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Almost Contact Manifolds with B-Metric

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In this paper eleven basic classes of almost contact manifolds with B-metric are introduced. Examples of almost contact manifolds with B-metric are constructed on odd dimensional spheres.

1. Introduction

On an almost contact manifold (M, J) there can be considered two kinds of metrics g compatible with the almost complex structure J. If the almost complex structure J induces an isometry in each tangent fibre, then (M, J, g) has the structure of an almost Hermitian manifold. In the case when J induces an antiisometry in each tangent fibre, then (M, J, g) has the structure of an almost complex Riemannian manifold (almost complex manifold with B-metric).

Geometry of almost contact metric manifolds is a natural extension of almost Hermitian geometry to the odd dimensional case. Similarly, geometry of almost contact manifolds with B-metric can be considered as a natural extension of geometry of almost complex Riemannian manifolds to the odd dimensional case.

In this paper we give a classification of almost contact manifolds with B-metric with respect to the covariant derivative of the fundamental tensor of type $(1,1)$. We obtain eleven basic classes of almost contact manifolds with B-metric and construct some examples. We show that the isotropic sphere in R^{2n+2} carries in a natural way almost contact structure with B-metric, which can be considered as an analogue to the Sasakian structure.

2. Preliminaries

A $(2n+1)$-dimensional real deifferentiable manifold M is said to have a (φ,ξ,η)-structure, or an almost contact structure, if it admits a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions:

$$\eta(\xi) = 1, \tag{1}$$

$$\varphi^2 = -I + \eta\otimes\xi, \tag{2}$$

where I denotes the identity transformation.

We denote by $\mathcal{H}M$ the Lie algebra of C^∞-vector fields on M.

Definition. If a manifold M with a (φ,ξ,η)-structure admits a metric g such that

$$g(\varphi X,\varphi Y) = -g(X,Y) + \eta(X)\eta(Y), \tag{3}$$

where $X,Y\in\mathcal{H}M$, then M is said to be an almost contact manifold with B-metric.

From (1), (2) and (3) it follows immediately $\eta\circ\varphi = 0$, $\varphi\xi = 0$, $\eta(X) = g(X,\xi)$, $g(\varphi X,Y) = g(X,\varphi Y)$; $X,Y\in\mathcal{H}M$.

The tensor $\tilde{g}$ given by

$$\tilde{g}(X,Y) = g(\varphi X,Y) + \eta(X)\eta(Y); \quad X,Y\in\mathcal{H}M \tag{4}$$

is a B-metric associated with the metric g. The two metrics g and $\tilde{g}$ are indefinite of signature $(n+1,n)$.

Let ∇ be the Riemannian connection of the metric g. For all vectors x, y, z in the tangential space T_pM, $p\in M$, we denote

$$F(x,y,z) = g((\nabla_x\varphi)y,z).$$

Because of (1), (2) and (3) the tensor F has the following properties:

$$\begin{aligned} &F(x,y,z) = F(x,z,y),\\ &F(x,\varphi y,\varphi z) = F(x,y,z) - \eta(y)F(x,\xi,z) - \eta(z)F(x,y,\xi),\\ &F(x,\xi,\xi) = 0,\end{aligned} \tag{5}$$

for all vectors x, y, z in T_pM.

The following 1-forms are associated with F:

$$\theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^\star(x) = g^{ij}F(e_i, \varphi e_j, x), \tag{6}$$
$$\omega(x) = F(\xi, \xi, x),$$

where $x \in T_pM$, $\{e_i, \xi\}$, $(i = 1, \dots 2n)$ is a basis of T_pM, and (g^{ij}) is the inverse matrix of (g_{ij}).

Let $\tilde{\nabla}$ be be the Riemannian connection of the metric $\tilde{g}$. For all vector fields X, Y in $\mathcal{H}M$ we define

$$\Phi(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y.$$

Since $\tilde{\nabla}$ and ∇ are symmetric linear connections, then Φ is a symmetric tensor of type $(1,2)$, i. e.

