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Homogenization of Integral Functionals with Extreme Local Properties

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We consider homogenization of sequences of integral functionals, where the integrands are large on a subset $V \subset \mathbf{R}^n$. Our results are used to study and compare new and classical non-linear bounds for the corresponding limit-functional. Moreover, we compare the macroscopic behavior of the chessboard type structure with that of disperse structures.

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Introduction

Many problems in the homogenization theory are devoted to the study of the asymptotic behavior as h goes to ∞ of integral functionals of the form

$$\mathcal{F}_h(u) = \int_\Omega f(hx, Du(x))\, dx, \ \Omega \subset \mathbf{R}^n$$

defined on some (subset of a) Sobolev space $\mathrm{H}^{1,p}(\Omega)$, where the function f is periodic in the first variable and satisfies natural growth conditions of order p in the second variable. Such functionals appear naturally in connection with boundary value problems described by some minimum energy principle of the form

$$E_h = \min\left\{\mathcal{F}_h(u) + \int_\Omega gu dx : u \in \mathrm{H}^{1,p}(\Omega),\ \mathrm{u} = \phi \text{ on } \gamma_0 \subset \partial\Omega\right\}.$$

In several important cases it happens that the "energy" E_h converges (as h goes to $+\infty$) to

$$E_{hom} = \min\left\{\mathcal{F}_{hom}(u) + \int_\Omega gu dx : u \in \mathrm{H}^{1,p}(\Omega),\ \mathrm{u} = \phi \text{ on } \gamma_0\right\},$$

where $\mathcal{F}_{hom}$ is of the form

$$\mathcal{F}_{hom}(u) = \int_{\Omega} f_{hom}(Du(x))\, dx.$$

Hence, E_{hom} approximates E_h for cases where the characteristic length of the oscillations is small compared with the size of Ω. Concerning this fact and other basic information in homogenization theory and some of its applications we refer to the literature e.g. the books [3, 4, 14] and the paper [12].

The nonlinear extension of the classical Hashin-Shtrikman bounds yield important information of the range of effective properties achievable through homogenization of structures with predescribed volume-fractions. Explicit formulae for these bounds are not available for nonlinear problems.

In this paper we consider cases where the integrand $f(\cdot,\xi)$ is relatively large on a subset $V \subset \mathbf{R}^n$ and derive estimates for the nonlinear bounds of Wiener and Hashin-Shtrikman type. For such problems these bounds are often far to wide to give accurate information of f_{hom}. Therefore, we also study some sharp new nonlinear bounds which are particularly useful in cases where V or $\mathbf{R}^n \backslash \mathbf{V}$ is a disperse set. An other type of bounds is proved for chessboard type structures (for which neither V nor $\mathbf{R}^n \backslash \mathbf{V}$ is disperse). These bounds show that the macroscopic behavior corresponding to the chessboard type structure is similar to that of structures where V is disperse if $p < n$ and that of structures where $\mathbf{R}^n \backslash \mathbf{V}$ is disperse if $p \geq n$.

The paper is organized as follows. In Section 1 we have collected some necessary preliminaries and notations. In Section 2 we state and prove a few results connected to some classical and new nonlinear bounds. In Section 3 we consider in more detail the special case when $f(x,\cdot)$ grows quadratically and we present some numerical experiments which illustrate our theoretical results. The bounds connected to the structure of chessboard type are presented in Section 4.

1. Preliminaries

Let us start with some notations. If Y is a cube in $\mathbf{R}^n$ we consider the usual $L^p(Y)$, $[L^p(Y)]^n$ spaces of measurable functions defined on Y with values in $\mathbf{R}$ and $\mathbf{R}^n$, respectively, and the usual Sobolev space $H^{1,p}(Y)$ of real measurable functions defined on Y. The real number $p \in]1,+\infty[$ is fixed and we let p' be the number such that $1/p + 1/p' = 1$. Moreover, we denote with $\mathcal{C}^{\infty}_{per}(\mathcal{Y})$ the space of $\mathcal{C}^{\infty}$ Y-periodic functions, and with $H^{1,p}_{per}(Y)$ the closure of $\mathcal{C}^{\infty}_{per}(\mathcal{Y})$ in $H^{1,p}(Y)$. With $|E|$ we denote the Lebesgue measure of a measurable set E. Let $f : \mathbf{R}^n \times \mathbf{R}^n \to [0,+\infty[$ have the following properties:

- $f(\cdot,\xi)$ is a measurable function for all $\xi \in \mathbf{R}^n$;
- $f(x,\cdot)$ is a convex function for all $x \in \mathbf{R}^n$;
- the function $f(\cdot,\xi)$ is Y-periodic, i.e. we have that

$$f(x + |Y|^{\frac{1}{n}} e_h, \xi) = f(x,\xi)$$

  for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$ and $h = 1,\ldots,n$ ($e_1,\ldots,e_n$ is the canonical basis of $\mathbf{R}^n$);
- f satisfies natural growth conditions of power p, i.e. there exists a constant C such that

$$|\xi|^p \le f(x,\xi) \le C(1 + |\xi|^p)$$

  for all $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$.

