (intervals from $\mathfrak{I}=I(R) \cup \mathcal{I}$) and the normal outer intervals (the intervals from $\mathcal{O}$) are intervals from different type. The interval set $I(R)$ extended by both types of infinite intervals will be denoted by

$$I(R^*)=\mathfrak{I} \cup \mathcal{O}=I(R) \cup \mathcal{I} \cup \mathcal{O}.$$

In the course of a computation involving normal o-intervals it may sometimes happen that the computed left endpoint a^- of an o-interval becomes less than the right endpoint a^+. Such a situation is called overlapping and can be expected by certain operations. The result of an operation which possibly leads to overlapping and therefore needs normalisation will be further marked by an asterik; during normalisation the interval changes its type from an $\mathcal{O}$-interval to an $\mathcal{I}$-interval:

$$A^*=\langle a^-, a^+\rangle^*=\begin{cases}\langle a^-, a^+\rangle, & \text{if } a^->a^+,\\ R^*, & \text{otherwise.}\end{cases}$$

Remark. For inner and outer intervals some authors are using same notations since a normal o-interval can be recognized by the relation $a^->a^+$ [14], [21], [30]. We are using special notation for the outer intervals in order to preserve the notation $[\alpha, \beta]$ with $\alpha>\beta$ for the improper intervals considered in section 6, which have a completely different meaning. E. Kaucher uses three different types of special notations: $(\alpha, \beta, 0)$ for inner intervals, $(\alpha, \beta, 1)$ for outer proper intervals and $(\alpha, \beta, -1)$ for outer improper intervals.

We now introduce the extended interval structure $\mathcal{S}^*=(I(R^*), +, \times, /, \subseteq)$. We first extend the definition domain of the relation $\subseteq$ from $I(R) \times I(R)$ into $I(R^*) \times I(R^*)$ by means of the following table (in what follows we denote $A=[a^-, a^+]$ if $A \in \mathfrak{I}$ and $A=\langle a^-, a^+\rangle$ if $A \in \mathcal{O}$):

Relation inclusion $A \subseteq B \leftrightarrow$

$B\in$ \ $A\in$	$\mathfrak{I}=I(R)\cup\mathcal{I}$	$\mathcal{O}$
$\mathfrak{I}$	$b^-\leqq a^-$ and $a^+\leqq b^+$	$B=R^*$
$\mathcal{O}$	$b^-\leqq a^-$ or $a^+\leqq b^+$	$b^-\leqq a^-$ and $a^+\leqq b^+$

Standard interval arithmetic operations involving o-intervals are introduced by means of the basic relation (9). The endpoint expressions for the operations involving (normal) o-intervals can be obtained by i) presenting the o-intervals as union of two intervals; ii) using the fact that the standard arithmetic operations and the operation "$\cup$" are distributive, and iii) using the endpoint expressions for i-intervals. The endpoint expressions for the standard operations involving o-intervals obtained in the above mentioned manner are presented below. For convenience we give tables for the standard operations involving both types of intervals.

Addition. If both intervals A, B are o-intervals ($A, B \in \mathcal{O}$), then $\mathcal{S}$-addition is defined by

$$A+B=\langle a^-, a^+\rangle+\langle b^-, b^+\rangle=R^*.$$

If the intervals A and B are from different type, say $A\in\mathfrak{I}$, $B\in\mathcal{O}$ then

$$A+B=[a^-, a^+]+\langle b^-, b^+\rangle=\langle a^-+b^-, a^++b^+\rangle^*$$

$$=\begin{cases}\langle a^-+b^-, a^++b^+\rangle, & \text{if } \omega(A)<\omega(B),\\ R^*, & \text{otherwise.}\end{cases}$$

In the above formula $\omega(\langle a^-, a^+\rangle)=|a^--a^+|=a^--a^+$. In particular, if $A\in\mathcal{S}$, then $\omega(A)=\infty$, and $A+B=R^*$.

The complete definition of addition for intervals from $I(R^*)$ can be summarised in the following table

Operation addition $A+B=B+A$

$B\in$ \\ $A\in$	$\mathfrak{I}=I(R)\cup\mathcal{S}$	$\mathcal{O}$
$\mathfrak{I}$	$[a^-+b^-, a^++b^+]$	$\langle a^-+b^-, a^++b^+\rangle^*$
$\mathcal{O}$	$\langle a^-+b^-, a^++b^+\rangle^*$	R^*

Examples. $\langle 1, -1\rangle+[-1, 1]=\langle 0, 0\rangle^*=R^*$, $\langle 1, -1\rangle+[0, 1]=\langle 1, 0\rangle$, $\langle 1, -1\rangle+\langle 1, -1\rangle=R^*$.

Multiplication. In order to define multiplication denote by $Z_{\mathcal{O}}$ the set of outer intervals which does not contain zero, that is $Z_{\mathcal{O}}=\{\langle a^-, a^+\rangle\,|\,a^+<0<a^-\}$; then the set $\mathcal{O}\backslash Z_{\mathcal{O}}=\{\langle a^-, a^+\rangle\,|\,a^+<a^-<0 \text{ or } 0<a^+<a^-\}$ is the set of outer intervals, which contain zero. The expressions for the interval product $A\times B$ in $I(R^*)$ is given by the following table

Operation multiplication $A\times B=B\times A$

B \ A	$\mathfrak{I}\backslash Z_{\mathfrak{I}}$	$Z_{\mathfrak{I}}$	$\mathcal{O}\backslash Z_{\mathcal{O}}$	$Z_{\mathcal{O}}$
$\mathfrak{I}\backslash Z_{\mathfrak{I}}$	$[a^{-\sigma(B)}b^{-\sigma(A)}, a^{\sigma(B)}b^{\sigma(A)}]$	$[a^{-\delta}b^{\delta}, a^{\delta}b^{\delta}]$, $\delta=\sigma(B)$	$\langle a^{-\sigma(B)}b^{-\sigma(A)}, a^{\sigma(B)}b^{\sigma(A)}\rangle^*$	$\langle a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}\rangle$, $\delta=\sigma(B)$
$Z_{\mathfrak{I}}$	$[a^{\delta}b^{-\delta}, a^{\delta}b^{\delta}]$, $\delta=\sigma(A)$	$[\min\{a^-b^+, a^+b^-\}, \max\{a^-b^-, a^+b^+\}]$	R^*	R^*
$\mathcal{O}\backslash Z_{\mathcal{O}}$	$\langle a^{-\sigma(B)}b^{-\sigma(A)}, a^{\sigma(B)}b^{\sigma(A)}\rangle^*$	R^*	R^*	R^*
$Z_{\mathcal{O}}$	$\langle a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}\rangle$, $\delta=\sigma(A)$	R^*	R^*	$\langle\min\{a^-b^-, a^+b^+\}, \max\{a^-b^+, a^+b^-\}\rangle$

Examples. $\langle 5/3, 1\rangle\times[1/5, 1/4]=\langle 1/3, 1/4\rangle$, $\langle 5/4, 1\rangle\times[1/5, 1/4]=\langle 1/4, 1/4\rangle^*=R^*$.