$$\Phi(X,Y) = \Phi(Y,X), \quad X, Y \in \mathcal{H}M.$$

We denote the corresponding tensor of type $(0,3)$ by the same letter:

$$\Phi(x,y,z) = g(\Phi(x,y),z), \quad x,y,z \in T_pM.$$

It is easy to check that for arbitrary x, y, z in T_pM

$$\begin{aligned}
\Phi(x,y,z) = \frac{1}{2} \{&\{-F(x,y,\varphi z) - F(y,x,\varphi z) + F(\varphi z, x, y)\} \\
&+ \eta(x)\{F(y,z,\xi) + F(\varphi z, \varphi y, \xi)\} \\
&+ \eta(y)\{F(x,z,\xi) + F(\varphi z, \varphi x, \xi)\} \\
&+ \eta(z)\{F(x,y,\xi) + F(y,x,\xi) + F(x,\varphi y,\xi) \\
&\quad + F(y,\varphi x,\xi) - F(\xi,x,y)\} \\
&- \eta(z)\{\eta(x)\omega(\varphi y) + \eta(y)\omega(\varphi x)\}\}, \\
\Phi(x,y,\xi) = \frac{1}{2} \{&\{F(x,y,\xi) + F(y,x,\xi) + F(x,\varphi y,\xi) \\
&\quad + F(y,\varphi x,\xi) - F(\xi,x,y)\} \\
&- \{\eta(x)\omega(\varphi y) + \eta(y)\omega(\varphi x)\}\}, \\
\Phi(x,\xi,\xi) = 0&, \\
\Phi(\xi,x,y) = \frac{1}{2} \{&\{F(x,y,\xi) + F(\varphi y,\varphi x,\xi) - F(x,\varphi y,\xi) \\
&\quad + F(\varphi y,x,\xi) - F(\xi,x,\varphi y)\} \\
&+ \eta(x)\omega(y)\}, \\
\Phi(\xi,\xi,x) = \omega&(x) - \omega(\varphi x).
\end{aligned} \tag{7}$$

We consider the following 1-forms associated with Φ:

$$f(x) = g^{ij}\,\Phi(e_i, e_j, x), \qquad f^{\star}(x) = g^{ij}\,\Phi(e_i, \varphi e_j, x), \qquad x \in T_pM.$$

From (7) we can compute the formula for F expressed by Φ:

$$\begin{aligned} F(x,y,z) = {} & \Phi(x,y,\varphi z) + \Phi(x,z,\varphi y) \\ & + \frac{1}{2}\,\eta(y)\,\{\Phi(x,z,\xi) - \Phi(x,\varphi z,\xi) + \Phi(\xi,x,z) - \Phi(\xi,x,\varphi z)\} \\ & + \frac{1}{2}\,\eta(z)\,\{\Phi(x,y,\xi) - \Phi(x,\varphi y,\xi) + \Phi(\xi,x,y) - \Phi(\xi,x,\varphi y)\} \end{aligned}$$

for arbitrary x, y, z in T_pM.

3. The space of covariant derivatives of the structure φ.

Let V be a $(2n+1)$-dimensional vector space with almost contact structure (φ,ξ,η) and B-metric g. We consider the vector space $\mathcal{F}$ of all tensors of type $(0,3)$ over V having the properties

$$(8) \qquad \begin{aligned} F(x,y,z) &= F(x,z,y), \\ F(x,\varphi y,\varphi z) &= F(x,y,z) - \eta(y)\,F(x,\xi,z) - \eta(z)\,F(x,y,\xi), \end{aligned}$$

for arbitrary vectors x, y, z in V.

The metric g induces on F an inner product $<,>$, defined by

$$< F_1, F_2 >= g^{iq}\,g^{jr}\,g^{ks}\,F_1(e_i,e_j,e_k)\,F_2(e_q,e_r,e_s)$$

for $F_1, F_2 \in \mathcal{F}$ and $\{e_i, e_{2n+1} = \xi\}$ $(i = 1,\dots,2n)$ – a basis of V.

Let G be the group $G = GL(n,C) \cap O(n,n)$, i. e. G consists of the real matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ which are in $O(n,n)$. (A, B are matrices of type $n \times n$). We consider the group $G \times I$. The standart representation of $G \times I$ in V induces a natural representation λ of $G \times I$ in $\mathcal{F}$:

$$(\lambda(a)F)(x,y,z) = F(a^{-1}x, a^{-1}y, a^{-1}z), \qquad a \in G \times I,\ F \in \mathcal{F},\ x, y, z \in V,$$

so that

$$< \lambda(a)F_1, \lambda(a)F_2 >=< F_1, F_2 >, \quad a \in G \times I, \quad F_1, F_2 \in \mathcal{F}.$$

Let $x \in V$, then $x = hx + \eta(x)\xi$, where $hx = -\varphi^2 x$.

Now we define the operator

$$\mathcal{P}_1 : \mathcal{F} \to \mathcal{F},$$

$$\mathcal{P}_1(F)(x,y,z) = F(hx,hy,hz), \quad x,y,z \in V,\ F \in \mathcal{F}.$$

We have

Lemma 3.1. *The operator $\mathcal{P}_1$ has the following properties:*
(i$_1$) $<\mathcal{P}_1 F_1, F_2> = < F_1, \mathcal{P}_1 F_2>$, $F_1, F_2 \in \mathcal{F}$.
(i$_2$) $\mathcal{P}_1 \circ \mathcal{P}_1 = \mathcal{P}_1$.

