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On a Generalization of the Quaternion Algebra

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1.Introduction

The quaternion algebra $\mathbb{H}$ (over the field of real numbers $\mathbb{R}$) is defined as 4-dimensional vector space

$$\mathbb{H} = \mathbb{R} + \mathbb{R}_i + \mathbb{R}_j + \mathbb{R}_k,$$

where the elements 1, i, j, k multiplied with ± 1 form the quaternion group $\mathbb{Q}$, known as a two – generated group – $\mathbb{Q} = < i, j >$, with defining relations (see [2])

$$-1 = i^2 = j^2, \ i^4 = 1, \ j = iji,$$

or equivalently see ([5] and [6])

$$i = jij, \quad j = iji.$$

We have $k = ij$.

By generalization from the latter presentation of the quaternion group we get groups $\mathbb{Q}_n$, containing the quaternion group $\mathbb{Q} = \mathbb{Q}_2$ as a subgroup and having some of its basic properties (see [7]). In particular, the group $\mathbb{Q}_n$ has an application in the theory of Hadamard matrices (see [8]).

In the present research (over the field $\mathbb{R}$ of real numbers) the 8-dimensional algebra $\mathbb{H}(\mathbb{Q}_3) = \mathbb{H}_3$ (associated with the 3 – generated quaternion group $\mathbb{Q}_3$) is constructed. It contains the quaternion algebra $\mathbb{H}$, the center of which is a two-dimensional subalgebra. The subset of its invertible elements form the uniquely non-trivial proper ideals I^- and I^+ such that $\mathbb{H}_3 = I^- + I^+$. All 4-dimensional subalgebras are given as well as all 2-dimensional subalgebras of the algebra $\mathbb{H}_3$. All idempotents of the algebra $\mathbb{H}_3$ are determined. By the formulation of the propositions and their proofs two bases are used – one is formed by elements of the group $\mathbb{Q}_3$, and the other – by the elements of the ideals I^- and I^+. The results depending on the bases are formulated for both of these bases, but the proofs are given only in one of the two cases.

The present investigation follows mostly the ideas due to A. Hurwitz [3], [4] and are known in mathematics as classical.

Non-defined terminology and notation may be found in [7] and in [8].

2. Definitions and Basic Properties of the Algebra of the 3-generated Quaternion Group

Denote by $\mathbb{Q}_3$ the tree – generated quaternion group –

$$\mathbb{Q}_3 =< g_1, g_2, g_3\, g_i = g_j g_i g_j,\ i \neq j,\ i, j = 1, 2, 3 >=$$

$$\{e_i = g_i,\ e_4 = g_1 g_2,\ e_5 = g_1 g_3,\ e_6 = g_2 g_3,\ e_7 = g_1 g_3 g_3\},$$

where $g_i^2 = -1 = -e_0$ and the multiplication of the elements e_i is given by the following

Table 1

$\diagdown$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_4	e_5	$-e_2$	$-e_3$	e_7	$-e_6$
e_2	$-e_4$	$-e_0$	e_6	e_1	$-e_7$	$-e_3$	e_5
e_3	$-e_5$	$-e_6$	$-e_0$	e_7	e_1	e_2	$-e_4$
e_4	e_2	$-e_1$	e_7	$-e_0$	e_6	$-e_5$	$-e_3$
e_5	e_3	$-e_7$	$-e_1$	$-e_6$	$-e_0$	e_4	e_2
e_6	e_7	e_3	$-e_2$	e_5	$-e_4$	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	$-e_4$	$-e_3$	e_2	$-e_1$	e_0

Here for brevity, the rows and the columns corresponding to $-e_i$ ($i = 0, 1, ..., 7$) are not presented.

From Table 1 it is easy to see by direct calculation that the group $\mathbb{Q}_3$ contains three subgroups isomorphic to the quaternion group $\mathbb{Q}$, namely:

$$\{\pm e_0, \pm e_1, \pm e_2, \pm e_4\} \cong \{\pm e_0, \pm e_1, \pm e_3, \pm e_5\} \cong \{\pm e_0, \pm e_2, \pm e_3, \pm e_6\}. \tag{1}$$

Moreover, its center is the subgroup

$$\mathbb{C} = \mathbb{C}(\mathbb{Q}_3) = \{\pm 1, \pm e_7\},$$

such that

$$\mathbb{Q}_3/\mathbb{C} \cong \mathbf{V},$$

where $\mathbf{V} = \{\bar{e}_0, \bar{e}_2, \bar{e}_2, \bar{e}_3\}$ is the Kleinean group.