According to the above definition for the special case of scalar multiplication involving outer intervals we have

$$\alpha.\langle a^-, a^+\rangle=[\alpha,\alpha]\times\langle a^-, a^+\rangle=\begin{cases}\langle\alpha a^-, \alpha a^+\rangle\in\mathcal{O}, & \text{if } \alpha>0,\\ 0\in\mathfrak{I}, & \text{if } \alpha=0,\\ \langle\alpha a^+, \alpha a^-\rangle\in\mathcal{O}, & \text{if } \alpha<0;\end{cases}$$

in particular negation of an o-interval is given by

$$-1.\langle a^-, a^+\rangle=\langle -a^+, -a^-\rangle\in\mathcal{O}.$$

Division. Division is defined by $A/B=A\times(1/B)$, where inversion $1/B$ is defined by the following table

Operation inversion $1/B$

$B\in$	$1/B$	$1/B\in$
$\mathfrak{I}\backslash Z_{\mathfrak{I}}$	$[1/b^+, 1/b^-]$	$\mathfrak{I}$
$Z_{\mathfrak{I}}$	$\langle 1/b^+, 1/b^-\rangle$	$\mathcal{O}$
$\mathcal{O}\backslash Z_{\mathcal{O}}$	$\langle 1/b^+, 1/b^-\rangle$	$\mathcal{O}$
$Z_{\mathcal{O}}$	$[1/b^+, 0]\cup[0, 1/b^-]$	$I(R^*)$

In the first situation $B\in\mathfrak{I}\backslash Z_{\mathfrak{I}}$ the result $1/B$ may not belong to $\mathfrak{I}\backslash Z_{\mathfrak{I}}$ because if B^- or B^+ is equal to ∞ or $-\infty$, then some of the endpoints of the result will be zero and therefore the result is not in $\mathfrak{I}\backslash Z_{\mathfrak{I}}$. In the second case if some of the endpoints B^- or B^+ is zero we obtain a degenerated o-interval of the form $\langle\infty, \beta\rangle$, $\langle\alpha, -\infty\rangle$, $\langle\infty, -\infty\rangle$. The result in the case $B\in\mathcal{O}\backslash Z_{\mathcal{O}}$ may not belong to $\mathcal{O}\backslash Z_{\mathcal{O}}$ because if B^- or B^+ is equal to $-\infty$ or to ∞ then some of the endpoints of the result will be zero. In the last situation $B\in Z_{\mathcal{O}}$ the result can be either an inner interval or an outer interval. Indeed we have

$$1/B=[1/b^+, 0]\cup[0, 1/b^-]=\begin{cases}[1/b^+, 1/b^-]\in\mathfrak{I}, & \text{if } b^-b^+\neq 0,\\ \langle\infty, 1/b^-\rangle\in\mathcal{O}, & \text{if } b^+=0,\ b^-\neq 0,\\ \langle 1/b^+, -\infty\rangle\in\mathcal{O}, & \text{if } b^-=0,\ b^+\neq 0,\\ \langle\infty, -\infty\rangle\in\mathcal{O}, & \text{if } b^-=b^+=0.\end{cases}$$

By showing that the results of the arithmetic operations are always elements of $I(R^*)$ we thus proved that $\mathcal{S}^*$ is a closed interval space. The properties of $\mathcal{S}^*$ are studied in [9], [14].

We now introduce the extended interval structure $\mathcal{M}^*=(I(R^*), +, +^-, \times, \times^-, \subseteq, \leq)$ by defining a relation $\leq$ and two operations $+^-$ and $\times^-$.

Relation $A\leq B$

$B\in$ \ $A\in$	$\mathfrak{I}=I(R)\cup$	$\mathcal{O}$
$\mathfrak{I}$	$a^-\leq b^-$ and $a^+\leq b^+$	$B=[b^-, \infty]$ and $b^-\geq a^-$
$\mathcal{O}$	$A=[-\infty, a^+]$, $a^+\leq b^+$	$a^-\leq b^-$ and $a^+\leq b^+$

$\mathcal{M}$-addition is defined as follows. For two outer intervals, $A, B\in\mathcal{O}$ we set ($\alpha=\varphi(A, B)$):

$$A+^-B=\langle a^-, a^+\rangle+^-\langle b^-, b^+\rangle=\langle a^{-\alpha}+b^{\alpha}, a^{\alpha}+b^{-\alpha}\rangle$$
$$=\begin{cases}\langle a^-+b^+, a^++b^-\rangle, & \text{if } \omega(A)\geq\omega(B),\\ \langle a^++b^-, a^-+b^+\rangle, & \text{if } \omega(A)\geq\omega(B).\end{cases}$$

If A and B are from different type, then we set:

$$A+^-B=\begin{cases}\langle a^-, a^+\rangle+^-[b^-, b^+]=\langle a^-+b^+, a^++b^-\rangle, & \text{for } A\in\mathcal{O},\ B\in\mathfrak{I};\\ [a^-, a^+]+^-\langle b^-, b^+\rangle=\langle a^++b^-, a^-+b^+\rangle, & \text{for } A\in\mathfrak{I},\ B\in\mathcal{O}.\end{cases}$$

The above definitions are summarized in the following table ($\alpha=\varphi(A, B)$):

Operation $\mathscr{M}$-addition $A+^{-}B=B+^{-}A$

$B\in$ \ $A\in$	$\mathfrak{I}=I(R)\cup\mathscr{I}$	$\mathcal{O}$
$\mathfrak{I}$	$[a^{-\alpha}+b^{\alpha}, a^{\alpha}+b^{-\alpha}]$	$\langle a^{-}+b^{+}, a^{+}+b^{-}\rangle$
$\mathcal{O}$	$\langle a^{+}+b^{-}, a^{-}+b^{+}\rangle$	$\langle a^{-\alpha}+b^{\alpha}, a^{\alpha}+b^{-\alpha}\rangle$

The nonstandard multiplication is introduced as follows ($\varepsilon=\psi(A, B)$ below):