If we denote $W_1 = Im\mathcal{P}_1$, then Lemma 3.1 implies

$$W_1 = \{F \in \mathcal{F} \mid \mathcal{P}_1 F = F\},$$
$$W_1^{\perp} = Ker\,\mathcal{P}_1 = \{F \in \mathcal{F} \mid \mathcal{P}_1 F = 0\}.$$

Further we consider the operator

$$\mathcal{P}_2 : W_1^{\perp} \to W_1^{\perp},$$

defined by

$$\mathcal{P}_2(F)(x,y,z) = \eta(y)\,F(hx,\xi,hz) + \eta(z)\,F(hx,hy,\xi), \qquad x,y,z \in V, \quad F \in W_1^{\perp}.$$

Then we obtain

Lemma 3.2. *The operator $\mathcal{P}_2$ has the following properties:*
(i$_1$) $<\mathcal{P}_2 F_1, F_2> = < F_1, \mathcal{P}_2 F_2>$, $F_1, F_2 \in W_1^{\perp}$,
(i$_2$) $\mathcal{P}_2 \circ \mathcal{P}_2 = \mathcal{P}_2$.

If we denote $W_2 = Im\,\mathcal{P}_2$, then Lemma 3.2 implies

$$W_2 = \{F \in W_1^{\perp} \mid \mathcal{P}_2 F = F\},$$
$$W_2^{\perp} = Ker\,\mathcal{P}_2 = \{F \in W_1^{\perp} \mid \mathcal{P}_2 F = 0\}.$$

Finally, let us consider the operator

$$\mathcal{P}_3 : W_2^{\perp} \to W_2^{\perp},$$

defined by

$$\mathcal{P}_3(F)(x,y,z) = \eta(x)\,F(\xi,hy,hz), \quad x,y,z \in V, \quad F \in W_2^{\perp}.$$

In a straightforward way we get

Lemma 3.3. *The operator $\mathcal{P}_3$ has the following properties:*
(i$_1$) $<\mathcal{P}_3 F_1, F_2> = <F_1, \mathcal{P}_3 F_2>, \quad F_1, F_2 \in W_2^{\perp}$,
(i$_2$) $\mathcal{P}_3 \circ \mathcal{P}_3 = \mathcal{P}_3$.

If we denote $W_3 = Im\, \mathcal{P}_3$ and $W_4 = Ker\, \mathcal{P}_3$, then Lemma 3.3 implies

$$W_3 = \{F \in W_2^{\perp} \mid \mathcal{P}_3 F = F\},$$
$$W_4 = \{F \in W_2^{\perp} \mid \mathcal{P}_3 F = 0\}.$$

From Lemma 3.1, Lemma 3.2 and Lemma 3.3 we obtain immediately

Proposition 3.4. (Partial decomposition). *The decomposition*

$$\mathcal{F} = W_1 \oplus W_2 \oplus W_3 \oplus W_4$$

is orthogonal and invariant under the action of $G \times I$.

Proposition 3.5. *The classes W_i $(i = 1, \dots, 4)$ are characterized as follows*

$$W_1 = \{F \in \mathcal{F} \mid F(\xi, x, y) = F(x, \xi, y) = 0\},$$
$$W_2 = \{F \in \mathcal{F} \mid F(\xi, y, z) = F(x, hy, hz) = 0\},$$
$$W_3 = \{F \in \mathcal{F} \mid F(x, y, \xi) = F(hx, y, z) = 0\},$$
$$W_4 = \{F \in \mathcal{F} \mid F(hx, y, z) = F(x, hy, hz) = 0\},$$

for arbitrary vectors x, y, z in V.

Now, let $\overline{V}$ be the orthogonal complement of the subspace spanned by ξ. The endomorphism φ induces on $\overline{V}$ a complex structure and g is the complex Riemannian metric (a B-metric) on $\overline{V}$. Ganchev and Borisov decomposed the vector space [3]

$$W = \{\alpha \in \bigotimes^3 \overline{V}^* \mid \alpha(x, y, z) = \alpha(x, z, y) = \alpha(x, \varphi y, \varphi z),\ x, y, z \in \overline{V}\}$$

into three orthogonal and invariant under the action of G subspaces.