Consider a subspace $\mathbb{J}$ of the group algebra $\mathbb{RQ}_3$ spanned by all the elements $[e_i + (-e_i)]$, $i = 0, 1, ..., 7$. The following holds:

Lemma 1. *$\mathbb{J}$ is an ideal of $\mathbb{RQ}_3$, and $\mathbb{H}_3 \cong \mathbb{RQ}_3/\mathbb{J}$. Thus $\mathbb{H}_3$ is an 8-dimensional associative) algebra over $\mathbb{R}$; clearly $\mathbb{H}_3$, contains the quaternion algebra (denoted here and on by) $\mathbb{H}$.*

Every element $a = \sum_{i=0}^{i=7} a_i e_i \in \mathbb{H}_3$ is a divisor of zero iff

$$a_7 = \epsilon a_0,\ a_6 = -\epsilon a_1,\ a_5 = \epsilon a_2,\ a_4 = -\epsilon a_3, \tag{2}$$

where $\epsilon = \pm 1$. If

$$a = a_0(1 + \epsilon e_7) + a_1(e_1 - \epsilon e_6) + a_2(e_2 + \epsilon e_5) + a_3 e_3 - \epsilon e_4)$$

and

$$a^* = b_0(1 - \epsilon e_7) + b_1(e_1 + \epsilon e_6) + b_2(e_2 - \epsilon e_5) + b_3(e_3 + \epsilon e_4).$$

then

$$aa^* = a^*a = 0.$$

Proof. The direct calculation shows that

$$\mathbb{J} = \mathbb{R}(e_0 + (-e_0)) + \mathbb{R}(e_1 + (-e_1)) + ... + \mathbb{R}(e_7 + (-e_7))$$

is really an ideal of $\mathbb{RQ}_3$. It is enough to prove that, for all $j, j, (e_i + (-e_i))e_j \in \mathbb{J}$. For example:

$$(e_1 + (-e_1))e_2 = e_1 e_2 + (-e_1)e_2 = e_4 + (-e_4) \in \mathbb{J}.$$

The next part of the lemma follows immediately from the list of the properties of the 3-generated quaternion group $\mathbb{Q}_3$ and from the general theory of the (group) algebras over a field [9]. Especially, from the properties of the elements of the group $\mathbb{Q}_3$ and from the fact that the conjugacy class K_a of every element $a \in \mathbb{Q}_3 \setminus \mathbb{C}$ is two-element as $K_a = \{\pm a\}$, it follows that as a basis of the algebra $\mathbb{H}_3$ the set of the representatives of all conjugacy classes plus the representative of the sets $\{\pm 1\}$, $\{\pm a_1 a_2 a_3\}$ can be considered. Again from ([9], Theorem 2.1) it is known that any element of an algebra over a field is either invertible or a divisor of zero.

Further, let $a = \sum_{i=0}^{i=7} a_i e_i$ and $x = \sum_{i=0}^{i=7} x_i e_i$ be two non-zero elements of the algebra $\mathbb{H}_3$ such that $ax = 0$. Then the coordinates $\mathbb{R}_i$ of the product $ax = \sum_{i=0}^{i=7} \mathbb{R}_i e_i$ form yield a homogeneous system of equations in the unknowns $x_o, \ldots, x_7$ with matrix $\mathbb{D}$:

$$\mathbb{D}(a_0, \ldots, a_7) = \mathbb{D} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & a_7 \\ a_1 & a_0 & a_4 & -a_5 & a_2 & a_3 & -a_7 & -a_6 \\ a_2 & a_4 & a_0 & -a_6 & -a_1 & a_7 & a_3 & a_5 \\ a_3 & a_5 & a_6 & a_0 & -a_7 & -a_1 & -a_2 & -a_4 \\ a_4 & -a_2 & a_1 & -a_7 & a_0 & -a_6 & a_5 & -a_3 \\ a_5 & -a_3 & a_7 & a_1 & a_6 & a_0 & -a_4 & a_2 \\ a_6 & -a_7 & -a_3 & a_2 & -a_5 & a_4 & a_0 & -a_1 \\ a_7 & a_6 & -a_5 & a_4 & a_3 & -a_2 & a_1 & a_0 \end{pmatrix}.$$

Its j-th column $(c_0, \ldots, c_7)$, containing the coefficients of the unknown x_j, is determined according to the identity

$$(\sum_{i=0}^{i=7} a_i e_i) e_j = \sum_{k=0}^{k=7} c_k e_k.$$

The matrix $\mathbb{D}$ for $a_i = 1$ $(i = 0, \ldots, 7)$ is a Hadamard matrix [8].