Operation $\mathscr{M}$-multiplication $A\times^{-}B=B\times^{-}A$

B \ A	$\mathfrak{I}\setminus Z_{\mathfrak{I}}$	$Z_{\mathfrak{I}}$	$\mathcal{O}\setminus Z_{\mathcal{O}}$	$Z_{\mathcal{O}}$
$\mathfrak{I}\setminus Z_{\mathfrak{I}}$	$[a^{\sigma(B)\varepsilon}b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}b^{\sigma(A)\varepsilon}]$	$[a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}]$, $\delta=\sigma(B)$	$\langle a^{\sigma(B)\varepsilon}b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}b^{\sigma(A)\varepsilon}\rangle$	$\langle a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}\rangle$, $\delta=\sigma(B)$
$Z_{\mathfrak{I}}$	$[a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}]$, $\delta=\sigma(A)$	$[\max\{a^{-}b^{+}, a^{+}b^{-}\}, \min\{a^{-}b^{-}, a^{+}b^{+}\}]$	$\langle a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}\rangle$, $\delta=\sigma(A)$	$\langle\max\{a^{-}b^{+}, a^{+}b^{-}\}, \min\{a^{-}b^{-}, a^{+}b^{+}\}\rangle$
$\mathcal{O}\setminus Z_{\mathcal{O}}$	$\langle a^{\sigma(B)\varepsilon}b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}b^{\sigma(A)\varepsilon}\rangle$	$\langle a^{-\delta}b^{-\delta}, a^{\delta}b^{-\delta}\rangle$, $\delta=\sigma(B)$	$\langle a^{\sigma(B)\varepsilon}b^{-\sigma(A)\varepsilon}, a^{-\sigma(B)\varepsilon}b^{\sigma(A)\varepsilon}\rangle$	$\langle a^{-\delta}b^{\delta}, a^{\delta}b^{\delta}\rangle$, $\delta=\sigma(B)$
$Z_{\mathcal{O}}$	$\langle a^{-\delta}b^{-\delta}, a^{-\delta}b^{\delta}\rangle$, $\delta=\sigma(A)$	$\langle\max\{a^{-}b^{+}, a^{+}b^{-}\}, \min\{a^{-}b^{-}, a^{+}b^{+}\}\rangle$	$\langle a^{\delta}b^{-\delta}, a^{\delta}b^{\delta}\rangle$, $\delta=\sigma(A)$	$\langle\max\{a^{-}b^{-}, a^{+}b^{+}\}, \min\{a^{-}b^{+}, a^{+}b^{-}\}\rangle$

Due to the overlapping effect not all of the properties S1 − S7 and M1 − M7 remain true in the extended interval structure $\mathscr{M}^{*}=(I(R^{*}), +, +^{-}, \times, \times^{-}, \subseteq, \leqq)$ defined above. In Section 8 we briefly discuss a possibility for the validity of all these properties.

6. The interval spaces $\mathscr{K}=(\mathscr{H}, +, \times, \subseteq)$, $\mathscr{K}^{*}=(\mathscr{H}^{*}, +, \times, \subseteq)$

In this section we consider an extension of the definition domains of the interval-arithmetic relations and operations from $I(R)$ into the set $\mathscr{H}=\{[a, b]\,|\,a, b\in R\}\cong R^{2}$ of all ordered couples of real numbers [14]−[19], [31]. The first component of $A\in\mathscr{H}$ is denoted by a^{-} or A^{-}, the second one by a^{+} or A^{+}, so that

$A=[a^-, a^+]=[A^-, A^+]$. The elements of $\mathscr{H}$ are called generalized intervals; a generalized interval $A=[a^-, a^+]\in\mathscr{H}$ is a proper (regular) one if $a^-\leqq a^+$, and improper one if $a^-\geqq a^+$ (degenerated intervals with $a^-=a^+$ will be considered both as proper and improper). The set of all elements of $\mathscr{H}$, which are proper intervals is equivalent to $I(R)$ and is further denoted again by $I(R)$; the set of all improper intervals is denoted by $\overline{I(R)}$, so that $\mathscr{H}=I(R)\cup\overline{I(R)}$. Define for $A\in\mathscr{H}\backslash\mathscr{T}$, $\mathscr{T}=Z\cup\bar{Z}$, $\bar{Z}=\{A\,|\,a^+\leqq 0\wedge a^-\geqq 0\}$

$$\sigma(A)=\begin{cases}+, & \text{if } 0\leqq a^-,\ 0\leqq a^+;\\ -, & \text{if } a^-\leqq 0,\ a^+\leqq 0 \ (\text{but } A\neq[0, 0]).\end{cases}$$

The interval arithmetic structure $\mathscr{K}=(\mathscr{H}, +, \times, \subseteq)$ is obtained by extending the definition domains of $+$, $\times$ and $\subseteq$ as defined in $\mathscr{S}$ from $I(R)$ to $\mathscr{H}$. A formal substitution of $I(R)$ by $\mathscr{H}$ and of Z by $\mathscr{T}$ in (2)–(4) yields the definitions of $+$, $\times$ and $\subseteq$ in $\mathscr{H}$:

$$A\subseteq B\leftrightarrow(b^-\leqq a^-)\wedge(a^+\leqq b^+), \quad \text{for } A,\ B\in\mathscr{H},$$

$$\text{(22)}\qquad A+B=[a^-+b^-,\ a^++b^+], \quad \text{for } A,\ B\in\mathscr{H},$$

$$\text{(23)}\qquad A\times B=\begin{cases}[a^{-\sigma(B)}b^{-\sigma(A)},\ a^{\sigma(B)}b^{\sigma(A)}], & \text{for } A,\ B\in\mathscr{H}\backslash\mathscr{T},\\ [a^{\delta}b^{-\delta},\ a^{\delta}b^{\delta}],\ \delta=\sigma(A), & \text{for } A\in\mathscr{H}\backslash\mathscr{T},\ B\in\mathscr{T},\\ [a^{-\delta}b^{\delta},\ a^{\delta}b^{\delta}],\ \delta=\sigma(B), & \text{for } A\in\mathscr{T},\ B\in\mathscr{H}\backslash\mathscr{T}.\end{cases}$$

Using the above definition of „$\subseteq$“ we can write $\bar{Z}=\{A\in\overline{I(R)}\,|\,A\subseteq 0\}$. In addition, let us assume [17], [31] the following extension of (5), which accomplishes the definition of $A\times B$ for the situation when both A, $B\in\mathscr{T}$:

$$\text{(24)}\qquad A\times B=\begin{cases}[\min\{a^-b^+,\ a^+b^-\},\ \max\{a^-b^-,\ a^+b^+\}], & \text{for } A,\ B\in Z,\\ [\max\{a^-b^-,\ a^+b^+\},\ \min\{a^-b^+,\ a^+b^-\}], & \text{for } A,\ B\in\bar{Z},\\ 0, \quad \text{for } A\in Z\wedge B\in\bar{Z} \text{ or } A\in\bar{Z}\wedge B\in Z.\end{cases}$$

From (23) for $A=[a, a]=a$, $B\in\mathscr{H}$, we have $a.B=[a, a]\times B=[ab^{-\sigma(a)}, ab^{\sigma(a)}]$. Substituting $a=-1$ we obtain $(-1)\times B=-B=[-b^+, -b^-]$. The compound operation $A+(-1)\times B=A+(-B)=[a^--b^+,\ a^+-b^-]$, for A, $B\in\mathscr{H}$ is an extension of the $\mathscr{S}$-subtraction (7) into $\mathscr{H}$ and will be further denoted $A-B$.