For arbitrary x, y, z in $\overline{V}$ we define

$$\mathcal{F}_1 = \Big\{F \in \mathcal{F} \mid F(x, y, z) = \frac{1}{2n}\{g(x, \varphi y)\theta(\varphi z) + g(x, \varphi z)\theta(\varphi y) - g(\varphi x, \varphi y)\theta(z) - g(\varphi x, \varphi z)\theta(y)\}\Big\}$$

$$\mathcal{F}_2 = \{F \in \mathcal{F} \mid F(x, y, \varphi z) + F(y, z, \varphi x) + F(z, x, \varphi y) = 0,\ \theta = 0\},$$
$$\mathcal{F}_3 = \{F \in \mathcal{F} \mid F(x, y, z) + F(y, z, x) + F(z, x, y) = 0\}.$$

Since W_1 is naturally isomorphic to W, we have

Theorem 3.6. *The decomposition*

$$W_1 = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_3$$

is orthogonal and invariant under the action of G.

Let W' and W'' be the following subspaces of W_2:

$$W' = \{F \in \mathcal{F} \mid F(x,y,z) + F(\varphi x, \varphi y, z) + F(\varphi x, y, \varphi z) = 0\},$$
$$W'' = \{F \in \mathcal{F} \mid F(x,y,z) - F(\varphi x, \varphi y, z) - F(\varphi x, y, \varphi z) = 0\}.$$

It is easy to check

Theorem 3.7. *The decomposition*

$$W_2 = W' \oplus W''$$

is orthogonal and invariant under the action of $G \times I$.

Further, for arbitrary x, y, z in V we define

$$\mathcal{F}_7 = \{F \in W' \mid F(x,y,z) + F(y,z,x) + F(z,x,y) = 0\}.$$

For the orthogonal complement $\mathcal{F}_7^{\perp}$ of $\mathcal{F}_7$ in W' we obtain the characterization

$$\mathcal{F}_7^{\perp} = \{F \in W' \mid F(x,y,z) + F(y,z,x) - F(z,x,y) + 2\,F(\varphi x, \varphi y, z) = 0\}.$$

Let us now consider the subspaces of $\mathcal{F}_7^{\perp}$:

$$\mathcal{F}_4 = \left\{F \in \mathcal{F}_7^{\perp} \mid F(x,y,z) = -\frac{\theta(\xi)}{2n}\,\{\eta(y)\,g(\varphi x, \varphi z) + \eta(z)\,g(\varphi x, \varphi y)\}\right\},$$
$$\mathcal{F}_5 = \left\{F \in \mathcal{F}_7^{\perp} \mid F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\,\{\eta(y)\,g(\varphi x, z) + \eta(z)\,g(\varphi x, y)\}\right\},$$
$$\mathcal{F}_6 = \left\{F \in \mathcal{F}_7^{\perp} \mid \theta^*(\xi) = \theta(\xi) = 0\right\}.$$

Then we obtain

Theorem 3.8. *The decomposition*

$$W' = \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6 \oplus \mathcal{F}_7$$

is orthogonal and invariant under the action of $G \times I$.

Further, the subspaces $\mathcal{F}_8$ and $\mathcal{F}_9$ of W'' are defined as follows:

$$\mathcal{F}_8 = \{F \in W'' \mid F(x,y,z) + F(y,z,x) - F(z,x,y) - 2\,F(\varphi x, \varphi y, z) = 0\},$$
$$\mathcal{F}_9 = \{F \in W'' \mid F(x,y,z) + F(y,z,x) + F(z,x,y) = 0\}.$$

In a straightforward way we have

Theorem 3.9. *The decomposition*

$$W'' = \mathcal{F}_8 \oplus \mathcal{F}_9$$

is orthogonal and invariant under the action of $G \times I$.

Finally, we denote $\mathcal{F}_{10} = W_3$ and $\mathcal{F}_{11} = W_4$. Taking into account Proposition 3.5 and Theorems 3.6–3.9, we obtain

Theorem 3.10. *The decomposition*

$$\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_{11}$$

is orthogonal and invariant under the action of $G \times I$.

Next we summarize the characterization conditions for the factors $\mathcal{F}_i$ $(i = 1, \ldots, 11)$.

