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Solution of Dual Integral Equations by Fractional Calculus

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In this paper we give an explicit solution of a class of dual integral equations whose kernel is Fox's H-function, namely:

$$\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}(a_j,A_j)_1^{n+p}\\(b_j,B_j)_1^{m+q}\end{matrix}\right.\right] f(u)\,du=\varphi(x),\quad 0<x<1$$

$$\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\left|\begin{matrix}(c_j,C_j)_1^{n+p}\\(d_j,D_j)_1^{m+q}\end{matrix}\right.\right] f(u)\,du=\psi(x),\qquad x>1.$$

The solution is obtained in a closed form, together with the corresponding conditions for the parameters, by application of multiple Erdélyi-Kober fractional integration operators. As special cases it contains solutions of dual integral equations involving many special functions of mathematical physics.

1. Introduction

There are many techniques available for the solution of mixed boundary value problems arising in mathematical physics [1]. In certain cases, an application of integral transforms, reduces such problems to dual integral equations.

In general, the pair of equations

$$\int_0^\infty W(u)K(x,u)f(u)du=\varphi(x),\quad 0<x<1$$

$$\int_0^\infty K(x,u)f(u)du=\psi(x),\quad x>1,$$

where the kernel $K(x,u)$, the weight $W(u)$ and the functions $\varphi(x)$, $\psi(x)$ are known and $f(u)$ is to be determined, is known as a pair of dual integral equations.

A.M. Mathai and R.K. Saxena [2] proposed a method to obtain the solution of dual integral equations associated with an arbitrary special function having a Mellin-Barnes type integral representation. The kernel function is taken to be H-function of Fox or Meijer's G-function and compositions of Erdélyi-Kober operators of fractional integration are applied to reduce the given equations into two others with one and the same kernel. In this way, the solution is given in a closed but rather complicated form because these compositions of fractional integrals are not calculated explicitly. Moreover, conditions on the parameters of H- and G-functions in the kernel are not stated, as these authors present only a formal solution.

In this paper we give an explicit solution of a class of dual integral equations whose kernel is Fox's H-function. It contains as special cases the solutions of dual integral equations involving many special functions of mathematical physics. We illustrate this by an example with Bessel function as kernel. The solution is obtained by application of Multiple Erdélyi-Kober Fractional Integration Operators [3,6].

2. Multiple Erdélyi-Kober fractional integration operators

S.L. Kalla and L. Galué L. [3] defined Multiple Erdélyi-Kober Fractional Integration Operators of Riemann-Liouville and Weyl type.

Definition 1. *Let $m \geq 1$ be an integer, $\beta_k > 0$, $\lambda_k > 0$, $\delta_k \geq 0$, γ_k, $k = 1, 2, \ldots, m$ be real numbers. We define a multiple Erdélyi-Kober fractional integration operator of Riemann-Liouville type by the integral,*

$$I\, f(x) = I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m} f(x) =$$

$$(1)\qquad = \begin{cases} \int_0^1 H^{m,0}_{m,m}\left[\sigma \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\lambda_k}, \frac{1}{\lambda_k}\right)_1^m \end{matrix} \right.\right] f(x\sigma)\, d\sigma, & \text{if } \sum_1^m \delta_k > 0 \\ f(x), & \text{if } \delta_k = 0,\ \lambda_k = \beta_k,\ k = 1, \ldots, m \end{cases}$$

$$(2)\qquad = x^{-1} \int_0^x H^{m,0}_{m,m}\left[\frac{t}{x} \left| \begin{matrix} \left(\gamma_k+\delta_k+1-\frac{1}{\beta_k}, \frac{1}{\beta_k}\right)_1^m \\ \left(\gamma_k+1-\frac{1}{\lambda_k}, \frac{1}{\lambda_k}\right)_1^m \end{matrix} \right.\right] f(t)\, dt, \quad \text{if } \sum_1^m \delta_k > 0,$$

where $\sum_{k=1}^m \frac{1}{\lambda_k} \geq \sum_{k=1}^m \frac{1}{\beta_k}$ and $f(x) \in C_\alpha$ being

$$(3)\qquad C_\alpha = \left\{ f(x) = x^p \tilde{f}(x);\ p > \alpha,\ \tilde{f} \in C[0,\infty) \right\} \text{ with } \alpha \geq \min_k[-\lambda_k(\gamma_k+1)]$$

and

$$(4)\qquad H^{m,n}_{p,q}\left[x \left| \begin{matrix} (a_j, A_j)_1^p \\ (b_j, B_j)_1^q \end{matrix} \right.\right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \theta(s) x^s\, ds$$

with

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)},$$

($\mathcal{L}$ a suitably chosen contour in $\mathbb{C}$) is the Fox's H-function [4].

Some properties of these operators are:

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}\, x^{\rho} = \prod_{k=1}^{m}\frac{\Gamma\left(\gamma_k + 1 + \dfrac{\rho}{\lambda_k}\right)}{\Gamma\left(\gamma_k + \delta_k + 1 + \dfrac{\rho}{\beta_k}\right)} x^{\rho}, \tag{5}$$

$$I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}\, f(x) = \left(\prod_{k=1}^{m} I^{\gamma_k,\delta_k}_{\beta_k,\lambda_k,1}\right) f(x), \tag{6}$$

$$\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}\right\}^{-1} f(x) = I^{(\gamma_k+\delta_k),(-\delta_k)}_{(\lambda_k),(\beta_k),m}\, f(x), \tag{7}$$

$$\mathcal{M}\left\{I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),m}\, f(x); s\right\} = \prod_{k=1}^{m}\frac{\Gamma\left(\gamma_k + 1 - \dfrac{s}{\lambda_k}\right)}{\Gamma\left(\gamma_k + \delta_k + 1 - \dfrac{s}{\beta_k}\right)}\mathcal{M}\{f(x); s\}, \tag{8}$$

where $\mathcal{M}\{f(x); s\}$ denote the Mellin transform of $f(x)$.

Special case: When $\lambda_k = \beta_k$, $k = 1, \ldots, m$ operators (1), (2) coincide with the generalized fractional integrals of Kiryakova [6].