The substructures $(\mathscr{H}, +, \subseteq)$ and $(\mathscr{H}\backslash\mathscr{T}, \times, \subseteq)$ of $\mathscr{K}$ are isotone groups [17], [19]; hence there exist inverse elements with respect to the operations (22) and (23). Denote the inverse additive element of $A\in\mathscr{H}$ by $-_hA$, and the inverse element of $A\in\mathscr{H}\backslash\mathscr{T}$ w. r. t. "$\times$" by $1/_hA$. For the inverse elements we obtain the end-point presentations $-_hA=[-a^-, -a^+]$, for $A\in\mathscr{H}$, and $1/_hA=[1/a^-, 1/a^+]$, for $A\in\mathscr{H}\backslash\mathscr{T}$.

The inverse additive element $-_h A=[-a^-, -a^+]$ in $\mathscr{H}$ should not be confused with the negative element $-A$ obtained by multiplication with -1, i. e. $-A=(-1)\times A=[-a^+, -a^-]$. Using the monadic operators $-A=[-a^+, -a^-]$ and $-_h A=[-a^-, -a^+]$ we can compose the monadic operator $-_h(-A)=-(-_h A)$, which is called [17, 19] conjugation and is further denoted by $\bar{A}$ or A_-. We can write

$$\bar{A}=A_-=-_h(-A)=-(-_h A)=[a^+, a^-]. \tag{25}$$

Note that conjugation in $\mathscr{H}$ is a compound operation derived from the basic operations $+$, $\times$ and their inverse elements.

Equalities (25) suggest to check if there exists a monadic operator $\gamma(A)$ in $\mathscr{H}\backslash\mathscr{T}$ which possibly satisfies the relations $1/_h\gamma(A)=\gamma(1/_h A)=\bar{A}$. It is easy to see that such is the unique operator $\gamma(A)=\overline{1/_h A}=1/_h\bar{A}=[1/a^+, 1/a^-]$, for $A\in\mathscr{H}\backslash\mathscr{T}$; denoting $\gamma(A)=1/A$ (since $\gamma(A)$ is an extension of $1/A$ for $A\in I(R)$) we have

$$1/_h(1/A)=1/(1/_h A)=\bar{A}. \tag{26}$$

We can now compose the operation $A\times(1/B)$ for $A\in\mathscr{H}$, $B\in\mathscr{H}\backslash\mathscr{T}$. This operation, which will be further denoted by A/B, is an extension in $\mathscr{H}$ of the $\mathscr{S}$-operation A/B defined by (6):

$$A/B=A\times(1/B)=\begin{cases}[a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}] & \text{for } A, B\in\mathscr{H}\backslash\mathscr{T},\\ [a^{-\delta}/b^{-\delta}, a^{\delta}/b^{-\delta}],\ \delta=\sigma(B) & \text{for } A\in\mathscr{T},\ B\in\mathscr{H}\backslash\mathscr{T}.\end{cases}$$

From $\bar{A}=-(-_h A)$ and $\bar{A}=1/(1/_h A)$ (see (25) and (26) above) we obtain the following easy-to-memorize expressions for the inverse elements: $-_h A=-\bar{A}$, $1/_h A=1/\bar{A}$. The inverse elements $-_h A$, $1/_h A$ generate operations $A+(-_h B)$, $A\times(1/_h B)$, which are inverse to the operations $A+B$ and $A\times B$, respectively. Denoting these two operations by $A-_h B$ and $A/_h B$, resp., we have

$$A-_h B=A+(-_h B)=[a^- - b^-, a^+ - b^+],\quad A, B\in\mathscr{H},$$

$$A/_h B=A\times(1/_h B)$$
$$=\begin{cases}[a^{-\sigma(B)}/b^{-\sigma(A)}, a^{\sigma(B)}/b^{\sigma(A)}], & A, B\in\mathscr{H}\backslash\mathscr{T},\\ [a^{-\delta}/b^{\delta}, a^{\delta}/b^{\delta}],\ \delta=\sigma(B), & A\in\mathscr{T},\ B\in\mathscr{H}\backslash\mathscr{T}.\end{cases}$$

The inverse operations can be expressed by means of the conjugation (25) in the following way:

$$A-_h B=A+(-\bar{B})=A-\bar{B},$$

$$A/_h B=A\times(1/\bar{B})=A/\bar{B}.$$

From the last equality we obtain $A/B=A/_h\bar{B}=A/_h(-_h((-1)\times B))$, showing that the operation for division "/" can be composed by means of the operations "+", "×" and their inverse operations $-_h$ and $/_h$ (alternatively to the situation in

$\mathscr{S}$ where "/" is an independent operation). Since division "/" in $\mathscr{H}$ can be derived from operations (22)−(24) and their inverse the symbol "/" should not be necessarily included[2] in the set of basic operations in $\mathscr{K}$, resp. in the notation for $\mathscr{K}=(\mathscr{H}, +, \times, \subseteq)$.

Summarizing, the interval space $\mathscr{K}=(\mathscr{H}, +, \times, \subseteq)$ involves the operations subtraction "−", division "/", conjugation and the inverse operations $A-\bar{B}$, $A/\bar{B}$. Other useful compound operations are $A+\bar{B}$ and $A\times\bar{B}$; their end-point presentations are resp.: $A+\bar{B}=[a^-+b^+,\ a^++b^-]$ for $A, B\in\mathscr{H}$, and

$$A\times\bar{B}=\begin{cases}[a^{-\sigma(B)}\, b^{\sigma(A)},\ a^{\sigma(B)}\, b^{-\sigma(A)}] & \text{for } A,\ B\in\mathscr{H}\backslash\mathscr{T},\\ [a^{\delta}\, b^{\delta},\ a^{\delta}\, b^{-\delta}],\ \delta=\sigma(A), & \text{for } A\in\mathscr{H}\backslash\mathscr{T},\ B\in\mathscr{T},\\ [a^{-\delta}\, b^{-\delta},\ a^{\delta}\, b^{\delta}],\ \delta=\sigma(B), & \text{for } A\in\mathscr{T},\ B\in\mathscr{H}\backslash\mathscr{T} \text{ (note that } \sigma(B)=\sigma(\bar{B})!).\end{cases}$$

More generally, denoting $A_-=\bar{A}$, $A_+=A$ for $t, s, \in\{+,-\}$, we can write (22), (23) in the form

$$A_t+B_s=[a^{-t}+b^{-s},\ a^t+b^s], \text{ for } A,\ B\in\mathscr{H}, \tag{27}$$

$$A_t\times B_s=\begin{cases}[a^{-t\sigma(B)}\, b^{-s\sigma(A)},\ a^{t\sigma(B)}\, b^{s\sigma(A)}], & \text{for } A,\ B\in\mathscr{H}\backslash\mathscr{T},\\ [a^{t\delta}\, b^{-s\delta},\ a^{t\delta}\, b^{s\delta}],\ \delta=\sigma(A), & \text{for } A\in\mathscr{H}\backslash\mathscr{T},\ B\in\mathscr{T},\\ [a^{-t\delta}\, b^{s\delta},\ a^{t\delta}\, b^{s\delta}],\ \delta=\sigma(B), & \text{for } A\in\mathscr{T},\ B\in\mathscr{H}\backslash\mathscr{T}.\end{cases} \tag{28}$$

Formulae (27), (28) allow to compute the following expressions: $A+B$, $\bar{A}+B$, $A+\bar{B}$, $\bar{A}+\bar{B}$, $A\times B$, $\bar{A}\times B$, $A\times\bar{B}$, $\bar{A}\times\bar{B}$.