Let x, y, z be arbitrary vectors in V. Then

$$\begin{aligned}
\mathcal{F}_1 : F(x,y,z) = {} & \frac{1}{2n}\{g(x,\varphi y)\,\theta(\varphi z) + g(x,\varphi z)\,\theta(\varphi y) \\
& - g(\varphi x,\varphi y)\,\theta(hz) - g(\varphi x,\varphi z)\,\theta(hy)\}, \\
\mathcal{F}_2 : F(\xi,y,z) = {} & F(x,\xi,z) = 0; \\
& F(x,y,\varphi z) + F(y,z,\varphi x) + F(z,x,\varphi y) = 0; \quad \theta = 0, \\
\mathcal{F}_3 : F(\xi,y,z) = {} & F(x,\xi,z) = 0; \\
& F(x,y,z) + F(y,z,x) + F(z,x,y) = 0, \\
\mathcal{F}_4 : F(x,y,z) = {} & -\frac{\theta(\xi)}{2n}\{\eta(y)\,g(\varphi x,\varphi z) + \eta(z)\,g(\varphi x,\varphi y)\}, \\
\mathcal{F}_5 : F(x,y,z) = {} & -\frac{\theta^*(\xi)}{2n}\{\eta(y)\,g(\varphi x,z) + \eta(z)\,g(\varphi x,y)\}, \\
\mathcal{F}_6 : F(x,y,z) = {} & -F(\varphi x,\varphi y,z) - F(\varphi x,y,\varphi z) \\
= {} & -F(y,z,x) + F(z,x,y) - 2F(\varphi x,\varphi y,z), \\
& \theta(\xi) = \theta^*(\xi) = 0, \\
\mathcal{F}_7 : F(x,y,z) = {} & -F(\varphi x,\varphi y,z) - F(\varphi x,y,\varphi z) \\
= {} & -F(y,z,x) - F(z,x,y), \\
\mathcal{F}_8 : F(x,y,z) = {} & F(\varphi x,\varphi y,z) + F(\varphi x,y,\varphi z) \\
= {} & -F(y,z,x) + F(z,x,y) + 2F(\varphi x,\varphi y,z),
\end{aligned}$$

$$\begin{aligned}\mathcal{F}_9 : F(x,y,z) &= F(\varphi x, \varphi y, z) + F(\varphi x, y, \varphi z)\\ &= -F(y,z,x) - F(z,x,y),\\ \mathcal{F}_{10} : F(x,y,z) &= \eta(x)\, F(\xi, \varphi y, \varphi z),\\ \mathcal{F}_{11} : F(x,y,z) &= \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.\end{aligned}$$

Taking into account the characterization symmetries of $F(x,y,z)$ in the factors $\mathcal{F}_i$ $(i = 1, \dots 11)$, we obtain

Proposition 3.11. *The dimensions of the factors in the decomposition of the space $\mathcal{F}$ are given in the following table:*

dim $\mathcal{F}_1$	2	$2n$
dim $\mathcal{F}_2$	0	$n(n-1)(n+2)$
dim $\mathcal{F}_3$	0	$n^2(n-1)$
dim $\mathcal{F}_4$	1	1
dim $\mathcal{F}_5$	1	1
dim $\mathcal{F}_6$	0	$(n-1)(n+2)$
dim $\mathcal{F}_7$	0	$n(n-1)$
dim $\mathcal{F}_8$	1	n^2
dim $\mathcal{F}_9$	1	n^2
dim $\mathcal{F}_{10}$	1	n^2
dim $\mathcal{F}_{11}$	2	$2n$

4. Basic classes of almost contact manifolds with B-metric and some examples

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric. Using the decomposition of the space $\mathcal{F}$ over $V = T_pM$, $p \in M$, we define the corresponding subclasses of the class of almost contact manifolds with B-metric with respect to the covariant derivative of the structure tensor field φ.

An almost contact manifold with B-metric is said to be in the class $\mathcal{F}_i$ $(i = 1, \dots, 11)$ if the tensor $F(x,y,z) = g((\nabla_x \varphi)y, z)$ belongs to the class $\mathcal{F}_i$ over $V = T_pM$ for each $p \in M$.

Thus, we define the eleven basic classes of almost contact manifolds with B-metric.

In a similar way we define the classes $\mathcal{F}_i \oplus \mathcal{F}_j$, etc. It is clear that 2^{11} classes of almost contact manifolds with B-metric are possible.

The class $\mathcal{F}_0$ of almost contact manifolds with B-metric is defined by the condition $F(x,y,z)=0$. This special class belongs to everyone of the defined classes.

First we give an example of a manifold in the class $\mathcal{F}_0$.

Example 1. Let $R^{2n+1} = \{(u^1,\ldots,u^n;v^1,\ldots,v^n;t) \mid u^i,v^i,t \in R\}$. We define the structure (φ,ξ,η,g) on R^{2n+1} in the following way:

$$\begin{aligned}
\xi &= \frac{\partial}{\partial t}, \qquad \eta = dt;\\
\varphi\left(\frac{\partial}{\partial u^i}\right) &= \frac{\partial}{\partial v^i}, \quad \varphi\left(\frac{\partial}{\partial v^i}\right) = -\frac{\partial}{\partial u^i}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0;\\
g(x,x) &= -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j + \nu^2,
\end{aligned}$$

where $x = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i} + \nu \frac{\partial}{\partial t}$ and δ_{ij} are the Kronecker's symbols. It follows from this definition that

$$\begin{aligned}
g(\xi,x) &= \eta(x),\\
g(\varphi x,\varphi x) &= -g(x,x) + \eta(x)\eta(x)
\end{aligned}$$

for an arbitrary vector x.