We recall the minimal energy principle

$$f_{hom}(\xi) = \min_{u \in \mathrm{H}^{1,p}_{per}(\mathrm{Y})} \frac{1}{|Y|} \int_Y f(x, \xi + Du(x))\, dx,\ \xi \in \mathbf{R}^n,$$

where f_{hom} is the homogenized integrand corresponding to the functions f_h, $h = 1,2,3,\ldots$, defined by $f_h(x,\xi) = f(hx,\xi)$. We also recall the minimal principle of the complementary energy:

$$f^*_{hom}(\eta) = \min_{v \in \mathrm{V}^{p'}_{sol}(\mathrm{Y})} \frac{1}{|Y|} \int_Y f^*(x, \eta + v)\, dx,\ \xi \in \mathbf{R}^n. \tag{1}$$

Here,

$$\mathrm{V}^{p'}_{sol}(\mathrm{Y}) = \left\{ \mathrm{v} \in \left[\mathrm{L}^{p'}(\mathrm{Y})\right]^n : \int_{\mathrm{Y}} \mathrm{v}\, \mathrm{dx} = 0 \text{ and } \int_{\mathrm{Y}} \mathrm{v} \cdot \mathrm{D}\varphi\, \mathrm{dx} = 0\ \forall \varphi \in \mathrm{H}^{1,p}_{per}(\mathrm{Y}) \right\},$$

and f^* is the Legendre-Fenchel dual (convex polar) of f defined by

$$f^*(x,\eta) = \sup_{E \in \mathbf{R}^n} \{E \cdot \eta - f(x,E)\}.$$

2. Nonlinear bounds

In this section we present and discuss a few results connected to some classical and new nonlinear bounds.

Let a be a m-tuple of the form $a = (a_1, ..., a_m)$ such that

$$0 < a_i < 1 \text{ and } \sum_{i=1}^{m} a_i = 1.$$

Similarly, let $g = (g_1, ..., g_m)$, where g_i is a convex function $g_i : \mathbf{R}^n - [0, +\infty[$

$$\lambda_i^- |\xi|^p \leq g_i(\xi) \leq \lambda_i^+ (|\xi|^p + l), \ l \geq 0, \ g_i(0) = 0,$$

and $\lambda_i^-, \lambda_i^+ > 0$. We let $\Im_{a,g}$ denote the family of functions f of the form

$$f(x, \xi) = \sum_{i=1}^{m} g_i(\xi) \chi_{A_i}(x),$$

where $\cup A_i = \mathbf{R}^n$, $A_i \cap A_j = \emptyset$ for $i \neq j$, and such that the characteristic function of the set A_i, χ_{A_i} is Y-periodic and $a_i = \int_Y \chi_{A_i} dx$. We recall the well known nonlinear Wiener bounds:

$$f_{W-}(\xi) \leq f_{hom}(\xi) \leq f_{W+}(\xi),$$

where

$$f^*_{W-}(\xi) = \int_Y f^*(x, \xi) dx \text{and} \ \ f_{W+}(\xi) = \int_Y f(x, \xi) dx.$$

In the case when $f(\cdot, \xi)$ satisfies the property of cubic symmetry, we have the following sharper nonlinear bounds of Hashin-Shtrikman type (denoted the HS-bounds):

$$f_{HS-}(\xi) \leq f_{hom}(\xi) \leq f_{HS+}(\xi),$$

where

$$f_{HS-}(\xi) = \inf \left\{ h_{hom}(\xi) \ : \ h \in \Im^{\text{cub}}_{a,g} \right\},$$

$$f_{HS+}(\xi) = \sup \left\{ h_{hom}(\xi) \ : \ h \in \Im^{\text{cub}}_{a,g} \right\},$$

$$\Im^{\text{cub}}_{a,g} = \{ h \in \Im_{a,g} : h(\cdot, \xi) \text{ satisfies the property of cubic symmetry } \}$$

We recall that $f(\cdot, \xi)$ satisfies the property of cubic symmetry iff

$$f(x, \xi) = f(\sigma \mathbf{x}, \xi),$$

where σ is the rotation by $\pi/2$ in the plane of coordinates x_i, x_j, $i \neq j$, $i, j = 1, ..., n$.

Remark 1. By the definition, the HS-bounds are best possible among the bounds that can be obtained for the class $\Im^{\text{cub}}_{a,g}$ without taking into account other geometrical properties than the volume-fractions $\{a_i\}$. For the simple case of linear two-phase problems it turns out that the HS-bounds coincide with the well known Hashin-Shtrikman bounds. Generally, explicit formulae for these bounds, in terms of $\{a_i\}$ and $\{g_i\}$, are not available.

Definition 2.1. We say that $V \subset \mathbf{R}^n$ is Y-periodic iff the characteristic function of V is Y-periodic.

Definition 2.2. Let $V \subset \mathbf{R}^n$ be a Y-periodic set. We say that V is a disperse set if is the union $\cup_{i=1}^{\infty}\overline{O}_i$ of mutually disjoint components $\overline{O}_i$, each being the closure of a smooth bounded domain O_i, and such that at most a finite collection of these components intersects Y.

In order to be able to make further study of the nonlinear Wiener and Hashin-Shtrikman bounds, we now present the following statement of independent interest.

Theorem 2.3. *Suppose that* $f \in \Im_{a,g}$ *is of the form*

$$f(x,\xi) = \sum_{i=1}^{m} g_i(\xi)\,\chi_{A_i}(x),$$

such that for a fixed $j \in \{1,...,m\}$ *we have that the closure of* $\mathbf{R}^n \backslash \mathbf{A_j}$ *is a disperse set. Then,*

$$\lambda_j^- c^- \, |\xi|^p \leq f_{hom}(\xi) \leq \lambda_j^+ c^+ \left(|\xi|^p + l\right),$$

where $0 < c^-, c^+ < +\infty$ *are only dependent on* A_j *and* p.

Proof. Let $\mathbf{R}^n \backslash \mathbf{A_j} = \cup_{i=1}^{\infty}\overline{\mathbf{O}}_i$ and let $\cup_{i=1}^{\infty}\overline{O}'_i$ be a Y-periodic union of mutually disjoint components $\overline{O}'_i$, each being the closure of a smooth bounded domain O'_i such that $\overline{O}_i \subset O'_i$. Let $u_i \in H^{1,s}_{\mathrm{per}}(\mathrm{Y})$, $s > 1$, be such that $u_i = 0$ on $\mathbf{R}^n \backslash \cup_{i=1}^{\infty}\overline{\mathbf{O}}'_i$ and $Du_i = 1$ on $\cup_{i=1}^{\infty}\overline{O}_i$ (the extension $u_i|_{\overline{O}_i} \to u_i|_{\overline{O}'_i}$ is possible according to [1, A. 5.12]).