Definition 2. *Let $n \geq 1$ be an integer, $\varepsilon_k > 0$, $\xi_k > 0$, $\alpha_k \geq 0$, τ_k, $k = 1, 2, \ldots, n$ be real numbers. We define a multiple Erdélyi-Kober fractional integration operator of Weyl type by the integral:*

$$Kf(x) = K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x) =$$

$$(9) \qquad \begin{cases} \int_1^\infty H^{n,0}_{n,n}\left[\frac{1}{\sigma}\left|\begin{matrix}\left(\tau_k+\alpha_k+\frac{1}{\varepsilon_k},\frac{1}{\varepsilon_k}\right)_1^n\\ \left(\tau_k+\frac{1}{\xi_k},\frac{1}{\xi_k}\right)_1^n\end{matrix}\right.\right] f(\sigma x)d\sigma, & \text{if } \sum_1^n \alpha_k > 0\\ f(x), & \text{if } \alpha_k = 0, \varepsilon_k = \xi_k, k = 1,2,...,n \end{cases}$$

$$(10) \qquad = x^{-1}\int_x^\infty H^{n,0}_{n,n}\left[\frac{x}{t}\left|\begin{matrix}\left(\tau_k+\alpha_k+\frac{1}{\varepsilon_k},\frac{1}{\varepsilon_k}\right)_1^n\\ \left(\tau_k+\frac{1}{\xi_k},\frac{1}{\xi_k}\right)_1^n\end{matrix}\right.\right] f(t)dt, \text{ if } \sum_1^n \alpha_k > 0,$$

where $\sum_{k=1}^n \frac{1}{\xi_k} > \sum_{k=1}^n \frac{1}{\varepsilon_k}$ *and* $f(x) \in C^*_{\alpha^*}$ *being*

$$(11) \qquad C^*_{\alpha^*} = \{f(x) = x^q \tilde{\tilde{f}}(x); q < \alpha^*, \tilde{\tilde{f}}(x) \in C(0,\infty), |\tilde{\tilde{f}}(x)| \le A_{\tilde{\tilde{f}}}\}$$

with $\alpha^* \le \max_k(\xi_k, \tau_k)$.

Some properties of these operators are:

$$(12) \qquad K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} x^\lambda = \prod_{k=1}^n \frac{\Gamma\left(\tau_k - \frac{\lambda}{\xi_k}\right)}{\Gamma\left(\tau_k + \alpha_k - \frac{\lambda}{\varepsilon_k}\right)} x^\lambda,$$

$$(13) \qquad K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x) = \left(\prod_{k=1}^n K^{\tau_k,\alpha_k}_{\varepsilon_k,\xi_k,1}\right) f(x),$$

$$(14) \qquad \left\{K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n}\right\}^{-1} f(x) = K^{(\tau_k+\alpha_k),(-\alpha_k)}_{(\xi_k),(\varepsilon_k),n} f(x),$$

$$(15) \qquad \mathcal{M}\left\{K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),n} f(x); s\right\} = \prod_{k=1}^n \frac{\Gamma\left(\tau_k + \frac{s}{\xi_k}\right)}{\Gamma\left(\tau_k + \alpha_k + \frac{s}{\varepsilon_k}\right)} \mathcal{M}\{f(x); s\}.$$

Special case: For $\lambda_k = \beta_k = 1$, $k = 1, ..., m$ operators (9) and (10) coincide with the generalized fractional integrals of Weyl type involving Meijer's G-function, see Kiryakova [5].

3. The problem and its reduction by Mellin transform.

Let $m \geq 0$, $n \geq 0$, $p \geq 0$, $q \geq 0$ be integers such that $\delta = (m+n) - (p+q) \geq 0$. As in [2] we consider the case $\delta = 0$, i.e. $m+n = p+q$, although the same approach applies to the general case $\delta \geq 0$.

We consider a pair of dual integral equations of the form

$$(16) \qquad \int_0^\infty H^{m,n}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix}(a_j, A_j)_1^{n+p}\\(b_j, B_j)_1^{m+q}\end{matrix}\right] f(u)du = \varphi(x), \quad 0 < x < 1$$

$$(17) \qquad \int_0^\infty H^{m,n}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix}(c_j, C_j)_1^{n+p}\\(d_j, D_j)_1^{m+q}\end{matrix}\right] f(u)du = \psi(x), \quad x > 1$$

with Fox's H-functions as kernels.

Using the Parseval theorem for the Mellin transform, the equations (16) and (17) are transformed in:

$$\mathcal{M}^{-1}\{X_1(-s)F(1-s)\} = \varphi(x), \quad 0 < x < 1$$

$$\mathcal{M}^{-1}\{X_2(-s)F(1-s)\} = \psi(x), \quad x > 1,$$

where

$$(18) \qquad X_1(-s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{j=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{j=m+1}^{q+m} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p+n} \Gamma(a_j + A_j s)},$$

$$(19) \qquad X_2(-s) = \frac{\prod_{j=1}^m \Gamma(d_j + D_j s) \prod_{j=1}^n \Gamma(1 - c_j - C_j s)}{\prod_{j=m+1}^{q+m} \Gamma(1 - d_j - D_j s) \prod_{j=n+1}^{p+n} \Gamma(c_j + C_j s)},$$

$F(s)$ is the Mellin transform of $f(u)$ with normal restrictions in the definitions of the functions involved.

Applying the inversion theorem of Mellin, the new pair of equations is:

$$\frac{1}{2\pi i}\int_{\mathcal{L}} x^{-s} X_1(-s) F(1-s)\,ds = \varphi(x), \quad 0 < x < 1, \tag{20}$$

$$\frac{1}{2\pi i}\int_{\mathcal{L}} x^{-s} X_2(-s) F(1-s)\,ds = \psi(x), \quad x > 1. \tag{21}$$

4. Solution of the dual integral equations system by applying of multiple Erdélyi-Kober fractional integration operators.