By analogy with $A-_hB=A-\bar{B}$, $A/_hB=A/\bar{B}$ we may denote $A+_hB=A+\bar{B}$, $A\times_hB=A\times\bar{B}$. Then $\bar{A}+B=\overline{A+_hB}$, $\bar{A}\times B=\overline{A\times_hB}$.

We next summarize the main properties of the interval structure $\mathscr{K}=(\mathscr{H}, +, \times, \subseteq)$ (see [17], [19]). If not specified $A, B, C, \ldots$ denote elements of $\mathscr{H}$. With one exception (**K5.** i)) all properties can be found in [19].

K1. $A+B=B+A$, $A\times B=B\times A$.

K2. $(A+B)+C=A+(B+C)$, $(A\times B)\times C=A\times(B\times C)$.

K3. $X=[0, 0]=0$ and $Y=[1, 1]=1$ are the unique neutral elements with respect to addition and multiplication; that is,

$$A=X+A\leftrightarrow X=[0,0];\ A=Y\times A\leftrightarrow Y=[1,\ 1].$$

K4. Every element $A\in\mathscr{H}$ has an unique inverse element with respect to + and every element $A\in\mathscr{H}\backslash\mathscr{T}$ possesses an unique inverse element with respect to ×. These are the elements $-\bar{A}$, resp. $1/\bar{A}$, i. e.: $0=A+(-\bar{A})=A-\bar{A}$ and $1=A\times(1/\bar{A})=A/\bar{A}$.

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[2] This fact has been possibly not noticed by E. Kaucher who is using a notation for $\mathscr{K}$ of the form $(\mathscr{H}, +, \times, /, \subseteq)$.

K5. i) For A, B, C, $A+B\in\mathcal{K}\setminus\mathcal{T}$ (see Proposition 1 below)

$$(A+B)\times C=(A\times C_{\sigma(A)\sigma(A+B)})+(B\times C_{\sigma(B)\sigma(A+B)});$$

ii) For A, B, $C\in I(R)$ we have $A(B+C)\subseteq AB+AC$.

iii) For A, B, $C\in\overline{I(R)}$ we have $A(B+C)\supseteq AB+AC$.

K6. Let $*\in\{+,-,\times,/\}$. Then $X\subseteq X_1 \Rightarrow X*C\subseteq X_1*C$. As a corollary we have $X\subseteq X_1$, $Y\subseteq Y_1\Rightarrow X*Y\subseteq X_1*Y_1$.

K7. $\mathcal{K}$ is a lattice w. r. t. $\subseteq$. The lattice operations w. r. t. $\subseteq$ are the intersection (the meet) and the connected union (the joint) of two $\mathcal{K}$-intervals:

$$\inf_{\subseteq}(A,\ B)=[\max\{A^-,\ B^-\},\ \min\{A^+,\ B^+\}]=[A\wedge B],$$

$$\sup_{\subseteq}(A,\ B)=[\min\{A^-,\ B^-\},\ \max\{A^+,\ B^+\}]=[A\vee B].$$

The lattice operations satisfy the following properties

i) For A, $B\in\mathcal{K}$: $(A\vee B)+C=(A+B)\vee(B+C)$, $(A\wedge B)+C=(A+B)\wedge(B+C)$

ii) For A, $B\in\mathcal{K}\setminus\mathcal{T}$: $(A\vee B)\times C=(A\times B)\vee(B\times C)$, $(A\wedge B)\times C=(A\times B)\wedge(B\times C)$.

K8. Conjugation satisfies the following properties

$$\overline{A+B}=\bar{A}+\bar{B},\quad \overline{A\times B}=\bar{A}\times\bar{B},\quad \overline{A\wedge B}=\bar{A}\wedge\bar{B},$$

$$A+\bar{A}=a^- - a^+\in R,\ A\times\bar{A}=\{a^-\ a^+,\ \text{if } A\in\mathcal{K}\setminus\mathcal{T};\ 0,\ \text{if } A\in\mathcal{T}\}$$

K9. For A, $B\in\mathcal{K}\setminus\mathcal{T}$, $1/(A\times B)=(1/A)\times(1/B)$, $1/(A/B)=B/A$.

Proposition 1. *We have for* A, B, C, $A+B\in\mathcal{K}\setminus\mathcal{T}$,

$$(A+B)\times C=\begin{cases}(A\times C)+(B\times C) & \text{if } \sigma(A)=\sigma(B)\ \ (=\sigma(A+B)),\\ (A\times C)+(B\times\bar{C}) & \text{if } \sigma(A)=-\sigma(B)=\sigma(A+B),\\ (A\times\bar{C})+(B\times C) & \text{if } \sigma(A)=-\sigma(B)=-\sigma(A+B).\end{cases}$$

Proof. Using the definitions for multiplication and addition, we obtain

$$(A+B)\times C=[(A+B)^-,\ (A+B)^+]\times C=[(A+B)^{-\sigma(C)}\,C^{-\sigma(A+B)},\ (A+B)^{\sigma(C)}\,C^{\sigma(A+B)}]$$

$$=[(A^{-\sigma(C)}+B^{-\sigma(C)})\,C^{-\sigma(A+B)},\ (A^{\sigma(C)}+B^{\sigma(C)})\,C^{\sigma(A+B)}]$$

$$=[A^{-\sigma(C)}\,C^{-\sigma(A+B)}+B^{-\sigma(C)}C^{-\sigma(A+B)},\ A^{\sigma(C)}\,C^{\sigma(A+B)}+B^{\sigma(C)}C^{\sigma(A+B)}].$$

Computing in a similar way the expressions in the right-hand side of Proposition 1 we obtain the proof.

The interval structure $\mathcal{K}$ can be extended in the outlined in sections 4 and 5 manner by infinite intervals. The structure $\mathcal{K}^*$ thus obtained is considered in detail in [9], [14] and will not be discussed here.