If ∇ is the Levi-Civita connection of the metric g, it is easy to check that $\nabla\varphi = 0$.

Hence, $(R^{2n+1},\varphi,\xi,\eta,g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_0$.

It is well known [2,4,5,6] that any real hypersurface of an almost Hermitian manifold carries in a natural way an almost contact metric structure.

In a similar way it can be shown that on every real nonisotropic hypersurface of a complex Riemannian manifold there arises an almost contact structure with B-metric.

Let $R^{2n+2} = \left\{(u^1,\ldots,u^{n+1};v^1,\ldots,v^{n+1}) \mid u^i,v^i \in R\right\}$. We consider R^{2n+2} as a complex Riemannian manifold with the canonical complex structure J and the metric g, defined by

$$g(x,x) = -\delta_{ij}\lambda^i\lambda^j + \delta_{ij}\mu^i\mu^j,$$

where $x = \lambda^i \frac{\partial}{\partial u^i} + \mu^i \frac{\partial}{\partial v^i}$.

Identifying the point $p = (u^1,\ldots,u^{n+1};v^1,\ldots,v^{n+1})$ in R^{2n+2} with its position vector Z, we study two hypersurfaces of R^{2n+2}.

Example 2. Let

$$S:\ g(Z,Z) = -1$$

be the time-like sphere of the metric g. The position vector Z is the unit normal to the tangent space T_pS at $p \in S$. We determine the structure vector field ξ on S with the conditions

$$\xi = \lambda Z + \mu JZ, \quad g(Z,\xi) = 0, \quad g(\xi,\xi) = 1.$$

At every point $p \in S$ we set $g(Z,JZ) = tgt$ for $t \in (-\pi/2, \pi/2)$. Then

$$\xi = \sin t\,.\,Z + \cos t\,.\,JZ, \quad J\xi = -\cos t\,.\,Z + \sin t\,.\,JZ. \tag{9}$$

The last equality implies that $g(J\xi, Z) = 1/\cos t$. Hence, $J\xi$ is not in T_pS.

To define the structure tensor φ for an arbitrary vector x in T_pS, we consider the vector Jx and denote by φx its projection into T_pS with respect to $J\xi$. Then we have the unique decomposition

$$Jx = \varphi x + \eta(x) J\xi, \tag{10}$$

where η is a 1-form in T_pS.

It follows from (10) that

$$\begin{aligned} &\varphi^2 x = -x + \eta(x)\xi, \\ &\eta(\varphi x) = 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad x \in T_pS. \end{aligned} \tag{11}$$

Using (9), (10) and (11), we find

$$g(x,\xi) = \eta(x), \quad x \in T_pS. \tag{12}$$

Further, the equalities (10) and (11) imply

$$g(\phi x, \phi y) = -g(x,y) + \eta(x)\eta(y), \quad x, y \in T_pS. \tag{13}$$

Thus, we obtained that (φ, ξ, η, g) is an almost contact structure with B-metric on the unit time-like sphere S^{2n+1}.

Now, we shall study the covariant derivative of the structure tensor field φ on S.

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of the metric g in R^{2n+2} and S, respectively. Then the formulas of Gauss and Weingarten are

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y - g(AX,Y)Z, \quad X,Y \in \mathcal{H}S; \\ &\overline{\nabla}_X Z = -AX, \quad X \in \mathcal{H}S. \end{aligned} \tag{14}$$

Since $\overline{\nabla}$ is flat, then $\overline{\nabla}_X Z = X$, Z being the position vector field and X being an arbitrary vector field on S. Hence, $A = -I$, where I is the identity transformation. Then the formulas (14) become

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + g(X,Y)Z, \quad X,Y \in \mathcal{H}S; \\ &\overline{\nabla}_X Z = X, \quad X \in \mathcal{H}S. \end{aligned} \tag{15}$$

From (9), (10) and (15) we find

$$\begin{aligned} \nabla_x \xi \ &= \ \cos t \,.\, \varphi x \ - \ \sin t \,.\, \varphi^2 x, \\ F(x,y,\xi) \ &= \ g((\nabla_x \varphi)\xi, y) \ = \ -\cos t \,.\, g(\varphi x, \varphi y) \ - \ \sin t \,.\, g(\varphi x, y), \end{aligned} \tag{16}$$

for arbitrary vectors x, y in T_pS.