The computation of its determinant $|\mathbb{D}|$ could be carried out, for instance, by calculating at first $|\mathbb{D}|^2 = |\mathbb{D}\mathbb{D}'|$ and then by expanding the received determinant on the minors of 2, 3, 4 and 5 columns. The basic role in this calculations play the terms:

$$s = \sum_{i=0}^{i=7} a_i^2 \text{ and } t = 2(-a_0 a_7 + a_1 a_6 - a_2 a_5 + a_3 a_4). \tag{3}$$

Finally, we find that

$$\begin{aligned} |\mathbb{D}| = {} & (s^2 - t^2)^2 = \\ = {} & [(a_0 - a_7)^2 + (a_1 + a_6)^2 + (a_2 - a_5)^2 + (a_3 + a_4)^2]^2 \times \\ & [(a_0 + a_7)^2 + (a_1 - a_6)^2 + (a_2 + a_5)^2 + (a_3 - a_4)^2]^2. \end{aligned} \tag{4}$$

Consequently, the system $ax = 0$ has a nonzero solution iff the determinant $|\mathbb{D}| = 0$. This condition is true iff the conditions (2) are true.

We state: An element $a = \sum_{i=0}^{i=7} a_i e_i \in \mathbb{H}_3$ is invertible iff $|\mathbb{D}|| \neq 0$, and it is a divisor of zero iff $|\mathbb{D}| = 0$.

A direct calculation shows that in this case every non zero element $a \in \mathbb{H}_3$ of the form $(\epsilon = \pm 1)$

$$\begin{aligned} a & = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + \epsilon(-a_3 e_4 + a_2 e_5 - a_1 e_5 + a_0 e_7) \\ & = a_0(1 + \epsilon e_7) + a_1(e_1 - \epsilon e_6) + a_2(e_2 + \epsilon e_5) + a_3(e_3 - \epsilon e_4) \end{aligned}$$

is a divisor of zero and that

$$aa^* = a^*a = 0,$$

where

$$\begin{aligned} a^* &= a_0 + a_1e_1 + a_2e_2 + a_3e_3 - \epsilon(-a_3e_4 + a_2e_5 - a_1e_5 + a_0e_7) \\ &= a_0(1 - \epsilon e_7) + a_1(e_1 + \epsilon e_6) + a_2(e_2 - \epsilon e_5) + a_3(e_3 + \epsilon e_4). \end{aligned}$$

This completes the proof. ■

By Frobenius's theorem ([9], Theorem 6.1; [1]), the only finite-dimensional algebras over the field of all real numbers $\mathbb{R}$ are the following: the field of all real numbers, the field of all complex numbers, and the algebra of real quaternions. Here, in connection with this theorem, the structure of the algebra $\mathbb{H}_3$ is investigated. The next propositions hold.

Let us put

$$\mathbb{I}^- = \{q^- = a_0(e_0 - e_7) + a_1(e_1 + e_6) + a_2(e_2 - e_5) + a_3(e_3 + e_4)\}, \leq qno(5)$$

$$\mathbb{I}^+ = \{q^+ = b_0(e_0 + e_7) + b_1(e_1 - e_6) + b_2(e_2 + e_5) + b_3(e_3 - e_4)\}, \leq qno(6)$$

where a_i, $b_j \in \mathbb{R}$. $i, j = 0, 1, 2, 3$.

Theorem 1. *The following holds:*

1) $\mathbb{I}^-$ and $\mathbb{I}^+$ are ideals of the algebra $\mathbb{H}_3$, and $\mathbb{H}_3$ is a direct sum of $\mathbb{I}^-$ and $\mathbb{I}^+$.

2) $\mathbb{I}^-$ and $\mathbb{I}^+$ are the only proper ideals of the algebra $\mathbb{H}_3$.

3) $\mathbb{I}^- \cong \mathbb{H} \cong \mathbb{I}^+$.