The main idea in this section is to transform each of the kernels $X_1(-s)$, $X_2(-s)$ of equations (20), (21) to a common kernel $X(-s)$ of the same form, using fractional integration operators. In this way the equations (20), (21) can be reduced to a pair of equations with a common kernel:

$$X(-s) = \frac{\prod_{k=1}^{m}\Gamma(b_k + B_k s)\prod_{k=1}^{n}\Gamma(1 - c_k - C_k s)}{\prod_{k=1}^{p}\Gamma(a_{n+k} + A_{n+k} s)\prod_{k=1}^{q}\Gamma(1 - d_{m+k} - D_{m+k} s)}, \tag{22}$$

and their definition can be written by one formula.

Definition 3. *Let*

$$I' = \prod_{k=1}^{n} I^{-c_k, c_k - a_k}_{\frac{1}{A_k}, \frac{1}{C_k}, 1} = I^{(-c_k),(c_k - a_k)}_{(\frac{1}{A_k}),(\frac{1}{C_k}),n} \tag{23}$$

$$I^* = \prod_{k=1}^{q} I^{-b_{m+k}, b_{m+k} - d_{m+k}}_{\frac{1}{D_{m+k}}, \frac{1}{B_{m+k}}, 1} = I^{(-b_{m+k}),(b_{m+k} - d_{m+k})}_{(\frac{1}{D_{m+k}}),(\frac{1}{B_{m+k}}),q} \tag{24}$$

be multiple Erdélyi-Kober fractional integration operators of Riemann-Liouville type with $c_k > a_k$, $k = 1, \dots, n$ and $b_{m+k} > d_{m+k}$, $k = 1, \dots, q$, then

$$I = I^* I' = I^{(\gamma_k),(\delta_k)}_{(\beta_k),(\lambda_k),n+q}, \tag{25}$$

where

$$\begin{aligned} &\gamma_k = -c_k, \quad \delta_k = c_k - a_k, \\ &\beta_k = \frac{1}{A_k}, \qquad \lambda_k = \frac{1}{C_k}, \end{aligned} \qquad k = 1, \dots, n, \tag{26}$$

$$\begin{aligned} &\gamma_{n+k} = -b_{m+k}, \quad \delta_{n+k} = b_{m+k} - d_{m+k}, \\ &\beta_{n+k} = \frac{1}{D_{m+k}}, \qquad \lambda_{n+k} = \frac{1}{B_{m+k}}, \end{aligned} \qquad k = 1, \dots, q. \tag{27}$$

Theorem 1. *For $\varphi(x) \in C_\alpha$ and $C_k\alpha + 1 > c_k > a_k$, $k = 1, \dots, n$; $B_{m+k}\alpha + 1 > b_{m+k} > d_{m+k}$, $k = 1, \dots, q$, the operator I transforms the integral equation* (20)*:*

$$\frac{1}{2\pi i} \int_{\mathcal{L}} x^{-s} X_1(-s) F(1-s) ds = \varphi(x), \quad 0 < x < 1,$$

into

$$\frac{1}{2\pi i} \int_{\mathcal{L}} x^{-s} X(-s) F(1-s) ds = I\varphi(x), \quad 0 < x < 1, \tag{28}$$

where $X(-s)$ is given in (22) and $I\varphi(x) \in C_\alpha$ for $\varphi(x) \in C_\alpha$.

Proof. . Applying the operator I as defined in (25) to the integral equation (20), we obtain:

$$\frac{1}{2\pi i} \int_{\mathcal{L}} I^* I'\{x^{-s}\} X_1(-s) F(1-s) ds = I\varphi(x), \quad 0 < x < 1. \tag{29}$$

From (5) and (23):

$$I'\{x^{-s}\} = \prod_{k=1}^{q} \frac{\Gamma(-c_k + 1 - C_k s)}{\Gamma(-a_k + 1 - A_k s)} x^{-s}.$$

From (5) and (24):

$$I^*I'\{x^{-s}\} = \prod_{k=1}^{n} \frac{\Gamma(-b_{m+k}+1-B_{m+k}s)}{\Gamma(-d_{m+k}+1-D_{m+k}s)} \prod_{k=1}^{n} \frac{\Gamma(-c_k+1-C_ks)}{\Gamma(-a_k+1-A_ks)} x^{-s},$$

substituting this expression in (29) and using (18) we obtain the desired result. ∎

Definition 4. *Let*

$$K' = \prod_{k=1}^{m} K^{b_k, d_k-b_k}_{\frac{1}{D_k}, \frac{1}{B_k}, 1} = K^{(b_k),(d_k-b_k)}_{(\frac{1}{D_k}),(\frac{1}{B_k}),m} \tag{30}$$

$$K^* = \prod_{k=1}^{p} K^{c_{n+k}, a_{n+k}-c_{n+k}}_{\frac{1}{A_{n+k}}, \frac{1}{C_{n+k}}, 1} = K^{(c_{n+k}),(a_{n+k}-c_{n+k})}_{(\frac{1}{A_{n+k}}),(\frac{1}{C_{n+k}}),p} \tag{31}$$

be multiple Erdélyi-Kober fractional integration operators of Weyl type with $d_k > b_k$, $k = 1,\ldots,m$ *and* $a_{n+k} > c_{n+k}$, $k = 1,\ldots,p$, *then*

$$K = K^*K' = K^{(\tau_k),(\alpha_k)}_{(\varepsilon_k),(\xi_k),m+p}, \tag{32}$$

where

$$\tau_k = b_k, \quad \alpha_k = d_k - b_k, \quad k = 1,2,\ldots,m, \quad \varepsilon_k = \frac{1}{D_k}, \quad \xi_k = \frac{1}{B_k}, \tag{33}$$

$$\tau_{m+k} = c_{n+k}, \quad \alpha_{m+k} = a_{n+k} - c_{n+k}, \quad k = 1,2,\ldots,p. \tag{34}$$

$$\varepsilon_{m+k} = \frac{1}{A_{n+k}}, \quad \xi_{m+k} = \frac{1}{C_{n+k}},$$

Theorem 2. *For $\psi(x) \in C^*_{\alpha^*}$ and $B_k\alpha^* < b_k < d_k$, $k = 1, 2, ..., m$; $C_{n+k}\alpha^* < c_{n+k} < a_{n+k}$, $k = 1, ..., p$, the operator K transforms the integral equation (21):*

$$\frac{1}{2\pi i}\int_{\mathcal{L}} x^{-s} X_2(-s) F(1-s) ds = \psi(x), \quad x > 1,$$

into

$$\frac{1}{2\pi i}\int_{\mathcal{L}} x^{-s} X(-s) F(1-s) ds = K\psi(x), \quad x > 1, \tag{35}$$

where $X(-s)$ is given in (22) and $K\psi(x) \in C^*_{\alpha^*}$ for $\psi(x) \in C^*_{\alpha^*}$.