Using (6) and (16) we compute

$$\cos t \ = \ \frac{\theta(\xi)}{2n}; \quad \sin t \ = \ \frac{\theta^*(\xi)}{2n}. \tag{17}$$

Taking into account (15), (16) and (17) we find

$$\begin{aligned} (\nabla_x \varphi)\, y = &- \frac{\theta(\xi)}{2n} \,\{\eta(y)\,\varphi^2 x \,+\, g(\varphi x, \varphi y)\,\xi\} \\ &- \frac{\theta^*(\xi)}{2n} \,\{\eta(y)\,\varphi x \,+\, g(\varphi x, y)\,\xi\}\,. \end{aligned}$$

Hence,

$$\begin{aligned} F(x,y,z) = &- \frac{\theta(\xi)}{2n} \,\{\eta(y)\, g(\varphi x, \varphi z) \,+\, \eta(z)\, g(\varphi x, \varphi y)\} \\ &- \frac{\theta^*(\xi)}{2n} \,\{\eta(y)\, g(\varphi x, z) \,+\, \eta(z)\, g(\varphi x, y)\}\,. \end{aligned}$$

Thus, $(S^{2n+1}, \varphi, \xi, \eta, g)$ is an almost contact manifold with B-metric in the class $\mathcal{F}_4 \oplus \mathcal{F}_5$.

Example 3. Let M be the hypersurface of R^{2n+2} determined by

$$M \,:\, g(Z, JZ) \ = \ 0; \ \ g(Z,Z) > 0.$$

At every point $p \in M$ we can put $g(Z,Z) = \operatorname{ch}^2 t,\ t>0$. The vector field JZ is normal to M. We choose the unit normal $N = (1/\operatorname{ch} t)\,JZ$, which is time-like, i. e. $g(N,N) = -1$.

We define the structure vector field on M by the equality

$$\xi = -JN = \frac{1}{\operatorname{ch} t} Z. \tag{18}$$

For an arbitrary vector x in T_pM we consider the vector Jx. Denoting the orthogonal projection of Jx into T_pM by φx, we have the unique decomposition

$$Jx = \varphi x + \eta(x)\,N, \tag{19}$$

where η is a 1-form in T_pM.

From (18) and (19) it follows that

$$\begin{aligned} \varphi^2(x) &= -x + \eta(x)\,\xi, \\ \eta(\varphi x) &= 0, \quad \varphi\xi = 0, \quad \eta(\xi) = 1, \quad x \in T_pM. \end{aligned} \tag{20}$$

Using (18), (19) and (20) we find

$$\begin{aligned} g(\xi,x) &= \eta(x), \\ g(\varphi x,\varphi y) &= -g(x,y) + \eta(x)\eta(y), \quad x,y \in T_pM. \end{aligned} \tag{21}$$

Taking into account (18), (20) and (21), we can conclude that (M,φ,ξ,η,g) is an almost contact manifold with B-metric.

Denoting by $\overline{\nabla}$ and ∇ the Levi-civita connections of the metric g in R^{2n+2} and M, respectively, the formulas of Gauss and Weiergarten are

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y - g(AX,Y)N, \quad X,Y \in \mathcal{H}M; \\ &\overline{\nabla}_X N = -AX, \quad X \in \mathcal{H}M. \end{aligned} \tag{22}$$

From (22), the definition of N and $\overline{\nabla}_X Z = X$, Z being the position vector field, we get $A = -(1/\operatorname{ch} t)\,\varphi$. Then the formulas (22) become

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + \frac{1}{\operatorname{ch} t} g(\varphi X,Y)N, \quad X,Y \in \mathcal{H}M; \\ &\overline{\nabla}_X N = \frac{1}{\operatorname{ch} t}\varphi X, \quad X \in \mathcal{H}M. \end{aligned} \tag{23}$$

To compute $(\nabla_x\varphi)\,y$, first we find

$$\overline{\nabla}_x \xi = -\frac{1}{\operatorname{ch} t}\,\varphi^2 x, \quad x \in T_pM.$$

Then

$$F(x,y,\xi) = -\frac{1}{\operatorname{ch} t}\, g(\varphi x, y).$$

From this equality and (6) we compute

$$\theta(\xi) = 0; \qquad \frac{\theta^*(\xi)}{2n} = \frac{1}{\operatorname{ch} t}.$$

Finally, using (19) and (23), we calculate

$$(\nabla_x \varphi)\, y = -\frac{\theta^*(\xi)}{2n}\,\{\eta(y)\,\varphi x + g(\varphi x, y)\,\xi\}, \quad x, y \in T_pM.$$

Hence

$$F(x,y,z) = -\frac{\theta^*(\xi)}{2n}\,\{\eta(y)\, g(\varphi x, z) + \eta(z)\, g(\varphi x, y)\}.$$

Thus, (M,φ,ξ,η,g) is an almost contact manifold with B-metric in the class $\mathcal{F}_5$.