Proof. 1) As *it* is known (Lemma 1.), the quaternions q^- and q^+, forming the sets $\mathbb{I}^-$ and $\mathbb{I}^+$, exhaust all divisors of zero of the algebra $\mathbb{H}_3$. Also, we have $\mathbb{I}^- \neq 0$, $\mathbb{I}^+ \neq 0$ and

$$q^-q^+ = q^+q^- = 0. \tag{7}$$

Evidently, the sets $\mathbb{I}^-$ and $\mathbb{I}^+$ are closed under scalar multiplication and they are abelian groups under addition in $\mathbb{H}_3$. We shall prove, that these two subsets are ideals of $\mathbb{H}_3$.

For this aim, let us put

$$2f_i = \begin{cases} e_i - e_{7-i}, & if \quad i = 0, 2, \\ e_i + e_{7-i}, & if \quad i = 1, 3, 5, 7 \\ e_{i-7} - e_i, & if \quad i = 4, 5. \end{cases} \tag{8}$$

So, in matrix notation we get

$$(9) \qquad \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{pmatrix} = (2^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}) \times \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \end{pmatrix}.$$

In other words, for the vectors

$$f = (f_0, ..., f_7) \text{ and } e = (e_0, ..., e_7)$$

holds

$$(10) \qquad f = e(2^{-1}T'),$$

A direct calculation shows that

$$(11) \qquad TT' = T'T = 2E, \quad \text{i.e. } |T| \neq 0 \text{ is.}$$

Since the vectors e_i, $i = 0, 1, ..., 7$ are linearly independent, linearly independent are the vectors $f_0, ..., f_7$ too. Also, the vectors $f_0, ..., f_7$ form a basis of the space $\mathbb{H}_3$. Moreover, there holds the following multiplication

Table 2

\ f_0 \	f_1	f_2	f_3
f_1	$-f_0$	f_3	$-f_2$
f_2	$-f_3$	$-f_0$	f_1
f_3	f_2	$-f_1$	$-f_0$

\ f_7 \	f_6	f_5	f_4
f_6	$-f_7$	$-f_4$	f_5
f_5	f_4	$-f_7$	$-f_6$
f_4	$-f_5$	f_6	$-f_7$

and

$$f_i f_j = f_j f_i = 0 \text{ for } i \in \{0, ..., 3\} \text{ and } j \in \{4, ..., 7\}.$$

Further, from the identities (5) and (6), according to the identities (8), we get

$$\mathbb{I}^- = \{q^- = a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3\}, \tag{12}$$

$$\mathbb{I}^+ = \{q^+ = b_0 f_7 + b_1 f_6 + b_2 f_5 + b_3 f_4\}. \tag{13}$$

The last identities, according to the Table 2, show that the vectors $f_0, ...$ $,f_3$ $(f_4, ..., f_7)$ form a basis of the space $\mathbb{I}^-(I^+)$.

Table 2 shows that the products $f_i q^-$, $q^- f_i$, $i = 0, ..., 7$ are quaternions of the type (12), too. Therefore, for every quaternion $q \in \mathbb{H}_3$ the products qq^- and $q^- q$ will be again of the type (12). Also

$$\mathbb{H}_3 \mathbb{I}^- \subseteq \mathbb{I}^- \text{ and } \mathbb{I}^- \mathbb{H}_3 \subseteq \mathbb{I}^-,$$

i.e. the set $\mathbb{I}^-$ is an ideal of $\mathbb{H}_3$.

By analogy, the following two inclusions hold as well

$$\mathbb{H}_3 \mathbb{I}^+ \subseteq \mathbb{I}^+ \text{ and } \mathbb{I}^+ \mathbb{H}_3 \subseteq \mathbb{I}^+,$$

showing that the set $\mathbb{I}^+$ is an ideal of $\mathbb{H}_3$ too.

It remains to prove that $\mathbb{I}^- + \mathbb{I}^+ = \mathbb{H}_3$. For every quaternion q contained in the intersection of the ideals $\mathbb{I}^-$ and $\mathbb{I}^+$ we have

$$\begin{aligned} q &= q^- = a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 \\ &= q^+ = b_0 f_7 + b_1 f_6 + b_2 f_5 + b_3 f_4. \end{aligned}$$

The last equalities imply

$$a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 - (b_0 f_7 + b_1 f_6 + b_2 f_5 + b_3 f_4) = 0.$$

Since the vectors f_i, $i = 0, ..., 7$, are linearly independent it follows that

$$a_0 = ... = a_3 = b_0 = ... = b_3 = 0.$$

Hence, the zero is the unique element of the intersection $\mathbb{I}^- \cap \mathbb{I}^+$ of the ideals $\mathbb{I}^-$ and $\mathbb{I}^+$.