Now, we can write the pair of equations (20), (21) in the following concise form:

$$\frac{1}{2\pi i}\int_{\mathcal{L}} x^{-s} X(-s) F(1-s) ds = g(x), \quad 0 < x < \infty, \tag{36}$$

with

$$g(x) = \begin{cases} I\varphi(x), & 0 < x < 1 \\ K\psi(x), & x > 1 \end{cases} \tag{37}$$

and $g(x) \in C_{\alpha\alpha^*}$, where
$C_{\alpha,\alpha^*} = \{g(x) \in C[0,1] \cap C[1,\infty); |g| \le Ax^{\alpha}, 0 < x < 1; |g| \le Ax^{\alpha^*}, x > 1\}$, $\alpha^* \le \alpha$.

From (36), using the inversion theorem of Mellin:

$$G(s) = X(-s)F(1-s), \tag{38}$$

where $G(s) = \mathcal{M}\{g(x); s\}$.

Therefore,

$$F(s) = \frac{G(1-s)}{X(s-1)} = G(1-s)\mathcal{H}(s) \tag{39}$$

and

$$f(x) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}(s) G(1-s) x^{-s} ds \tag{40}$$

where $\mathcal{H}(s) = \dfrac{1}{X(s-1)} =$

$$= \frac{\prod_{k=1}^{q} \Gamma(1 - d_{m+k} - D_{m+k} + D_{m+k}s) \prod_{k=1}^{p} (a_{n+k} + A_{n+k} - A_{n+k}s)}{\prod_{k=1}^{n} \Gamma(1 - c_k - C_k + C_k s) \prod_{k=1}^{m} \Gamma(b_k + B_k - B_k s)}.$$

Let $h(x) = \mathcal{M}^{-1}[\mathcal{H}(s)]$. From the result for the Mellin transform of the H-function [4, eq.(2.4.1)], we obtain:

$$h(x) = H^{q,p}_{p+n,q+m}\left[x \,\middle|\, \begin{matrix} (1-a_{n+k}-A_{n+k}, A_{n+k})_1^p, & (1-c_k-C_k, C_k)_1^n \\ (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q, & (1-b_k-B_k, B_k)_1^m \end{matrix}\right] \tag{41}$$

which surely exists for $\delta = (m+n)-(p+q) = 0$ and under the conditions on the parameters a_j, c_j, C_j ($j = 1, ..., n+p$), b_j, B_j, d_j ($j = 1, ..., m+q$) accepted here.

Using in (39) the Parseval theorem and from the result (37) we have:

$$f(x) = \int_0^1 h(xu) I\varphi(u) du + \int_1^\infty h(xu) K\psi(u) du$$

with $h(x)$ as defined in (41).

Theorem 3. *The function*

(42)

$$f(x) = \int_0^1 H^{q,p}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix} (1-a_{n+k}-A_{n+k}, A_{n+k})_1^p, & (1-c_k-C_k, C_k)_1^n \\ (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q, & (1-b_k-B_k, B_k)_1^m \end{matrix}\right]$$
$$\left\{\int_0^1 H^{n+q,0}_{n+q,n+q}\left[\sigma \,\middle|\, \begin{matrix} (1-a_k-A_k, A_k)_1^n, & (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q \\ (1-c_k-C_k, C_k)_1^n, & (1-b_{m+k}-B_{m+k}, B_{m+k})_1^q \end{matrix}\right] \varphi(u\sigma) d\sigma\right\} du +$$
$$\int_1^\infty H^{q,p}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix} (1-a_{n+k}-A_{n+k}, A_{n+k})_1^p, & (1-c_k-C_k, C_k)_1^n \\ (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q, & (1-b_k-B_k, B_k)_1^m \end{matrix}\right]$$
$$\left\{\int_1^\infty H^{m+p,0}_{m+p,m+p}\left[\frac{1}{\sigma} \,\middle|\, \begin{matrix} (d_k+D_k, D_k)_1^m, & (a_{n+k}+A_{n+k}, A_{n+k})_1^p \\ (b_k+B_k, B_k)_1^m, & (c_{n+k}+C_{n+k}, C_{n+k})_1^p \end{matrix}\right] \psi(u\sigma) d\sigma\right\} du$$

satisfies the dual integral equations:

$$\begin{aligned} &\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix} (a_j, A_j)_1^{n+p} \\ (b_j, B_j)_1^{m+q} \end{matrix}\right] f(u) du = \varphi(x), \quad 0 < x < 1 \\ &\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu \,\middle|\, \begin{matrix} (c_j, C_j)_1^{n+p} \\ (d_j, D_j)_1^{m+q} \end{matrix}\right] f(u) du = \psi(x), \quad x > 1 \end{aligned} \tag{43}$$

with the conditions,

$$
\begin{aligned}
&C_k\alpha+1>c_k>a_k,\quad k=1,\dots,n;\quad B_{m+k}\alpha+1>b_{m+k}>d_{m+k},\quad k=1,\dots,q\\
&d_k>b_k>B_k\alpha^*,\quad k=1,\dots,m;\quad a_{n+k}>c_{n+k}>C_{n+k}\alpha^*,\quad k=1,\dots,p
\end{aligned}
\tag{44}
$$

and $\varphi(x)\in C_\alpha$, $\psi(x)\in C^*_{\alpha^*}$, $\alpha^*\le\alpha$.