$f_{hom}(\xi) \leq \lambda_j^+ c^+ \left(|\xi|^p + l\right)$: Put $s = p$. It follows that the function $u = -\sum_{i=1}^{n} \xi_i u_i$ has the property $Du + \xi = 0$ on $\mathbf{R}^n \backslash \mathbf{A_j}$, and hence $f(\cdot, Du + \xi) = 0$ on $\mathbf{R}^n \backslash \mathbf{A_j}$. Moreover, on A_j we have

$$|Du + \xi| = \left|\xi - \sum_{i=1}^{n} \xi_i Du_i\right| \leq$$

$$\leq |\xi| + \sum_{i=1}^{n} |\xi_i|\,|Du_i| \leq |\xi| \left(1 + \sum_{i=1}^{n} |Du_i|\right).$$

Therefore,

$$f(\cdot, Du+\xi) \leq \lambda_j^+ (|Du+\xi|^p + l) \leq \lambda_j^+ \left(|\xi|^p \left(1 + \sum_{i=1}^n |Du_i|\right)^p + l \right)$$

on A_j. These facts give

$$\begin{aligned} f_{hom}(\xi) &\leq \int_Y f(x, Du+\xi)\, dx = \int_{A_j} f(x, \xi + Du)\, dx \leq \\ &\leq \lambda_j^+ \int_{A_j} (|Du+\xi|^p + l)\, dx \leq \lambda_j^+ c^+ (|\xi|^p + l), \end{aligned}$$

where $c^+ = \int_{A_j} (1 + \sum_{i=1}^n |Du_i|)^p \, dx$, and the upper bound for f_{hom} is proved.

$\lambda_j^- c^- |\xi|^p \leq f_{hom}(\xi)$: Put $s = p'$. First, we note that if $k\,|\xi|^q \leq h(\xi)$, where $k > 0$ and $q > 1$, then

$$h^*(\xi) \leq \frac{1}{q'} (qk)^{\frac{1}{1-q}} |\xi|^{q'} . \tag{2}$$

Indeed,

$$\begin{aligned} h^*(\xi) &= \sup_{E \in \mathbf{R}^n} \{E \cdot \xi - h(E)\} \leq \sup_{E \in \mathbf{R}^n} \{E \cdot \xi - k\,|E|^q\} = \\ &= \sup_{|E| \in \mathbf{R}^n} \{|E| \cdot |\xi| - k\,|E|^q\} = \frac{1}{q'} (qk)^{\frac{1}{1-q}} |\xi|^{q'} . \end{aligned}$$

Hence, for each $x \in A_j$ we have that

$$f^*(x, \xi) \leq \frac{1}{p'} \left(p\lambda_k^-\right)^{\frac{1}{1-p}} |\xi|^{p'} .$$

For each $i \in \{1, ..., n\}$ we choose $k \neq i$ and define

$$v_i = \frac{\partial u_k}{\partial x_k} e_i - \frac{\partial u_k}{\partial x_i} e_k .$$

Thus $v_i \in \left[\mathrm{L}^{p'}(\mathrm{Y})\right]^n$, $\int_Y v_i \, dx = 0$ (by the periodicity of u_k) and

$$\int_Y v_i \cdot D\varphi \, dx = \int_Y \frac{\partial u_k}{\partial x_k} \frac{\partial \varphi}{\partial x_i} - \frac{\partial u_k}{\partial x_i} \frac{\partial \varphi}{\partial x_k} dx = 0$$

$\forall \varphi \in C^\infty_{\mathrm{per}}(\mathcal{Y})$ (the last identity being due to Green's Theorem), hence for any $\varphi \in \mathrm{H}^{1,p}_{\mathrm{per}}(\mathrm{Y})$. Thus $v_i \in \mathrm{V}^{p'}_{sol}(\mathrm{Y})$ and we also have that $v_i = e_i$ on $Y \backslash A_j$. For any vector $\xi \in \mathbf{R}^n$ we let $v = -\sum_{i=1}^{n} \xi_i v_i$. Continuing similarly as above, we find that

$$f^*_{hom}(\xi) \leq k_0 \, |\xi|^{p'},$$

where

$$k_0 = \frac{1}{p'} \left(p\lambda_j^-\right)^{\frac{1}{1-p}} \int_{A_j} \left(1 + \sum_{i=1}^{n} |Du_i|\right)^{p'} dx.$$

Thus, using (2.1) once more, we deduce that

$$f_{hom}(\xi) = f^{**}_{hom}(\xi) \geq \frac{1}{p} \, (p' k_0)^{\frac{1}{1-p'}} \, |\xi|^p = \lambda_k^- c^- \, |\xi|^p,$$

where

$$c^- = \left(\int_{A_j} \left(1 + \sum_{i=1}^{n} |Du_i|\right)^{\frac{p}{p-1}} dx \right)^{\frac{1}{1-p}}$$

and the lower bound is also proved. The proof is complete. ∎

Corollary 2.4. *For every $k \in \{1, ..., m\}$ it holds that*

$$f_{W_-}(\xi) \leq f_{HS_-}(\xi) \leq \lambda_k^+ c^+ \left(|\xi|^p + l\right)$$

and

$$\lambda_k^- c^- \, |\xi|^p \leq f_{HS_+}(\xi) \leq f_{W_+}(\xi)$$

for all $\xi \in \mathbf{R}^n$, where $0 < c^-, c^+ < +\infty$ are only dependent on p, n and a_k.