Obviously, $\mathbb{I}^- + \mathbb{I}^+ \subseteq \mathbb{H}_3$. At the same time the vectors $f_0, ..., f_7$ form a basis of the space $\mathbb{H}_3$ and consequently for every $q \in \mathbb{H}_3$ we have

$$\sum_{j=0}^{j=7} a_j f_j = \sum_{j=0}^{j=3} a_j f_j + \sum_{j=4}^{j=7} a_j f_j \in \mathbb{I}^- + \mathbb{I}^+,$$

since the vectors $f_0, \ldots, f_3$ $(f_4, \ldots, f_7)$ form a basis of the space $\mathbb{I}^-(\mathbb{I}^+)$ too.

2) Let us assume that $\mathbb{I}$ is a non-trivial ideal of the algebra $\mathbb{H}_3$ such that $\mathbb{I} \subseteq \mathbb{I}^-$. Since every element $a \subseteq \mathbb{I} \setminus \{0\}$ is invertible in $\mathbb{I}^-$ it follows that for every element $b \in \mathbb{I}^-$ the equation $ax = b$ has a solution in $\mathbb{I}^-$. Hence, $\mathbb{I}^- \subseteq \mathbb{I}$. Also, $\mathbb{I} = \mathbb{I}^-$.

By analogy can be shown that the non-trivial ideal $\mathbb{I}^+$ is a minimal ideal of the algebra $\mathbb{H}_3$.

Further, let $\mathbb{J}$ be an arbitrary minimal non-trivial ideal of the algebra $\mathbb{H}_3$ and let put $\mathbb{J}^- = \mathbb{J} \cap \mathbb{I}$ and $\mathbb{J}^+ = \mathbb{J} \cap \mathbb{I}^+$. Since the sets $\mathbb{I}^-$ and $\mathbb{I}^+$ are minimal ideals, then (according to the proof of condition 1) of the theorem) either $\mathbb{J}^+ = \{0\}$ and consequently $\mathbb{J} = \mathbb{J}^- = \mathbb{I}^-$ or $\mathbb{J}^- = \{0\}$ and consequently $\mathbb{J} = \mathbb{J}^+ = \mathbb{I}^+$.

Also, the algebra $\mathbb{H}_3$ has no other non-trivial minimal ideals besides the indicated $\mathbb{I}^-$ and $\mathbb{I}^+$.

Thereto, the algebra $\mathbb{H}_3$ has no other non-trivial maximal ideals besides the indicated $\mathbb{I}^-$ and $\mathbb{I}^+$. Indeed, if we suppose that $\mathbb{I} \subset \mathbb{K}$ and $\mathbb{I} \neq \mathbb{K}$ for some non-trivial ideal $\mathbb{K}$ of the algebra $\mathbb{H}_3$, then according to the above condition 1) of the theorem it follows that $\mathbb{K}^+ = \mathbb{K} \cap \mathbb{I}^+ \neq 0$ and consequently $\mathbb{K}^+ = \mathbb{I}^+$. Hence $\mathbb{K} = \mathbb{H}_3$.

3) Each of the ideals $\mathbb{I}^-$ and $\mathbb{I}^+$ is a 4-space and as such it is isomorphic to the space $\mathbb{H} =< 1, i, j, k >$. It is easy to see that the mapping

$$\varphi\left(\sum_{i=0}^{i=3} c_i f_i\right) = c_0 + c_1 i + c_2 j + c_3 k$$

is an isomorphism of the algebra $\mathbb{I}^-$ onto the algebra $\mathbb{H}$.

In particular, testing whether the mapping φ is an isomorphism of the ring $\mathbb{I}^-$ onto the ring $\mathbb{H}$, it is sufficient to compare the products in the multiplication table of the basis vectors of the algebra $\mathbb{H}$ with Table 2.

This proves the Theorem. ■

Corollary 1. *The quaternions* $0, 1 = e_0$, f_0 *and* f_7 *are the uniquely idempotent of the algebra* $\mathbb{H}_3$.