Proof. Substituting (42) in (43) and changing the order of integration among the two external integrals, we have for $0<x<1$,

$$
\begin{aligned}
&\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\,\middle|\,\begin{matrix}(a_k,A_k)_1^{n+p}\\(b_k,B_k)_1^{m+q}\end{matrix}\right]f(u)du\\
&=\int_0^1\left\{\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\,\middle|\,\begin{matrix}(a_k,A_k)_1^{n+p}\\(b_k,B_k)_1^{m+q}\end{matrix}\right]\right.\\
&\times H^{q,p}_{p+n,q+m}\left[uw\,\middle|\,\begin{matrix}(1-a_{n+k}-A_{n+k},A_{n+k})_1^p, & (1-c_k-C_k,C_k)_1^n\\(1-d_{m+k}-D_{m+k},D_{m+k})_1^q, & (1-b_k-B_k,B_k)_1^m\end{matrix}\right]\\
&\times\left.\left(\int_0^1 H^{n+q,0}_{n+q,n+q}\left[\sigma\,\middle|\,\begin{matrix}(1-a_k-A_k,A_k)_1^n, & (1-d_{m+k}-D_{m+k},D_{m+k})_1^q\\(1-c_k-C_k,C_k)_1^n, & (1-b_{m+k}-B_{m+k},B_{m+k})_1^q\end{matrix}\right]\varphi(\sigma w)d\sigma\right)du\right\}dw\\
&+\int_1^\infty\left\{\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\,\middle|\,\begin{matrix}(a_k,A_k)_1^{n+p}\\(b_k,B_k)_1^{m+q}\end{matrix}\right]\right.\\
&\times H^{q,p}_{p+n,q+m}\left[uw\,\middle|\,\begin{matrix}(1-a_{n+k}-A_{n+k},A_{n+k})_1^p, & (1-c_k-C_k,C_k)_1^n\\(1-d_{m+k}-D_{m+k},D_{m+k})_1^q, & (1-b_k-B_k,B_k)_1^m\end{matrix}\right]\\
&\times\left.\left(\int_1^\infty H^{m+p,0}_{m+p,m+p}\left[\frac{1}{\sigma}\,\middle|\,\begin{matrix}(d_k+D_k,D_k)_1^m, & (a_{n+k}+A_{n+k},A_{n+k})_1^p\\(b_k+B_k,B_k)_1^m, & (c_{n+k}+C_{n+k},C_{n+k})_1^p\end{matrix}\right]\psi(\sigma w)d\sigma\right)du\right\}dw.
\end{aligned}
$$

Using the result [4, (5.1.1)],

$$
\begin{aligned}
T=&\int_0^\infty H^{m,n}_{p+n,q+m}\left[xu\,\middle|\,\begin{matrix}(a_k,A_k)_1^{n+p}\\(b_k,B_k)_1^{m+q}\end{matrix}\right]\times\\
&H^{q,p}_{p+n,q+m}\left[uw\,\middle|\,\begin{matrix}(1-a_{n+k}-A_{n+k},A_{n+k})_1^p, & (1-c_k-C_k,C_k)_1^n\\(1-d_{m+k}-D_{m+k},D_{m+k})_1^q, & (1-b_k-B_k,B_k)_1^m\end{matrix}\right]=\\
&u^{-1}H^{m+p,n+q}_{p+n+q+m,q+m+p+n}\left[\frac{x}{w}\,\middle|\,\begin{matrix}(a_k,A_k)_1^n, & (d_{m+k},D_{m+k})_1^q, & (b_k,B_k)_1^m,(a_k,A_k)_{n+1}^{p+n}\\(b_k,B_k)_1^m, & (a_{n+k},A_{n+k})_1^p, & (c_k,C_k)_1^n,(b_k,B_k)_{m+1}^{q+m}\end{matrix}\right].
\end{aligned}
$$

Using the definition of the H-function given in [4] and the following property of the H-function [4, (2.2.16)]:

$$
H^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_j,A_j)_1^p\\(b_j,B_j)_1^q\end{matrix}\right]=H^{n,m}_{q,p}\left[x^{-1}\,\middle|\,\begin{matrix}(1-b_j,B_j)_1^q\\(1-a_j,A_j)_1^p\end{matrix}\right],
$$

we obtain

$$T = w^{-1} H^{n+q,0}_{n+q,n+q} \left[\frac{w}{x} \middle| \begin{matrix} (1-b_{m+k}, B_{m+k})_1^q & , & (1-c_k, C_k)_1^n \\ (1-a_k, A_k)_1^n & , & (1-d_{m+k}, D_{m+k})_1^q \end{matrix} \right].$$

Therefore,

$$\int_0^\infty H^{m,n}_{p+n,q+m} \left[xu \middle| \begin{matrix} (a_k, A_k)_1^{n+p} \\ (b_k, B_k)_1^{m+q} \end{matrix} \right] f(u)du =$$

$$\int_0^1 w^{-1} H^{q+n,0}_{q+n,q+n} \left[\frac{w}{x} \middle| \begin{matrix} (1-b_{m+k}, B_{m+k})_1^q & , & (1-c_k, C_k)_1^n \\ (1-a_k, A_k)_1^n & , & (1-d_{m+k}, C_{m+k})_1^q \end{matrix} \right] \times$$

$$\left(\int_0^1 H^{n+q,0}_{n+q,n+q} \left[\sigma \middle| \begin{matrix} (1-a_k-A_k, A_k)_1^n, & (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q \\ (1-c_k-C_k, C_k)_1^n, & (1-b_{m+k}-B_{m+k}, B_{m+k})_1^q \end{matrix} \right] \varphi(\sigma w) d\sigma \right) dw +$$

$$\int_1^\infty w^{-1} H^{q+n,0}_{q+n,q+n} \left[\frac{w}{x} \middle| \begin{matrix} (1-b_{m+k}, B_{m+k})_1^q & , & (1-c_k, C_k)_1^n \\ (1-a_k, A_k)_1^n & , & (1-d_{m+k}, D_{m+k})_1^q \end{matrix} \right] \times$$