Proof. By letting $f \in \Im_{a,g}$, defined by

$$f(x, \xi) = \sum_{i=1}^{m} g_i(\xi) \, \chi_{A_i}(x),$$

be such that for a fixed $k \in \{1, ..., m\}$ $Y \backslash A_k$ is a cube, and the other sets $\{A_i\}_{i \neq k}$ are constructed such that $f(\cdot, \xi)$ satisfies the property of cubic symmetry, we get that

$$f_{W_-}(\xi) \leq f_{HS_-}(\xi) \leq f_{hom}(\xi) \leq f_{HS_+}(\xi) \leq f_{W_+}(\xi).$$

Hence, the result follows directly from Theorem 2.3. ∎

Remark 2. According to Corollary 2.4. we have that

$$\lim_{\lambda_k^+ \to 0} f_{W_-}(\xi) = \lim_{\lambda_k^+ \to 0} f_{HS_-}(\xi) = 0$$

and

$$\lim_{\lambda_k^- \to +\infty} f_{W+}(\xi) = \lim_{\lambda_k^- \to +\infty} f_{HS+}(\xi) = +\infty.$$

Hence, from Theorem 2.3 we conclude that the nonlinear bounds of Wiener and Hashin-Shtrikman type, $f_{W\pm}$ and $f_{HS\pm}$, are not well fitted for estimating f_{hom} in the case when $\mathbf{R}^n \backslash \mathbf{A_j}$ is a disperse set and λ_k^- is relatively large or λ_k^+ is relatively small for a fixed $k \neq j$. In such cases the bounds defined below turn out to be much more suitable.

We close this section by discussing some other type of bounds for the homogenized integrand f_{hom} for the special case when

$$f(x,\xi) = C(x,|\xi|)\,|\xi|^p\,,\ \xi \in \mathbf{R}^n. \tag{3}$$

Here, C is of the form $C(\cdot,t) = \sum_{i=1}^m \lambda_i(t)\chi_{A_i}$ and satisfies the property of cubic symmetry, i.e. $f \in \Im_{a,g}^{\text{cub}}$, where

$$\int_Y \chi_{A_i}\, dx = a_i \text{ and } g_i(\xi) = \lambda_i(|\xi|)\,|\xi|^p\,.$$

Moreover, the function $C : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}_+$ has the following additional properties: $\alpha \leq C(x,t) \leq \beta$ for all x and t for some constants $0 < \alpha \leq \beta$, $C(x,t)$ is $[0,1]^n$-periodic and Lebesgue measurable in x and differentiable and non-decreasing in t. Moreover, $C(x,t)t^p$ is convex in t and

$$\frac{d\,(C(x,t)t^p)}{dt} \leq p\beta t^{p-1} \tag{4}$$

for all x and t.

For every $t \geq 0$, let $c^+(t)$ and $c^-(t)$ be the numbers

$$c_p^+(t) = q_+^{r_1,\ r_2}(C(\cdot,t\left(\frac{\alpha}{\beta}\right)^{r_1})),$$

$$c_p^-(t) = q_-^{s_1,\ s_2}(C(\cdot,t\left(\frac{\alpha}{\beta}\right)^{\frac{1+s_1}{p-1}})).$$

Here,

$$r_1,\ r_2, s_1, s_2 = \begin{cases} -\frac{2}{p}, \frac{2}{p}, 1, \frac{1}{1-p} & \text{if } p \leq 2 \\ \frac{1}{1-p}, 1, \frac{2}{p}, -\frac{2}{p} & \text{if } 2 \leq p \end{cases}$$

and

$$\begin{aligned} q_-^{r,s}(\rho) &= \left(\tau_+^{[r]}(\left(\tau_-^{[s]}(\rho)\right))\right), \\ q_+^{r,s}(\rho) &= \left(\tau_-^{[r]}(\left(\tau_+^{[s]}(\rho)\right))\right), \end{aligned}$$

where

$$\tau_{-}^{[t]}(\rho) = \left(\int_0^1 \rho^t dx_1\right)^{\frac{1}{t}},$$
$$\tau_{+}^{[t]}(\rho) = \left(\int_0^1 \cdots \int_0^1 \rho^t \, dx_2 \cdots dx_n\right)^{\frac{1}{t}}.$$

We recall the following bounds (for the proof we refer to [7], see also [8, 9]):

$$|\xi|^p c_p^-(|\xi|) \leq f_{hom}(\xi) \leq |\xi|^p c_p^+(|\xi|) \tag{5}$$

for all $\xi \in \mathbf{R}^n$. The inequalities in 5 are sharp and equalities occur in many interesting cases (see [7, 8, 9])

Remark 3. By using the theory of power type means (see [2, 5, 11, 13]) it is possible to prove that $c_p^-(t)$ and $c_p^+(t)$ are continuous and non-decreasing in p and t.

Remark 4. Consider a fixed integer $j \in \{1, 2, ..., m\}$. If $Y \setminus A_j$ is a genuine subset of Y then

$$k^- \lambda_j^- \leq c_p^-(\cdot) \leq c_p^+(\cdot) \leq k^+ \lambda_j^+$$

where and $0 < k^-, k^+ < +\infty$. Moreover, k^- and $k^+$are only dependent of A_j and p. Hence, according to Corollary 2.4, we can in this case conclude that $c_p^-(\cdot)\,|\cdot|^p$ is a much sharper lower bound for f_{hom} than the Wiener and Hashin-Shtrikman lower bounds f_{W-} and f_{HS-} when λ_k^- is relatively large and $k \neq j$. Similarly, we see that $c_p^+(\cdot)\,|\cdot|^p$ is a much sharper upper bound for f_{hom} than the Wiener and Hashin-Shtrikman upper bounds f_{W+} and f_{HS+} when λ_k^+ is relatively small.