7. Generalized and directed intervals

For nondegenerate intervals define the operator $\tau:\mathcal{H}\backslash R\to\{+,\ -\}$ by $\tau(A)=\{+,\text{ if } A\in I(R);\ -,\text{ if } A\in\overline{I(R)}\}$; for degenerate intervals $\tau([a, a])$ can be either $+$ or $-$. Using τ we obtain for $A\in\mathcal{H}$ the presentation $A=(\tau(A),\ A_{\tau(A)})$, where $\tau(A)\in\{+,\ -\}$, $A_{\tau(A)}=\{A,\ \text{if } \tau(A)=+;\ \ \bar{A},\ \text{if } \tau(A)=-\}\in I(R)$. Obviously, $A=\{(+, A), \text{if } A\in I(R);\ (-, \bar{A}), \text{for } A\in\overline{I(R)}\}$. A couple of the form $(\pm, A)$, $A\in I(R)$, will be further referred as directed interval. The sets $\mathcal{H}$ and $\{+,\ -\}\times I(R)$ are equivalent and we shall seek relations between the spaces $\mathcal{K}$ and $\mathcal{S}$ and between $\mathcal{K}$ and $\mathcal{M}$, respectively. The next three propositions show how in certain cases the $\mathcal{K}$-operations $+$, $\times$ and the $\mathcal{K}$-relation $\subseteq$ can be interpreted in $\mathcal{S}$ in the situation $\tau(A)=\tau(B)$. Define the center of a generalized interval $A\in\mathcal{H}$ by setting again $\mu(A)=(a^-+a^+)/2$. Further for $A\in\mathcal{H}$ define $\omega(A)=|a^+-a^-|$ and for $A\neq[0, 0]$ define $\chi(A)=\{a^-/a^+\ \text{if } 0\leqq\tau(A)\mu(A);\ a^+/a^-\ \text{if } 0>\tau(A)\mu(A)\}$.

Proposition 2. *For A, $B\in\mathcal{H}$, such that $\tau(A)=\tau(B)=\tau$ we have*

$$A+B=(\tau,\ A_{\tau(A)}+B_{\tau(B)}). \tag{29}$$

Proposition 3. *For A, $B\in\mathcal{H}$, such that $\tau(A)=\tau(B)=\tau$ we have*

$$A\times B=(\tau,\ A_{\tau(A)}\times B_{\tau(B)}). \tag{30}$$

Proposition 4. *For A, $B\in\mathcal{H}$, $\tau(A)=\tau(B)$ we have*

$$A\subseteq B\leftrightarrow\begin{cases}A\subseteq B, & \text{if } A,\ B\in I(R),\\ \bar{A}\supseteq\bar{B}, & \text{if } A,\ B\in\overline{I(R)}.\end{cases}$$

Remark. In the situation $\tau(A)\neq\tau(B)$ we can not obtain similar correspondences between $\mathcal{K}$ and $\mathcal{S}$. However, propositions 2–4 can be generalized to give a full correspondence between $\mathcal{K}$ and $\mathcal{M}$ covering all possible cases. We shall thus show that $\mathcal{K}$ and $\mathcal{M}$ are in certain sense equivalent extensions of $\mathcal{S}$, inspite of the fact that the extended interval spaces $\mathcal{K}$ and $\mathcal{M}$ are obtained in completely different ways. Recall that $\mathcal{K}$ is obtained by: i) a generalization of the concept of interval (i. e. by an extension of the support $I(R)$ of $\mathcal{S}$ into the set $\mathcal{H}$), and ii) by an extension of the definition domains of the operations for addition and multiplication and of the relation for inclusion from $I(R)$ into $\mathcal{H}$. On the other side $\mathcal{M}$ is obtained by introducing two supplementary interval-arithmetic operations and a new relation, using thereby the conventional concept of interval (i. e. element of $I(R)$).

The $\mathcal{M}$-operations $+^-$, $\times^-$ and the relation $\leqq$ can be extended from $I(R)$ into $\mathcal{H}$ similarly to the extension of $+$, $\times$ that is by formally substituting $I(R)$ in (10), (11), (12) by $\mathcal{H}$ and Z by $\mathcal{T}$ (in the case $A, B\in\mathcal{T}$ expression (11) is extended by (24) where min and max exchange places).

The extended operation $+^-$ in $\mathcal{K}$ can be expressed as composite operation of the operations addition and conjugation in $\mathcal{K}$ by

$$A+^{-}B=\{C, \text{ if } \omega(A)\geqq\omega(B);\ \bar{C}, \text{ if } \omega(A)<\omega(B)\},\ C=A+\bar{B}.$$

We thus see that the extended arithmetic operation $+^{-}$ in $\mathscr{H}$ is a compound operation in $(\mathscr{H}, +, \times)$ for each one of the two cases $\omega(A)\geqq\omega(B)$ and $\omega(A)<\omega(B)$. Note that the operations $+$ and $+^{-}$ in $I(R)$ are independent, whereas their extensions $+$ and $+^{-}$ in $\mathscr{H}$ are interrelated. Each one of the three operations $+$, $+^{-}$ and conjugation in $\mathscr{K}$ can be expressed by means of the remaining two operations as follows:

$$A+B=\{A+^{-}\bar{B}, \quad \text{if } \omega(A)\geqq\omega(B);\ \bar{A}+^{-}B, \quad \text{if } \omega(A)<\omega(B)\},$$

$$\bar{A}=(-A)+(A+^{-}A).$$

Similarly the extended operation $\times^{-}$ in $\mathscr{H}$ can be expressed by the basic operations in $\mathscr{H}$. We have

$$A\times^{-}B=\{C, \quad \text{if } \chi(A)\geqq\chi(B);\ \bar{C}, \quad \text{if } \chi(A)<\chi(B)\},\ C=A\times\bar{B},$$

$$\bar{A}=(1/A)\times(A\times^{-}A).$$

The following propositions generalise formulae (29), (30) and show how any assertion in $\mathscr{K}$ can be formulated in terms of directed intervals using $\mathscr{M}$-operations.

Proposition 5. *For A, $B\in\mathscr{H}$, we have*

$$A+B=\begin{cases}(\tau(\max_{\omega}(A, B)),\ A_{\tau(A)}+B_{\tau(B)}) & \text{for } \tau(A)=\tau(B),\\ (\tau(\max_{\omega}(A, B)),\ A_{\tau(A)}+^{-}B_{\tau(B)}) & \text{for } \tau(A)\neq\tau(B);\end{cases}$$
$$=(\tau(\max_{\omega}(A, B)),\ A_{\tau(A)}+^{\tau(A)\tau(B)}B_{\tau(B)}),$$

where $\max_{\omega}(A, B)=\{A, \text{ if } \omega(A)\geqq\omega(B);\ B, \text{ if } \omega(A)<\omega(B)\}$.