$$\left(\int_1^\infty H^{m+p,0}_{m+p,m+p} \left[\frac{1}{\sigma} \middle| \begin{matrix} (d_k+D_k, D_k)_1^m, & (a_{n+k}+A_{n+k}, A_{n+k})_1^p \\ (b_k+B_k, B_k)_1^m, & (c_{n+k}+C_{n+k}, C_{n+k})_1^p \end{matrix} \right] \psi(\sigma w) d\sigma \right) dw,$$

as $H[\frac{w}{x}] \equiv 0$ for $w > x$ and $0 < x < 1$, this is, for $w > 1$ we have

$$\int_0^\infty H^{m,n}_{p+n,q+m} \left[xu \middle| \begin{matrix} (a_k, A_k)_1^{p+n} \\ (b_k, B_k)_1^{q+m} \end{matrix} \right] f(u)du =$$

$$\int_0^1 w^{-1} H^{q+n,0}_{q+n,q+n} \left[\frac{w}{x} \middle| \begin{matrix} (1-b_{m+k}, B_{m+k})_1^q & , & (1-c_k, C_k)_1^n \\ (1-a_k, A_k)_1^n & , & (1-d_{m+k}, D_{m+k})_1^q \end{matrix} \right] \times$$

$$\left(\int_0^1 H^{n+q,0}_{n+q,n+q} \left[\sigma \middle| \begin{matrix} (1-a_k-A_k, A_k)_1^n, & (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q \\ (1-c_k-C_k, C_k)_1^n, & (1-b_{m+k}-B_{m+k}, B_{m+k})_1^q \end{matrix} \right] \varphi(\sigma w) d\sigma \right) dw,$$

using the property of the H-function [4,(2.3.6)]:

$$(45) \qquad x^\sigma H^{m,n}_{p,q} \left[x \middle| \begin{matrix} (a_j, \alpha_j)_1^p \\ (b_j, \beta_j)_1^q \end{matrix} \right] = H^{m,n}_{p,q} \left[x \middle| \begin{matrix} (a_j+\sigma\alpha_j, \alpha_j)_1^p \\ (b_j+\sigma\beta_j, \beta_j)_1^q \end{matrix} \right],$$

and making a change of variable,

$$\int_0^\infty H^{m,n}_{p+n,q+m} \left[xu \middle| \begin{matrix} (a_k, A_k)_1^{p+n} \\ (b_k, B_k)_1^{q+m} \end{matrix} \right] f(u)du =$$

$$\int_0^{1/x} H^{q+n,0}_{q+n,q+n} \left[w \middle| \begin{matrix} (1-b_{m+k}-B_{m+k}, B_{m+k})_1^q & , & (1-c_k-C_k, C_k)_1^n \\ (1-a_k-A_k, A_k)_1^n & , & (1-d_{m+k}-D_{m+k}, D_{m+k})_1^q \end{matrix} \right] \times$$

$$\left(\int_0^1 H_{n+q,n+q}^{n+q,0}\left[\sigma \,\middle|\, \begin{matrix}(1-a_k-A_k,A_k)_1^n, & (1-d_{m+k}-D_{m+k},D_{m+k})_1^q\\ (1-c_k-C_k,C_k)_1^n, & (1-b_{m+k}-B_{m+k},B_{m+k})_1^q\end{matrix}\right]\varphi(xw\sigma)d\sigma\right)dw+ .$$

From the definition 1, result (1), and as

$$0<x<1 \Longrightarrow 1<\frac{1}{x}<\infty,$$

$$H[w]\equiv 0 \text{ for } w>1,$$

$$\int_0^\infty H_{p+n,q+m}^{m,n}\left[xu \,\middle|\, \begin{matrix}(a_k,A_k)_1^{p+n}\\ (b_k,B_k)_1^{q+m}\end{matrix}\right] f(u)du =$$

$$\int_0^1 w^{-1}H_{q+n,q+n}^{q+n,0}\left[\frac{w}{x} \,\middle|\, \begin{matrix}(1-b_{m+k}-B_{m+k},B_{m+k})_1^q, & (1-c_k-C_k,C_k)_1^n\\ (1-a_k-A_k,A_k)_1^n, & (1-d_{m+k}-D_{m+k},D_{m+k})_1^q\end{matrix}\right]\times$$

$$I^{(-c_k,-b_{m+k}),(-a_k+c_k,-d_{m+k}+b_{m+k})}_{(\frac{1}{A_k},\frac{1}{D_{m+k}}),(\frac{1}{C_k},\frac{1}{B_{m+k}}),n+q}\varphi(xw)dw.$$

Applying again the definition 1,

$$\int_0^\infty H_{p+n,q+m}^{m,n}\left[xu \,\middle|\, \begin{matrix}(a_k,A_k)_1^{p+n}\\ (b_k,B_k)_1^{q+m}\end{matrix}\right] f(u)du =$$

$$I^{(-a_k,-d_{m+k}),(-c_k+a_k,-b_{m+k}+d_{m+k})}_{(\frac{1}{C_k},\frac{1}{B_{m+k}}),(\frac{1}{A_k},\frac{1}{D_{m+k}}),n+q}\, I^{(-c_k,-b_{m+k}),(-a_k+c_k,-d_{m+k}+b_{m+k})}_{(\frac{1}{A_k},\frac{1}{D_{m+k}}),(\frac{1}{C_k},\frac{1}{B_{m+k}}),n+q}\varphi(x),$$

at last from (7) we obtain

$$\int_0^\infty H_{p+n,q+m}^{m,n}\left[xu \,\middle|\, \begin{matrix}(a_k,A_k)_1^{p+n}\\ (b_k,B_k)_1^{q+m}\end{matrix}\right] f(u)du = \varphi(x).$$

The proof for the case $x>1$ is analogous.

5. Example

Let us consider the dual integral equations

$$
(46)\qquad
\begin{aligned}
&\int_0^\infty u^{-\gamma} J_\mu(2\sqrt{xu}) f(u)du = g_1(x), \quad 0 < x < 1,\\
&\int_0^\infty u^{-\delta} J_\nu(2\sqrt{xu}) f(u)du = g_2(x), \quad x > 1,
\end{aligned}
$$

where $J_\nu(x)$ is the Bessel function, $f(u)$-unknown function, γ, δ, μ, ν are given parameters and $g_1(x)$, $g_2(x)$-given functions.