3. Further results for the case $p = 2$

In this section we discuss in more detail the case when f is of the form (3) and $p = 2$. Moreover, we assume that

$$\lambda_1(0) < \lambda_2(0) < \cdots < \lambda_m(0)$$

and

$$\lambda_1(\infty) < \lambda_2(\infty) < \cdots < \lambda_m(\infty).$$

We define the following constants:

$$\begin{aligned}
q_t^- &= c_2^-(t),\ q_t^+ = c_2^+(t),\\
w_t^- &= \left(\int_Y (C(\cdot,t))^{-1}\,dx\right)^{-1},\ w_t^+ = \int_Y C(\cdot,t)\,dx,\\
h_0^- &= -(n-1)\lambda_1(0) + \left(\int_Y \frac{1}{C(\cdot,0)+(n-1)\lambda_1(0)}dx\right)^{-1},\\
h_\infty^+ &= -(n-1)\lambda_m(\infty) + \left(\int_Y \frac{1}{C(\cdot,\infty)+(n-1)\lambda_m(\infty)}dx\right)^{-1},\\
t_0^+ &= \lambda_m(0)\left(1-\frac{n(1-a_m)(\lambda_m(0)-\lambda_1(0))}{n\lambda_m(0)+a_m(\lambda_1(0)-\lambda_m(0))}\right),\\
t_\infty^- &= \lambda_1(\infty)\left(1+\frac{n(1-a_1)(\lambda_m(\infty)-\lambda_1(\infty))}{n\lambda_1(\infty)+a_1(\lambda_m(\infty)-\lambda_1(\infty))}\right).
\end{aligned}$$

Remark 5. According to Remark 3, it yields that

$$q_0^- |\xi|^2 \le f_{hom}(\xi) \le q_\infty^+ |\xi|^2 .$$

In many cases q_0^- and q_∞^+ happen to be sufficiently close to each other to give a sharp estimate of f_{hom}. The general bounds (5) also show that

$$q_0^- |\xi|^2 \le f_{hom}(\xi) \le q_0^+ |\xi|^2$$

for small values of $|\xi|$ and

$$q_\infty^- |\xi|^2 \le f_{hom}(\xi) \le q_\infty^+ |\xi|^2$$

for large values of $|\xi|$.

The main result of this section reads:

Theorem 3.1. *There exist functions g, $h \in \Im_{a,g}^{cub}$ such that*

$$h_0^- |\xi|^2 \le f_{HS^-}(\xi) \le g_{hom}(\xi) \le h_\infty^- |\xi|^2,$$

$$h_0^+ |\xi|^2 \le h_{hom}(\xi) \le f_{HS^+}(\xi) \le h_\infty^+ |\xi|^2$$

for all $\xi \in \mathbf{R}^n$, where $h_0^+ = \max\left\{q_0^-, t_0^+\right\}$ and $h_\infty^- = \min\{q_\infty^+, t_\infty^-\}$. In addition, it holds that

$$w_0^+ |\xi|^2 \le f_{W^+}(\xi) \le w_\infty^+ |\xi|^2,$$

$$w_0^- |\xi|^2 \le f_{W^-}(\xi) \le w_\infty^- |\xi|^2$$

for all $\xi \in \mathbf{R}^n$.

Before continuing our discussion and proving the theorem, we present the following numerical experiments:

Let $g_i : \mathbf{R}^3 \to \mathbf{R}_+$ be defined by

$$g_i(\xi) = \begin{cases} k_1 \, |\xi|^2 & \text{for } i = 3 \\ \dfrac{k_2 + k_3 \, |\xi|}{1 + |\xi|} \, |\xi|^2 & \text{for } i = 2 \\ k_4 \, |\xi|^2 & \text{for } i = 1 \end{cases} .$$

Here, $\{k_i\}$ are positive constants with $k_2 \leq k_3$. Furthermore, let $a_1 = 0.970$, $a_2 = 0.015$ and $a_3 = 0.015$. Hence, $\{g_i\}$ and $\{a_i\}$ defines a family $\Im^{cub}_{a,g}$. Now, consider the concentric cubes B_1, B_2 and B_3 with volumes $0,970$, $0,985$ and 1, respectively, such that $B_3 = Y =]0,1[^3$. Furthermore, let A_i, $i = 1,2,3$, be a Y-periodic set such that $A_1 \cap Y = B_1$, $A_2 \cap Y = B_2 \backslash B_1$ and $A_3 \cap Y = B_3 \backslash B_2$, and let

$$f(x,\xi) = g_1(\xi)\chi_{A_1}(x) + g_2(\xi)\chi_{A_2}(x) + g_3(\xi)\chi_{A_3}(x).$$

Thus $f \in \Im^{cub}_{a,g}$ of the form

$$f(x,\xi) = C(x, |\xi|) \, |\xi|^2 .$$

In particular we note that $C(\cdot, 0)$ (resp. $C(\cdot, \infty)$) is equal to k_1, k_2 and k_4 (resp. k_1, k_2 and k_3) on A_3, A_2 and A_1, respectively.

In the table below we have listed the constants appearing in Remark 5 and Theorem 3.1 for four different combinations of $\{k_i\}$.

		case 1		case 2		case 3		case 4	
k_1		1		1		10^4		10^4	
k_2		1.1		10^2		10^2		$6{\cdot}10^3$	
k_3		1.5		10^3		10^3		$8{\cdot}10^3$	
k_4		10^4		10^4		1		1	
t_0^+	t_∞^-	9561	197	9561	197	101	1.10	101	1.10
q_0^-	q_0^+	102	104	191	194	102	102	160	162
q_∞^-	q_∞^+	116	118	194	196	111	112	180	182
h_0^-	h_∞^-	100	118	190	196	1.09	1.10	1.09	1.10
h_0^+	h_∞^+	9561	9561	9561	9591	102	115	160	187
w_0^+	w_∞^+	9703	9703	9704	9717	152	165	239	268
w_0^-	w_∞^-	35	40	67	66	1.03	1.03	1.03	1.03

Remark 6. First of all we observe that for the four cases we obtain very sharp estimates of f_{hom}, $f_{HS\pm}$ and $f_{W\pm}$. It is interesting to note that the bounds $q_0^- |\cdot|^2$ and $q_\infty^+ |\cdot|^2$ play an important role, not only for estimating f_{hom}, but also for estimating the nonlinear Hashin-Shtrikman bounds. Particularly we see that for all four cases h_0^- is close to h_∞^- and h_0^+ is close to h_∞^+. Hence, according to Theorem , we find a very good estimate of homogenized integrands g_{hom} and h_{hom} ($g, h \in \Im_{a,g}^{\text{cub}}$) that are close to f_{HS-} and f_{HS+}, respectively. Hence our approach yields important knowledge of the extreme effective properties of the class $\Im_{a,g}^{\text{cub}}$.