Proposition 6. *For A, $B\in\mathscr{H}$, we have*

$$A\times B=(\tau(\max_{\chi}(A, B)),\ A_{\tau(A)}\times^{\tau(A)\tau(B)}B_{\tau(B)}),$$

where $\max_{\chi}(A, B)=\{A, \text{ if } \chi(A)\geqq\chi(B);\ B, \text{ if } \chi(A)<\chi(B)\}$.

Proposition 7. *For A, $B\in\mathscr{H}$, the $\mathscr{K}$-relation $A\subseteq B$ is equivalent to one of the following $\mathscr{M}$-relations ("$\vee$" means "or" below):*

i) $A\subseteq B$ if A, $B\in I(R)$;

ii) $\bar{A}\supseteq\bar{B}$ if A, $B\in\overline{I(R)}$;

iii) $\bar{A}\supseteq B\vee\bar{A}\subseteq B\vee(\bar{A}\leqq B\wedge b^{-}\in\bar{A})\vee(\bar{A}\geqq B\wedge b^{+}\in\bar{A})$ if $A\in\overline{I(R)}$, $B\in I(R)$.

R e m a r k. The situation $A\in I(R)$, $B\in\overline{I(R)}$, $A\neq B$, contradicts to the assumption $A\subseteq B$ and hence is not possible.

E x a m p l e. As an example let us transform the $\mathscr{K}$-assertion: "For $X, X_1, Y, Y_1\in\mathscr{H}$, $X\subseteq X_1\wedge Y\subseteq Y_1\Rightarrow X+Y\subseteq X_1+Y_1$" into an $\mathscr{M}$-assertion using Proposition 5 and Proposition 7. According to Proposition 7 we should consider $6^2=36$

subcases. For instance in the subcase $X, X_1 \in I(R)$, $Y, Y_1 \in \overline{I(R)}$ the above assertion reads: "For $X, X_1, Y_-, Y_{1-} \in I(R)$, $X \subseteq X_1 \wedge Y_- \supseteq Y_{1-} \Rightarrow \{X +^- Y_- \subseteq X_1 +^- Y_{1-}$ if $\omega(X) \geqq \omega(Y_-)$; $X +^- Y_- \supseteq X_1 +^- Y_{1-}$ if $\omega(X) \leqq \omega(Y_{1-})\}$", taking into account that $\omega(X) \geqq \omega(Y_-) \Rightarrow \omega(X_1) \geqq \omega(Y_{1-})$ and $\omega(X_1) \leqq \omega(Y_{1-}) \Rightarrow \omega(X) \leqq \omega(Y_-)$. We can proceed in a similar way in the rest of the cases. Propositions 2–4 give an interpretation of the $\mathcal{K}$-structure in the space $\mathcal{S}$ only in the situation when $\tau(A) = \tau(B)$. The situation $\tau(A) \neq \tau(B)$ can not be interpreted in $\mathcal{S}$. A full interpretation of the space $\mathcal{K}$ is achieved in $\mathcal{M}$ by means of Propositions 5–7. These propositions show how any result in $\mathcal{K}$ can be formally transferred in $\mathcal{M}$. The opposite is also true, since the space $\mathcal{M}$ has a natural extension in $\mathcal{K}$.

8. Notes on applications

Finally let us mention several applications of the extended interval arithmetic structures considered in this paper. The basis of a differential calculus for discontinuous functions (complemented to so-called segment functions) is developed in [40] by essensially using the interval arithmetic algebraic structure $(\mathcal{I}, +, \times)$, which can be considered as a substructure of $\mathcal{S}^*$. An interval calculus for interval functions has been developed in [22, 26] on the basis of the substructure $(I(R), +, +^-, \times)$ and for a class of generalized interval functions (involving discontinuities and including segment functions as special case) based on $(\mathcal{I}, +, +^-, \times)$ in [2]. A theory of integration has been proposed in [4] founded on the subspace $(\mathcal{I}, +, \times, /)$. A method for nonlinear equations has been proposed by E. Hansen which is essensially based on the space $(I(R^*), +, \times, /)$.

The spaces $\mathcal{M}, \mathcal{M}^*, \mathcal{K}, \mathcal{K}^*$ can be applied to construct interval-arithmetic expressions for inner and outer inclusions of ranges of functions or for exact representation of ranges of monotone functions. To be more specific let us first briefly outline the application of $\mathcal{M}$ for the simplest case of continuous monotone functions of one variable.

Denote by $\mathrm{CM}(D)$ the set of all continuous functions which are monotone on $D \in I(R)$. For a function $f \in \mathrm{CM}(D)$ and $X \subseteq D$ denote $m(f; X) = \{+$, if f is nondecreasing in D; $-$, if f is nonincreasing in $D\}$. Then for $f, g \in \mathrm{CM}(D)$, $m(f; D) = m(g; D)$ means that both functions are nondecreasing (isotone) or both are nonicreasing (antitone) in D; $m(f; D) = -m(g; D)$ means that one of the functions is nondecreasing and the other is nonincreasing. Then the following proposition holds true [27]:

Proposition 8. *For $f, g \in \mathrm{CM}(D)$, $X \subseteq D$:*

$$(31)\quad f+g \in \mathrm{CM}(X) \Rightarrow (f+g)(X) = \begin{cases} f(X)+g(X), & \text{if } m(f;X)=m(g;X); \\ f(X)+^-g(X), & \text{if } m(f;X)=-m(g;X), \end{cases}$$

$$(32)\quad f-g \in \mathrm{CM}(X) \Rightarrow (f-g)(X) = \begin{cases} f(X)-^-g(X), & \text{if } m(f;X)=m(g;X); \\ f(X)-g(X), & \text{if } m(f;X)=-m(g;X). \end{cases}$$

If in addition f, g do not change sign in D, then

$$(33)\quad fg\in \mathrm{CM}(X)\Rightarrow (fg)(X)=\begin{cases} f(X)\times g(X), & \text{if } m(|f|;X)=m(|g|;X)\\ f(X)\times^{-} g(X), & \text{if } m(|f|;X)=-m(|g|;X),\end{cases}$$

$$(34)\quad f/g\in \mathrm{CM}(X),\ g(x)\neq 0\Rightarrow (f/g)(X)=\begin{cases} f(X)/^{-}g(X), & \text{if } m(|f|;X)=m(|g|;X);\\ f(X)/g(X), & \text{if } m(|f|;X)=-m(|g|;X).\end{cases}$$