We have [2, equation (1.7.9)]

$$
(47)\qquad J_\nu(x) = \left(\frac{x}{2}\right)^\nu H_{0,2}^{1,0}\left[\frac{x^2}{4}\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (0,1),(-\nu,1)\end{matrix}\right].
$$

Using (47) and (45), the system (46) transforms to

$$
\int_0^\infty H_{0,2}^{1,0}\left[xu\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (\frac{\mu}{2}-\gamma,1),(-\frac{\mu}{2}-\gamma,1)\end{matrix}\right] f(u)du = x^{-\gamma}g_1(x), 0 < x < 1,
$$

$$
\int_0^\infty H_{0,2}^{1,0}\left[xu\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (\frac{\nu}{2}-\delta,1),(-\frac{\nu}{2}-\delta,1)\end{matrix}\right] f(u)du = x^{-\delta}g_2(x), x > 1,
$$

applying the theorem 3 we obtain,

$$
\begin{aligned}
f(x) = &\int_0^1 H_{0,2}^{1,0}\left[xu\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (\frac{\nu}{2}+\delta,1),(\gamma-\frac{\mu}{2},1)\end{matrix}\right]\times\\
&\left\{\int_0^1 H_{1,1}^{1,0}\left[\sigma\,\middle|\,\begin{matrix}(\frac{\nu}{2}+\delta,1)\\ (\gamma+\frac{\mu}{2},1)\end{matrix}\right](u\sigma)^{-\gamma}g_1(u\sigma)d\sigma\right\}du+\\
&\int_0^\infty H_{0,2}^{1,0}\left[xu\,\middle|\,\begin{matrix}\overline{\qquad\qquad}\\ (\delta+\frac{\nu}{2},1),(\gamma-\frac{\mu}{2},1)\end{matrix}\right]\times
\end{aligned}
$$

$$\left\{\int_0^\infty H_{1,1}^{1,0}\left[\frac{1}{\sigma}\left|\begin{matrix}(\frac{\nu}{2}-\delta+1,1)\\(\frac{\mu}{2}-\gamma+1,1)\end{matrix}\right.\right](u\sigma)^{-\delta}g_2(u\sigma)d\sigma\right\}du. \tag{48}$$

From (45) and (47)

$$H_{0,2}^{1,0}\left[xu\left|\begin{matrix}\overline{\qquad\qquad}\\(\frac{\nu}{2}+\delta,1),(\gamma-\frac{\mu}{2},1)\end{matrix}\right.\right]=(xu)^{-\frac{\mu}{4}+\frac{\nu}{4}+\frac{\delta}{2}+\frac{\gamma}{2}}J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(2\sqrt{xu}). \tag{49}$$

Moreover, using the result [4, equation (2.2.4)]

$$H_{p,q}^{m,n}(x)=\sum_{h=1}^{m}\sum_{r=0}^{\infty}\prod_{\substack{j=1\\j\neq h}}^{m}\Gamma(b_j-\beta_j\xi_{h,r})\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi_{h,r})(-1)^r(x)^{\xi_{h,r}}\times$$

$$\left[\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi_{h,r})\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi_{h,r})r!\beta_h\right]^{-1}$$

with

$$\xi_{h,r}=\frac{(b_h+r)}{\beta_h},$$

we have that

$$\begin{aligned}H_{1,1}^{1,0}\left[\sigma\left|\begin{matrix}(\frac{\nu}{2}+\delta,1)\\(\gamma+\frac{\mu}{2},1)\end{matrix}\right.\right]&=\frac{\sigma^{\frac{\mu}{2}+\gamma}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)}\sum_{r=0}^{+\infty}\frac{\left(1-\frac{\nu}{2}-\delta+\frac{\mu}{2}+\gamma\right)_r\sigma^r}{r!}\\&=\frac{\sigma^{\frac{\mu}{2}+\gamma}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)}(1-\sigma)^{\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma-1}.\end{aligned} \tag{50}$$

Similarly

$$H_{1,1}^{1,0}\left[\frac{1}{\sigma}\left|\begin{matrix}(\frac{\nu}{2}-\delta+1,1)\\(\frac{\mu}{2}-\gamma+1,1)\end{matrix}\right.\right]=\frac{\sigma^{\delta-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma)}(\sigma-1)^{\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma-1}. \tag{51}$$

Substituting (49)-(51) in (48),

$$f(x)=x^{-\frac{\mu}{4}+\frac{\nu}{4}+\frac{\delta}{2}+\frac{\gamma}{2}}\left\{\int_0^1 u^{-\frac{\mu}{4}+\frac{\nu}{4}+\frac{\delta}{2}-\frac{\gamma}{2}}J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(2\sqrt{xu})\times\right.$$

$$\left[\int_0^1\frac{\sigma^{\frac{\mu}{2}}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)}(1-\sigma)^{\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma-1}g_1(u\sigma)d\sigma\right]du$$

$$(52)\qquad +\int_1^\infty u^{-\frac{\mu}{4}+\frac{\nu}{4}-\frac{\delta}{2}+\frac{\gamma}{2}}J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(2\sqrt{xu})$$

$$\left.\times\left[\int_0^\infty\frac{\sigma^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma)}(\sigma-1)^{\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma-1}g_2(u\sigma)d\sigma\right]du\right\},$$

where

$$(53)\qquad \begin{aligned}\varphi(x)&=x^{-\gamma}g_1(x),\\ \psi(x)&=x^{-\delta}g_2(x).\end{aligned}$$

If the right hand sides are $g_1(x)\in C_\alpha$, $g_2(x)\in C^*_{\alpha^*}$, then the corresponding conditions (44) on the parameters are equivalent to the conditions

$$(54)\qquad \nu>\lambda>\mu,\quad \frac{\mu}{2}+\alpha+1>0,\quad \frac{\mu}{2}+\delta-\gamma-\alpha^*>0.$$