Remark 7. In the definition of f we have that $Y \backslash A_3$ is a genuine subset of Y. Moreover, by Remark it holds that $q_0^- \leq c_p^-(|\xi|) \leq c_p^+(|\xi|) \leq q_\infty^+$, and, thus, we observe in particular that the numerical results fit very well to the theoretical results of the previous section (see Remark 4).

Remark 8. We see that for case 1 and case 2 f_{hom} is very close to f_{HS-}. This shows that f_{hom} is nearly lower optimal in the class $\Im_{a,g}^{\text{cub}}$. Correspondingly, we observe that f_{hom} is very close to f_{HS+} for case 3 and case 4. Hence, in these cases f_{hom} is nearly upper optimal in $\Im_{a,g}^{\text{cub}}$.

We note that $f(\cdot, \xi) = \lambda_2(|\xi|)\,|\xi|^2$ in A_2, where $\lambda_2\,(\cdot)$ ranges over the whole interval $]k_2, k_3[$. Therefore, the Euler equation corresponding to the minimum problem for finding $f_{hom}(\xi)$ is highly nonlinear in the region A_2, especially for case 2 and case 3. For a fixed ξ, any direct numerical treatment of this Euler equation leads to severe problems. In fact, if we for example choose the finite element method we will need a large number of finite elements in the very small subset of the unit-cube where $f(\cdot, \xi)$ varies non-quadratically between $|\xi|^2$ and $10^4\,|\xi|^2$. Moreover, the table values show that the properties of $f(\cdot, \xi)$ on A_2 are significant. Hence, any kind of averaging in this region will be misleading. Besides, in contrast to the case when $f(x, \cdot)$ is a quadratic form, the values $\{f_{hom}(e_i)\}$, $i = 1, ..., n$, alone will give no general information of f_{hom}. Therefore, we would have to compute $f_{hom}(\xi)$ numerically for a vast number of vectors $\xi \in \mathbf{R}^3$ in order to get a good picture of f_{hom}.

As discussed in the introduction, our ultimate goal is always to find the homogenized energy within Ω,

$$E(f_{hom}) = \min\left\{\int_\Omega f_{hom}(Du)\,dx + \int_\Omega gu\,dx : u \in \mathrm{H}^{1,\mathrm{p}}(\Omega),\ u = \phi \text{ on } \gamma_0 \subset \partial\Omega\right\}, \tag{6}$$

which approximates the energy $E(f_h)$ for large values of h, where $f_h(x, \xi) = f(hx, \xi)$. Since $q_0^- |\cdot|^2 \leq f_{hom} \leq q_\infty^+ |\cdot|^2$, it follows that

$$E(q_0^- |\cdot|^2) \leq E(f_{hom}) \leq E(q_\infty^+ |\cdot|^2),$$

and, thus, we obtain estimates of the homogenized energy.

Remark 9. We note that the Euler equations associated with the upper and lower bounds

$$E(q_0^- \, |\cdot|^2) \text{ and } E(q_\infty^+ \, |\cdot|^2)$$

are linear. Hence, these bounds can be computed numerically e.g. by standard FEM-algorithms, and since q_0^- is very close to q_∞^+, we get a good approximation of $E(f_{hom})$. Moreover, this shows that the Euler equation associated with $E_{hom}(f)$ is almost linear. Similarly we can obtain good approximations of the optimal bounds for the homogenized energy $E_{\sup} = E(f_{HS_-})$ and $E_{\inf} = E(f_{HS_+})$.

The constants q_0^- and q_∞^+ are very close to each other in all four cases discussed above. This is however not a general property for all $f \in \Im_{a,g}^{\text{cub}}$. In fact, let the concentric cubes B_1, B_2, B_3 have volumes 0,015, 0,985 and 1, respectively, and let $A_3 \cap Y = B_1$, $A_1 \cap Y = B_2 \backslash B_1$ and $A_2 \cap Y = B_3 \backslash B_2$. For the function $f(x,\xi) = \sum g_i(\xi)\chi_{A_i}(x)$ it turns out that $q_0^- \,/\, q_0^+ = 193 \,/\, 200$ and $q_\infty^- \,/\, q_\infty^+ = 1091 \,/\, 1095$ when the constants $\{k_i\}$ take values as in case 2. In order to estimate $E(f_{hom})$ in this case we have to apply the general bounds

$$E(f^-) \le E(f_{hom}) \le E(f^+),$$

where

$$f^-(\cdot) = c_2^-(|\cdot|)\,|\cdot|^2 \text{ and } f^+(\cdot) = c_2^+(|\cdot|)\,|\cdot|^2 .$$