Proof. Let $a = \sum_{i=0}^{i=7} c_i f_i$ be an idempotent of the algebra $\mathbb{H}_3$. Then, for its coordinates Table 2 implies

$$c_0^2 - \sum_{i=1}^{i=3} c_i^2 = c_0, \quad 2c_0 c_i = c_i, \quad i = 1, 2, 3; \tag{14}$$

$$c_7^2 - \sum_{i=4}^{i=6} c_i^2 = c_7, \quad 2c_7 c_i = c_i, \quad i = 4, 5, 6; \tag{15}$$

Further, the system (14) implies

$$c_0(c_0 - 1) = \sum_{i=1}^{i=3} c_2^2, \quad (2c_0 - 1)c_i = 0, \quad i = 1, 2, 3.$$

The first equation is possible only for $c_0 \leq 0$ or for $c_0 \geq 1$. The quadruples

$$g_0 = (0,0,0,0) \text{ and } h_0 = (1,0,0,0)$$

are evidently solutions of the system. Next, by $c_0 < 0$ the equations $(2c_0 - 1)c_i = 0$, $i = 1,2,3$ imply $c_i = 0$. But by $c_0 < 0$ and $c_i = 0$ the identity $c_0(c_0 - 1) = \sum_{i=1}^{i=3} c_i^2$ is incompatible.

In the same way we can say that the assumption $c_0 > 1$ is contradictory too.

Hence, the quadruples $(0,0,0,0)$ and $(1,0,0,0)$ are the unique solutions of the system (14).

Analogously it is proved that the quadruples

$$g_1 = (0,0,0,0) \text{ and } h = (0,0,0,1)$$

are the unique solutions of the system (15).

Also, the obtained results show that the generation q is an idempotent iff

either $q = g_0 + g_1 = 0$, i.e. when $c_i = 0$ for $i = 0, ..., 7$,
or $q = h_0 + g_1 = f_0$, i.e. when $c_0 = 1$ and $c_i = 0$ for $i = 0, ..., 7$,
or $q = g_0 + h_1 = f_7$, i.e. when $c_7 = 1$ and $c_i = 0$ for $i = 0, ..., 6$,
or $q = h_0 + h_1 = e_0$, i.e. when $c_0 = c_7 = 1$ and $c_i = 0$ for $i = 1, ..., 6$. ■

Using Corollary 1 and the above quoted results of Theorem 1 we get the following

Corollary 2. *The quaternions f_0 and f_7 are the uniquely non-zero idempotent of the algebra $\mathbb{I}^+$ and $\mathbb{I}^-$ correspondingly. There hold $\mathbb{I}^+ = \mathbb{H}_3 f_0$ and $\mathbb{I}^- = \mathbb{H}_3 f_7$.*

Let f be any isomorphism of $\mathbb{I}^-$ onto $\mathbb{I}^+$ (remind that $\mathbb{I}^- \cong \mathbb{I}^+ \cong \mathbb{H}$) and let g be any automorphism of $\mathbb{I}^+$. Denote by $\mathbf{H(g)}$ the set of all elements of $\mathbb{H}_3$ of the form

$$x + g(f(x)), \quad x \in \mathbb{I}^-. \tag{16}$$

Theorem 2. *The following holds:*

1) $\mathbf{H}(\mathbf{g})$ *is a subalgebra of* $\mathbb{H}_3$ *isomorphic of* $\mathbb{H}$*, and for any* g*,* $\mathbf{H}(\mathbf{g})$ *is different from* $\mathbb{I}^-$*,* $\mathbb{I}^+$*. If* g*,* h *are different isomorphisms of* $\mathbb{I}^+$*, then* $\mathbf{H}(\mathbf{g}) \neq \mathbf{H}(\mathbf{h})$*. In particular, there exists infinitely many subalgebras of* $\mathbb{H}_3$ *isomorphic to* $\mathbb{H}$*.*

2) If $\mathbb{A}$ *is a subalgebra of* $\mathbb{H}_3$ *such that* $\mathbb{A} \cong \mathbb{H}$*,* $\mathbb{A} \neq \mathbb{I}^-$*,* $A \neq \mathbb{I}^+$*, then* $\mathbb{A} \cong \mathbf{H}(\mathbf{g})$ *for some* g*.*

Proof. Direct calculation shows that $\mathbf{H}(\mathbf{g})$ is a subalgebra of $\mathbb{H}_3$. Next consider $\mathbf{H}(\mathbf{g})$, and define a map φ $\mathbf{H}(\mathbf{g}) \to \mathbb{I}^-$ by the identity $\varphi(x+g(f(x))) = x$.