Proposition 8 can be generalized for monotone functions which possibly tend to $\pm\infty$ at the endpoint(s) of D. In this case the interval-arithmetic structure $(\mathfrak{I}, +, +^{-}, \times, \times^{-})$ should be employed. It is also possible to generalise (31)–(34) for monotone functions tending to $\pm\infty$ at an inner point of D (such as the functions $f=1/x$, $g=-1/x^3$ in $D=[-1/2, 1/2]$); then the interval-arithmetic structure $\mathcal{M}^*$ should be employed. To this end we may extend the concept of a continuous and monotone function in the following manner. Let f be continuous on the set $D=[d^-, d^+]\backslash\{d\}=[d^-, d)\cup(d, d^+]$, where $d^-<d<d^+$. Assume that $\lim_{x\to d, x<d} f(x)\in\{-\infty, \infty\}$, and $\lim_{x\to d, x>d} f(x)\in\{-\infty, \infty\}$. In such case we say that f is defined and continuous at d (and, therefore on D) and write, according to the particular situation, $f(d)=\infty$, $f(d)=-\infty$ or $f(d)=\pm\infty=\langle\infty, -\infty\rangle$. Similarly, we extend the concept of monotonicity. Let f be defined in D. By definition, f is monotone at x, $d^-<x<d^+$, if $f(x)=\pm\infty$. Assume that $x\in D$ is such that $f(x)\notin\{-\infty, \infty, \pm\infty\}$. We say that f is monotone at x if there is a neighbourhood of x such that f is monotone in it in the usual sense. For example, the function $1/x$ is monotone in $[-1, 1]$, but $1/x^2$ is not monotone in $[-1, 1]$; the function $\mathrm{tg}\, x$ is continuous and monotone everywhere, $\mathrm{tg}\in \mathrm{CM}(R)$.

Relations (31)–(34) can serve for definitions of interval arithmetic operations. In the situation of standard arithmetic such approach is equivalent to using (9) whenever inner intervals are considered. However, both approaches are not equivalent in the case of outer intervals. To be more specific let us briefly consider a definition of interval arithmetic based on (31)–(34). We do this for the operations $+, +^{-}, -, -^{-}$; the operations $\times, \times^{-}, /, /^{-}$ can be defined similarly (for zero-free intervals).

Definition. Assume $A, B\in\mathcal{I}$. Let $D\in\mathcal{I}$ and $f, g, f+g, f-g\in \mathrm{CM}(D)$ and such that $m(f; D)=m(g; D)$, $f(D)=A$, $g(D)=B$. Then

$$(35)\quad \begin{aligned} A+B&=(f+g)(D),\\ A-^{-}B&=(f-g)(D).\end{aligned}$$

Let now $m(f; D)=-m(g; D)$ under the remaining assumptions. Then

$$(36)\quad \begin{aligned} A-B&=(f-g)(D),\\ A+^{-}B&=(f+g)(D).\end{aligned}$$

We can now check that (9) from one side, and (35), (36), from the other side, produce same endpoint expressions for the standard operations $+$, $-$ in the case of inner intervals but do not yield same expressions in the case of outer intervals. For example, for the sum of two outer intervals defined by (35) we obtain

$$\langle a^-, a^+\rangle + \langle b^-, b^+\rangle = \langle a^- + b^-, a^+ + b^+\rangle,$$

whereas (9) produces as result R^*. This can be explained as follows. Formula (35) implies that addition of two outer intervals is performed as addition of their corresponding components:

$$\langle a^-, a^+\rangle + \langle b^-, b^+\rangle = ([a^-, \infty] \cup [-\infty, a^+]) + ([b^-, \infty] \cup [-\infty, b^+])$$

$$= ([a^-, \infty] + [b^-, \infty]) \cup ([-\infty, a^+] + [-\infty, b^+]),$$

whereas according to (9) we have

$$\langle a^-, a^+\rangle + \langle b^-, b^+\rangle = ([a^-, \infty] \cup (+\infty, a^+]) + ([b^-, \infty] \cup [-\infty, b^+])$$

$$= ([a^-, \infty] + [b^-, \infty]) \cup ([-\infty, a^+] + [-\infty, b^+]) \cup ([a^-, \infty] + [-\infty, b^+])$$

$$\cup ([-\infty, a^+] + [b^-, \infty]).$$

This shows that interval-arithmetic operations can be defined by (31)−(34). Normalization of outer interval results by overlapping causes problems both for the algebraic theory and for the applications. Indeed, for one thing the failure of the associative law is due to normalization. On the other side normalization causes difficulties for getting exact expressions for the ranges and leads to rough inclusions. A simple way to overcome this problem is to generalize the concept of interval by considering intervals over $\mathfrak{I}$ instead of intervals over R. Recall that an interval over $\mathfrak{I}$ (as an ordered set w. r. t. $\leqq$) with "endpoints" A, $B \in \mathfrak{I}$, $A \leqq B$ according to the algebraic definition of an interval [3] is the set of i-intervals $\{X \in \mathfrak{I} \mid A \leqq X \leqq B\}$. Let us limit ourself to the special case when the endpoints are degenerate intervals A^-, A^+ and consider intervals of the form $\{X \in \mathfrak{I} \mid A^- \leqq X \leqq B^+\}$. Using for such intervals the previous notation $A = [A^-, A^+]$ we arrive to the same arithmetic as the arithmetic in $\mathfrak{I}$. Using this concept of interval in the definition of outer intervals, by setting

$$\langle A^-, A^+\rangle = [A^-, \infty] \cup [-\infty, A^+] = \{X \in \mathfrak{I} \mid A^- \leqq X\} \cup \{X \in \mathfrak{I} \mid X \leqq A^+\}$$

we see that overlapping by outer intervals does not need normalisation.

The theoretical tools and the applications mentioned above can be reformulated in terms of generalised (or directed) intervals considered in sections 6, 7 of the present paper. For instance, relations (31)−(34) have a simple analogue in the case of generalised intervals. Let $f \in \mathrm{CM}(X)$ and let $f[X] = [f(x^-), f(x^+)]$ be the directed range of F (see introduction). Then the following analogue of Proposition 8 holds true.

Proposition 9. *For f, $g \in \mathrm{CM}(D)$, $X \subseteq D$:*

$$f + g \in \mathrm{CM}(X) \Rightarrow (f+g)\,[X] = f[X] + g[X];$$

$$f-g\in \mathrm{CM}(X)\Rightarrow (f-g)\,[X]=f[X]-g[X].$$

If in addition to the above assumptions f, g do not change their sign in D, then

$$fg\in \mathrm{CM}(X)\Rightarrow (fg)\,[X]=f[X]_{\sigma(f(X))}\times g[X]_{\sigma(g(X))};$$

$$f/g\in \mathrm{CM}(X),\ g(x)\neq 0\Rightarrow (f/g)\,[X]=f[X]_{\sigma(f(X))}/g[X]_{\sigma(g(X))},$$

wherein $\sigma(f(X))=\sigma(f[X])$ is the sign of the interval $f(X)$ (or of the directed interval $f[X]$, which is the same), that is the sign of f on X.

Propositions 8, 9 can be further generalized for functions of many variables in a direction discussed in [7].

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