As particular case of (46), if $u=\dfrac{w^2}{4}$ and $\sqrt{x}$ is substituted by x, we get the following system

$$\int_0^\infty\left(\frac{w}{2}\right)^{-2\gamma}J_\mu(xw)f\left(\frac{w^2}{4}\right)\left(\frac{w}{2}\right)dw=g_1(x^2),\quad 0<x<1$$

(55)

$$\int_0^\infty\left(\frac{w}{2}\right)^{-2\delta}J_\nu(xw)f\left(\frac{w^2}{4}\right)\left(\frac{w}{2}\right)dw=g_2(x^2),\quad x>1.$$

Let $F(w) = \left(\frac{w}{2}\right) f\left(\frac{w^2}{4}\right)$, $G_1(x) = 2^{-2\gamma} g_1(x^2)$ and

$$G_2(x) = 2^{-2\delta} g_2(x^2). \tag{56}$$

Therefore, we obtain the solution of the classical dual integral equations of Titchmarsh [1, p.84], namely:

$$\int_0^\infty w^{-2\gamma} J_\mu(xw) F(w) dw = G_1(x), \quad 0 < x < 1$$

$$\int_0^\infty w^{-2\delta} J_\nu(xw) F(w) dw = G_2(x), \quad x > 1$$

and from (52),

$$f\left(\frac{w^2}{4}\right) = \left(\frac{w}{2}\right)^{-\frac{\mu}{2}+\frac{\nu}{2}+\delta+\gamma} \left\{ \int_0^1 u^{-\frac{\mu}{4}+\frac{\nu}{4}+\frac{\delta}{2}-\frac{\gamma}{2}} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(w\sqrt{u}) \times \right.$$

$$\left[\int_0^1 \frac{\sigma^{\frac{\mu}{2}}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)} (1-\sigma)^{\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma-1} g_1(u\sigma) d\sigma \right] du +$$

$$\int_1^\infty u^{-\frac{\mu}{4}+\frac{\nu}{4}-\frac{\delta}{2}+\frac{\gamma}{2}} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(w\sqrt{u}) \times$$

$$\left. \left[\int_1^\infty \frac{\sigma^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma)} (\sigma-1)^{\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma-1} g_2(u\sigma) d\sigma \right] du \right\}.$$

After a change of the variable,

$$f\left(\frac{w^2}{4}\right) = \left(\frac{w}{2}\right)^{-\frac{\mu}{2}+\frac{\nu}{2}+\delta+\gamma} \left\{ 4 \int_0^1 t^{1-\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma} \times \right.$$

$$\left[\int_0^t \frac{\tau^{\mu+1}t^{2\gamma-2\delta-\nu}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)}(t^2-\tau^2)^{\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma-1} g_1(\tau^2)d\tau\right] dt+$$

$$4\int_1^\infty t^{1-\frac{\mu}{2}+\frac{\nu}{2}-\delta+\gamma} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(wt)\times$$

$$\left.\left[\int_t^\infty \frac{\tau^{1-\nu}t^{2\delta+\mu-2\gamma}}{\Gamma(\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma)}(\tau^2-t^2)^{\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma-1} g_2(\tau^2)d\tau\right] dt\right\}.$$

From (56),

$$F(w)=\left(\frac{w}{2}\right)^{1-\frac{\mu}{2}+\frac{\nu}{2}+\delta+\gamma}\left\{2^{2\gamma+2}\int_0^1 t^{1-\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(wt)\times\right.$$

$$\left[\int_0^t \frac{\tau^{\mu+1}t^{2\gamma-2\delta-\nu}}{\Gamma(\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma)}(t^2-\tau^2)^{\frac{\nu}{2}+\delta-\frac{\mu}{2}-\gamma-1} G_1(\tau)d\tau\right] dt+$$

$$2^{2\delta+2}\int_1^\infty t^{1-\frac{\mu}{2}+\frac{\nu}{2}-\delta+\gamma} J_{\frac{\mu}{2}+\frac{\nu}{2}+\delta-\gamma}(wt)\times$$

$$\left.\left[\int_t^\infty \frac{\tau^{1-\nu}t^{2\delta+\mu-2\gamma}}{\Gamma(\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma)}(\tau^2-t^2)^{\frac{\nu}{2}-\delta-\frac{\mu}{2}+\gamma-1} G_2(\tau)d\tau\right] dt\right\}.$$

For $\gamma=-w/2, \delta=0$ and $\lambda=(\mu+\nu+w)/2$ we have the solution

$$F(w)=2^{\lambda-\nu}w^{1-\lambda+\nu}\left\{2^{1-w}\int_0^1 t^{1+\lambda-\mu}J_\lambda(wt)\times\right.$$

$$\left[\int_0^t \frac{\tau^{\mu+1}t^{-w-\nu}}{\Gamma(\lambda-\mu)}(t^2-\tau^2)^{\lambda-\mu-1}G_1(\tau)d\tau\right]+$$

$$2\int_1^{\infty} t^{1-\lambda+\nu}J_\lambda(wt)\left[\int_t^{\infty}\frac{\tau^{1-\nu}t^{\mu+w}}{\Gamma(\nu-\lambda)}(\tau^2-t^2)^{\nu-\lambda-1}G_2(\tau)d\tau\right]dt\Bigg\},$$

given by Peters [1, p.85-86].

The new right hand side, to which the fractional integrals I and K are to be applied, are

$$\varphi(x)=2^{-w}x^{w/2}G_1(\sqrt{x})\in \mathcal{C}_{\frac{w}{2}-\frac{1}{2}},$$
$$\psi(x)=G_2(\sqrt{x})\in \mathcal{C}^*_{-\frac{1}{2}}$$

with the corresponding conditions

$$\nu-\lambda>\mu>-1,\quad \mu+w>-1.$$

When $\lambda_k=\beta_k=1$, $k=1,\ldots,m$ formula (42) gives the solution of the corresponding system of dual integral equations involving Meijer's G-function, see Kiryakova [5].

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