It is clear that φ is a homomorphism. If

$$y = x + g(f(x)),\ x \in \mathbb{I}^-,$$

then $\varphi(y) = 0$ implies $x = 0$. But then $f(x) = 0$, therefore $g(f(x)) = 0$, hence $y = 0$. Thus $ker\varphi = 0$. By definition φ is onto, hence φ is isomorphism. Since $\mathbb{I}^- \cong \mathbb{H}$, we obtain $\mathbf{H}(\mathbf{g}) \cong \mathbb{H}$.

Consider $\mathbf{H}(\mathbf{g})$, $\mathbf{H}(\mathbf{h})$ with $g \neq h$. Suppose $\mathbf{H}(\mathbf{g}) = \mathbf{H}(\mathbf{h})$. Take

$$y = x + g(f(x)) \in \mathbf{H}(\mathbf{g}),$$

then there exists $x_1 \in \mathbb{I}^-$ such that

$$z = x_1 + h(f(x_1)) = y.$$

Thus

$$x + g(f(x)) = x_1 + h(f(x_1)), \tag{17}$$

where

$$x, x_1 \in \mathbb{I}^-,\ g(f(x)),\ h(f(x_1)) \in \mathbb{I}^+.$$

Since $\mathbb{H}_3 = \mathbb{I}^- + \mathbb{I}^+$, any element of $\mathbb{H}_3$ is uniquely presented in the form $a + b$ with $a \in \mathbb{I}^-$, $b \in \mathbb{I}^+$. Therefore (17) implies

$$x = x_1, \quad g(f(x)) = h(f(x)).$$

Since $\mathbb{H}(g) = \mathbb{H}(h)$, same is true for arbitrary $y = x + g(f(x))$. That means: $g(f(x)) = h(f(x))$ for all $x \in \mathbb{I}^-$ (since, for all $x \in \mathbb{I}^-$, there exists $y = x + g(f(x)) \in \mathbb{H}(g)$). But $\{f(x) | x \in \mathbb{I}^-\} = \mathbb{I}^+$. Hence $g(t) = h(t)$ for all $t \in \mathbb{I}^+$, and $g = h$. Thus $\mathbb{H}(g) = \mathbb{H}(h)$ implies $g = h$.

Since, for any $0 \neq y \in \mathbb{H}(g)$,

$$y = x + g(f(x)) \text{ with } x \neq 0, \quad g(f(x)) \neq 0,$$

we have

$$\mathbb{H}(g) \cap \mathbb{I}^- = \{0\}, \quad \mathbb{H}(g) \cap \mathbb{I}^+ = \{0\}.$$

Also

$$\mathbb{H}(g) \neq \mathbb{I}^-, \quad \mathbb{H}(g) \neq \mathbb{I}^+ \text{ for any } g.$$

2) Take $\mathbf{A}$ a subalgebra of $\mathbb{H}_3$, such that

$$\mathbf{A} \cong \mathbb{H}, \quad \mathbf{A} \neq \mathbb{I}^-, \quad \mathbf{A} \neq I^+.$$

Let

$$y \in \mathbf{A}, \quad y \neq 0, \quad y = x + z \text{ with } x \in I^-, \quad z \in \mathbb{I}^+.$$

Consider projection homomorphism of $\mathbf{A}$:

$$\phi : y = x + z \to x,$$
$$\lambda : y = x + z \to z, \quad x \in \mathbb{I}^-, \ z \in \mathbb{I}^+.$$

Since ϕ maps $\mathbf{A}$ into $\mathbb{I}^- \cong \mathbb{H}$ then either $ker\phi = 0$ or $ker\phi = \mathbf{A}$. If $ker\phi = \mathbf{A}$ then, for any

$$y = x + z \in \mathbf{A}, \quad x \in \mathbb{I}^-, \quad z \in \mathbb{I}^+$$

we have $x = 0$ and $y = z$; therefore $\mathbf{A} = \mathbb{I}^+$, a contradiction. If $ker\phi = 0$ then ϕ is an isomorphism (remind that $\mathbb{I}^- \cong \mathbb{H}$).

Same is true for $\lambda : ker\lambda = 0$, and $\lambda : \mathbf{A} \to \mathbb{I}^+$ is an isomorphism.