It is straightforward to express $c_2^-(t)$ and $c_2^+(t)$ in terms of t,$\{k_i\}$ and $\{a_i\}$. However, the Euler equations associated to the minimum problems for finding $E(f^-)$ and $E(f^+)$ are highly nonlinear and therefore more complicated to solve. Nevertheless, due to the facts that q_0^- is close to q_0^+ and q_∞^- is close to q_∞^+ we observe that the bounds $E(f^-)$ and $E(f^+)$ are close to each other, particularly when the variations on the boundary conditions ϕ and the values of the source g in (6) are such that the gradient of the solution, Du, is either large or small. In the latter case we can just use $E(q_0^- \, |\cdot|^2)$ as a lower bound for $E(f_{hom})$. As an upper bound for $E(f_{hom})$ we can use the value $E(f^+)'$ defined by

$$E(f^+)' = \int_\Omega f^+(Du')\,dx + \int_\Omega gu'\,dx,$$

where u' is the solution corresponding to the linear minimum problem for finding $E(q_\infty^- \, |\cdot|^2)$. Clearly, if $|Du|$ is small enough, then

$$E(q_0^+ \, |\cdot|^2) \simeq E(f^+)', \tag{7}$$

i.e. we find that

$$E(q_0^- |\cdot|^2) \leq E(f_{hom}) \leq E(f^+)' \simeq E(q_0^+ |\cdot|^2),$$

and, hence, we obtain a sharp estimate of the energy $E(f_{hom})$.

Proof of Theorem 3.1. Let $\varphi \in \Im_{a,g}^{cub}$ of the form

$$\varphi(x,\xi) = |\xi|^2 \sum_{i=1}^{m} \lambda_i(|\xi|)\chi_{A_i}.$$

We define the following functions associated with φ:

$$\mu^-(x) = \lambda_m(0)\chi_{A_m}(x) + \lambda_1(0)\chi_{\mathbf{R}^n \setminus \mathbf{A}_m}(x),$$

$$\mu^+(x) = \lambda_1(\infty)\chi_{A_1}(x) + \lambda_m(\infty)\chi_{\mathbf{R}^n \setminus \mathbf{A}_1}(x),$$

$$s^-(x,\xi) = |\xi|^2 \sum_{i=1}^{m} \lambda_i(0)\chi_{A_i}, \quad s^+(x,\xi) = |\xi|^2 \sum_{i=1}^{m} \lambda_i(\infty)\chi_{A_i},$$

$$g^-(x,\xi) = \mu^-(x)\,|\xi|^2 \text{ and } g^+(x,\xi) = \mu^+(x)\,|\xi|^2.$$

Since,

$$g^-(x,\xi) \leq s^-(x,\xi) \leq \varphi(x,\xi) \leq s^+(x,\xi) \leq g^+(x,\xi),$$

we find that

$$g_{hom}^-(\xi) \leq s_{hom}^-(\xi) \leq \varphi_{hom}(\xi) \leq s_{hom}^+(\xi) \leq g_{hom}^+(\xi). \tag{8}$$

Due to the fact that these functions satisfy the property of cubic symmetry it holds that $g_{hom}^{\pm}$ and $s_{hom}^{\pm}$ are of the forms $g_{hom}^{\pm}(\xi) = k_{hom}^{\pm}\,|\xi|^2$ and $s_{hom}^{\pm}(\xi) = l_{hom}^{\pm}\,|\xi|^2$ for some positive constants $k_{hom}^{\pm}$ and $l_{hom}^{\pm}$ (see e.g. [6, p. 39]). According to the Hashin-Shtrikman bounds for linear problems we have that $h_0^- \leq l_{hom}^-$ and $l_{hom}^+ \leq h_\infty^+$ (see e.g. [6, p. 188]). Hence,

$$h_0^-\,|\xi|^2 \leq \varphi_{hom}(\xi) \leq h_\infty^+\,|\xi|^2,$$

and it follows by the definition that

$$h_0^-\,|\xi|^2 \leq f_{HS-}(\xi) \leq f_{HS+}(\xi) \leq h_\infty^+\,|\xi|^2.$$

Moreover, if A_m is the classical Hashin-Shtrikman coated sphere assemblages, then it is well known that $k_{hom}^- = t_0^+$. Combined with (8) this shows that

$$t_0^+\,|\xi|^2 \leq \varphi_{hom}(\xi) \leq f_{HS+}(\xi).$$

Moreover, by Remark 5 and the definition of f_{HS+} we have

$$q_0^- |\xi|^2 \leq f_{hom}(\xi) \leq f_{HS+}(\xi).$$

Thus,

$$h_0^+ |\xi|^2 \leq h_{hom}(\xi) \leq f_{HS+}(\xi) \leq h_\infty^+ |\xi|^2,$$

where $h = f$ if $t_0^+ \leq q_0^-$, and $h = \varphi$ if $t_0^+ > q_0^-$. Similarly, if A_1 is the classical Hashin-Shtrikman coated sphere assemblages, we get that $k_{hom}^+ = t_\infty^-$, and, hence, by using (8) once more we obtain that

$$f_{HS-}(\xi) \leq \varphi_{hom}(\xi) \leq t_\infty^- |\xi|^2.$$

In addition, Remark 5 and the definition of f_{HS-} yield that

$$f_{HS-}(\xi) \leq f_{hom}(\xi) \leq q_\infty^+ |\xi|^2.$$

Hence,

$$h_0^- |\xi|^2 \leq f_{HS-}(\xi) \leq g_{hom}(\xi) \leq h_\infty^- |\xi|^2,$$

where $h = f$ if $q_\infty^+ \leq t_\infty^-$, and $h = \varphi$ if $q_\infty^+ > t_\infty^-$. The final part of the theorem follows directly by integrating the inequalities

$$C(x,0)\,|\xi|^2 \leq f(x,\xi) \leq C(x,\infty)\,|\xi|^2$$

and

$$(C(x,0))^{-1}\,|\xi|^2 \geq f^*(x,\xi) \geq (C(x,\infty))^{-1}\,|\xi|^2$$

over Y. This completes the proof. ∎

4. The chess board structure

We introduce the following generalization of the classical chess board structure: Let $Y = [-1,1]^n$ and let V be a Y-periodic set such that $V \cap Y = [-1,0]^n \cup [0,1]^n$. Furthermore, let

$$f_{chess}(x,\xi) = k\chi_{chess}(x) f_1(\xi) + (1 - \chi_{chess}(x))\, f_2(\xi),\ k > 0, \tag{9}$$

where χ_{chess} is the characteristic function of V. We assume that $f_i(0) = 0$ and that f_i is convex and satisfies the growth conditions

$$c_1 |\xi|^p \leq f_i(\xi) \leq c_2 (|\xi|^p + l), \tag{10}$$

for some constants $l \geq 0$, $0 < c_1, c_2 < +\infty$.