Remark. The class $\mathcal{F}_5$ is analoguous in some sense to the class of α-Sasakian manifolds in the theory of almost contact metric manifolds (e.g. see [1]).

5. Characterization of the classes in local coordinates

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric. At every point $p \in M$ we have

$$T_pM = D \oplus Im\ \eta,$$

where $D = Ker\ \eta$.

Let D^C be the complexification of the real vector space D with complex structure φ. Then we have $D^C = D^{1,0} \oplus D^{0,1}$. We put

$$T_p^C M = D^C \oplus Im\ \eta = D^{1,0} \oplus D^{0,1} \oplus Im\ \eta.$$

For an arbitrary point $p \in M$ we can choose locally vector frame fields of type $\{Z_\alpha, Z_{\overline{\beta}}, Z_0 = \xi\}$, where the complex vector fields $\{Z_\alpha\}$ $(\alpha = 1,\dots,n)$ form a basis for $D^{1,0}$ at every point $p \in M$ and $\{Z_{\overline{\beta}} = \overline{Z}_\beta\}$ $(\overline{\beta} = 1,\dots,n)$ is the conjugate basis for $D^{0,1}$ at $p \in M$.

In this section we shall characterize the basic classes of almost contact manifolds with B-metric in terms of the local frame fields introduced above.

Every one of the tensors over T_pM can be extended uniquely by a linearization to the corresponding tensor over $T_p^C M$.

Let $F(x,y,z)$, $x,y,z \in T_p^C M$, be the extended tensor F over $T_p^C M$ and $\{Z_\alpha, Z_{\overline{\beta}}, Z_0\}$ be a local frame field. We denote the components of F by

$$F_{ABC} = F(Z_A, Z_B, Z_C),$$

where the Latin capitals run through $1,\ldots,n;\overline{1},\ldots,\overline{n};0$.

Taking into account the symmetries of F, we find that its essential components (which may not be zero) are

$$F_{\alpha\beta\overline{\gamma}},\quad F_{\alpha\beta 0},\quad F_{\alpha\overline{\beta}0},\quad F_{0\beta\overline{\gamma}},\quad F_{00\gamma}$$

and their conjugates.

Using the characterization of the basic classes of almost contact manifolds with B-metric, we obtain characterization conditions of these manifolds in local components.

Here we give these conditions in the following table:

Class	Essential components	Characterization conditions
$\mathcal{F}_1$	$F_{\alpha\beta\overline{\gamma}}$	$F_{\alpha\beta\overline{\gamma}} = (\theta_{\overline{\gamma}}/n)\, g_{\alpha\beta}$
$\mathcal{F}_2$	$F_{\alpha\beta\overline{\gamma}}$	$F_{\alpha\beta\overline{\gamma}} = F_{\beta\alpha\overline{\gamma}},\ \theta_{\overline{\gamma}} = 0$
$\mathcal{F}_3$	$F_{\alpha\beta\overline{\gamma}}$	$F_{\alpha\beta\overline{\gamma}} = -F_{\beta\alpha\overline{\gamma}}$
$\mathcal{F}_4$	$F_{\alpha\beta 0}$	$F_{\alpha\beta 0} = (\theta_0/2n)\, g_{\alpha\beta}$
$\mathcal{F}_5$	$F_{\alpha\beta 0}$	$F_{\alpha\beta 0} = (i\theta_0^*/2n)\, g_{\alpha\beta}$
$\mathcal{F}_6$	$F_{\alpha\beta 0}$	$F_{\alpha\beta 0} = F_{\beta\alpha 0},\ \theta_0 = \theta_0^* = 0$
$\mathcal{F}_7$	$F_{\alpha\beta 0}$	$F_{\alpha\beta 0} = -F_{\beta\alpha 0}$
$\mathcal{F}_8$	$F_{\alpha\overline{\beta}0}$	$F_{\alpha\overline{\beta}0} = F_{\overline{\beta}\alpha 0}$
$\mathcal{F}_9$	$F_{\alpha\overline{\beta}0}$	$F_{\alpha\overline{\beta}0} = -F_{\overline{\beta}\alpha 0}$
$\mathcal{F}_{10}$	$F_{0\beta\overline{\gamma}}$	no conditions for $F_{0\beta\overline{\gamma}}$
$\mathcal{F}_{11}$	$F_{00\gamma} = \omega_\gamma$	no conditions for ω_γ

Using the relation (7), one can obtain characterization conditions for the basic classes $\mathcal{F}_i$ $(i = 1,\ldots,11)$ in terms of the tensor Φ (in global variables or in local components).

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