Thus for any

$$y = x + z \neq 0, \quad y \in \mathbf{A}, \quad x \in \mathbb{I}^-, \quad z \in \mathbb{I}^+$$

we have $x \neq 0$, $z \neq 0$. Let f be as in theorem. The map

$$\mu : x \to z$$

for any $y = x + z$ as above is an isomorphism of $\mathbb{I}^-$ onto $\mathbb{I}^+$. Thus we have two isomorphisms

$$f : \mathbb{I}^- \to \mathbb{I}^+, \quad \mu : \mathbb{I}^- \to \mathbb{I}^+.$$

Clearly $g = \mu f^{-1}$ is an automorphism of $\mathbb{I}^+$, and $\mu = gf$. Thus any element of $\mathbf{A}$ is of the form

$$y = x + z = x + \mu(x) = x + g(f(x)),$$

and $\mathbf{A} = \mathbb{H}(h)$.

This proves the theorem. ■

The analogous theorem may be proved about subfield of $\mathbb{H}_3$ isomorphic to the field $\mathbb{C}$ of all complex numbers.

Theorem 3. *Let be* $\mathbb{A}$ *an arbitrary algebra such that* $\mathbb{A} = \mathbb{I}_1 + \mathbb{I}_2$, *where:*

1) $\mathbb{I}_1$ *and* $\mathbb{I}_2$ *are proper ideals of the algebra* $\mathbb{A}$*;*

2) $\mathbb{I}_1 \cap \mathbb{I}_2 = \{0\}$;

3) $\mathbb{I}_1 \cong \mathbb{H} \cong \mathbb{I}_2$, *where* $\mathbb{H}$ *is the quaternion algebra.*

Then $\mathbb{A}$ *is an algebra isomorphic to* $\mathbb{H}_3$.

Proof. Evidently, $\mathbb{I}_1\mathbb{I}_2 \subseteq \mathbb{I}_2 \cap \mathbb{I}_2 = \{0\}$, i.e. $\mathbb{I}_1\mathbb{I}_2 = \{0\}$. By analogy, as well $\mathbb{I}_2\mathbb{I}_1 = \{0\}$. So we have $ab = ba = 0$ for every two elements $a \in \mathbb{I}^-$ and $b \in \mathbb{I}^+$. Further, let us assume according to property 3) that for the ideals

$$\mathbb{I}_1 =< f_o, ..., f_3 > \text{ and } \mathbb{I}_2 =< f_4, ..., f_7 >$$

holds the multiplication Table 2. Then we put (see (8)-(11))

$$e_i = \begin{cases} f_i - f_{7-i}, & if \quad i = 5, 7, \\ f_i + f_{7-i}, & if \quad i = 0, 1, 2, 3, \\ f_{7-i} - f_i, & if \quad i = 4, 6. \end{cases}$$

Since the vectors f_i, $i = 0, ..., 7$, are linearly independent, linearly independent are also the vectors $e_0, ..., e_7$. So, the vectors $e_0, ..., e_7$ form a basis of the space $\mathbb{A}$. The multiplication of the elements e_i, $i = 0, 1, ..., 7$ is determined according to Table 1. ■

Let us denote that the group $\mathbb{G}(\mathbb{A}) = \{\pm e_0, ..., \pm e_7\}$ is not two-generated. Indeed, from the table 1 we have:

$$< e_0, e_i >= \{\pm e_0, \pm e_i\} \neq \mathbb{G}(\mathbb{A}) \text{ for } i = 1, ..., 7$$

and

$$< e_i, e_j >= \{\pm e_0, \pm e_i, \pm e_j, \pm e_i e_j\} \neq \mathbb{G}(\mathbb{A}) \text{ for } i \neq j, \ i, j \in \{1, ..., 7\}.$$

Let $\mathbf{T} = \{\pm \mathbf{e_0}, \pm \mathbf{e_1}, ..., \pm \mathbf{e_7}\}$ and let the multiplication of the elements of the set $\mathbf{T}$ is given by Table 1. Then $\mathbf{T}$ is be a group isomorphic to $\mathbb{Q}_3$. In this case $\mathbb{A} \cong \mathbb{R}T/\mathbb{J}$ (See Lemma 1).

Corollary 3. *The above defined group* $\mathbb{T}$ *is a three generated group* $< a_1, a_2, a_3 >$ *subject only to the relations* $a_i = a_j a_i a_j$ *for* $i \neq j$, $i, j \in \{1, 2, 3\}$.

The proof of the corollary follows immediately from the proof of Theorem 3.

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