Theorem 4.1. *If φ_{chess} is the homogenized integrand corresponding to the sequence of integrands $\{f_{chess}(hx,\xi)\}_{h=1}^{\infty}$, then*

$$C_1(p,k)\,|\xi|^p \le \varphi_{chess}(\xi) \le C_2(p)\,(|\xi|^p + l)\,,$$

for all $\xi \in \mathbf{R}^n$, where $\lim_{k\to+\infty} C_1(p,k) = +\infty$ if $p \ge n$ and $C_2(p) < +\infty$ if $p < n$.

Remark 10. We note that for the chess board structure neither V nor $\mathbf{R}^n\backslash\mathbf{V}$ is disperse. According to Theorem 2.3, the above result shows that the macroscopic behavior corresponding to the chessboard type structure is bounded similar to that of structures where V is disperse if $p < n$ and that of structures where $\mathbf{R}^n\backslash\mathbf{V}$ is disperse if $p \ge n$.

Proof. Since the case $n = 1$ is trivial we assume $n \ge 2$.

$\varphi_{chess}(\xi) \le C_2(p)\,(\sum_{i=1}^n |\xi_i|^p + l)$: According to the minimum energy principle, the symmetry of V and the conditions on f_i it is easy to see that

$$\varphi_{chess}(\xi) \le \frac{1}{|Y|}\int_{Y\backslash V} c_2\left(|\xi|^p\, n^{p+1}\, |Du + e_1|^p + l\right)$$

for every $u \in \mathrm{H}$ where $\mathrm{H} = \left\{\mathrm{v} \in \mathrm{H}^{1,p}_{per}(\mathrm{Y}) : \mathrm{Dv} + \mathrm{e}_1 = 0 \text{ on } \mathrm{V}\right\}$. Hence, the upper

bound will follow as soon as we have proved that H is not empty. Let $I_1 = [0,1]$ and $I_{-1} = [-1,0]$. If $(l_1,\ldots,l_n)$ is a n-tuple of integers $l_i \in \{-1,1\}$ we let $V_{l_1,\ldots,l_n}$ be the set

$$V_{l_1,\ldots,l_n} = \{x = (x_1,\ldots,x_n) \in \mathbf{R}^n : \mathbf{x_i} \in \mathbf{I_{l_i}}\}\,.$$

In particular we observe that

$$Y = \cup_{l_i\in\{-1,1\}} V_{l_1,\ldots,l_n}.$$

Let $\mathbf{Z}^n$ denote the usual set of points in $\mathbf{R}^n$ with integer coordinates and let us define the function u on $Y\backslash\mathbf{Z}^n$ by

$$u(x) = 1 - l_1 + x_1 + (1 - l_1 x_1)\frac{l_1\sum_{i=2}^n x_i\,(1 - l_i x_i)\,(l_i - l_1)}{l_1 x_1 + \sum_{i=2}^n x_i\,(1 - l_i x_i)\,l_i},\quad x \in V_{l_1,\ldots,l_n}\backslash\mathbf{Z}^n.$$

We observe that u is well defined and takes the same values on opposite traces of $Y\backslash\mathbf{Z}^n$. Thus u can be extended to a Y-periodic function which is absolutely continuous on all line segments in $\mathbf{R}^n\backslash\mathbf{Z}^n$, parallel to the coordinate axes. Moreover, the classical partial derivatives of u belong to $\mathrm{L}^p(\mathrm{Y})$ for $1 < p < n$. Indeed, if $z = (z_1,\ldots,z_n) \in \mathbf{Z}^n$, $z_i \in \{-1,0,1\}$ then it is easy to show that

$$\left|\frac{\partial u(x)}{\partial x_i}\right| \le \frac{k_1}{|z - x|}$$

for all $x \in B_z = \{y : |z - y| \le 1/2\}$ and some constant k_1. Hence,

$$\int_{B_z} \left|\frac{\partial u(x)}{\partial x_i}\right|^p dx \le k_2 \int_0^{\frac{1}{2}} \left(\frac{1}{r}\right)^p r^{n-1} dr = k_2 \int_0^{\frac{1}{2}} r^{n-1-p} dr = \frac{k_2}{(n-p)} 2^{p-n}$$

for some constant k_2, and thereby it follows that

$$\int_Y \left|\frac{\partial u(x)}{\partial x_i}\right|^p dx < +\infty.$$

Now, by [16, Theorem 2.1.4.] we obtain that $u \in \mathrm{H}^{1,p}(\mathrm{Y})$, i.e. $u \in \mathrm{H}^{1,p}_{per}(\mathrm{Y})$, and, hence, since $Du + e_1 = 0$ on V, it follows that $u \in \mathrm{H}$.

$C_1(p,k)\,|\xi|^p \le \varphi_{chess}(\xi)$: Let $p \ge n$, let C be a positive number and let φ_k be the homogenized integrand corresponding to sequence of integrands $\{g_k(hx, \eta)\}_{h=1}^{\infty}$, where

$$g_k(\cdot, \eta) = k\,|\eta|^p\,\chi_V + (1 - \chi_V)\,|\eta|^p.$$

Furthermore, let A_k be the compact set in $\mathbf{R}^n$ given by

$$A_k = \{\xi \in \mathbf{R}^n : |\xi|^p = 1,\ \varphi_k(\xi) \le C\}$$

[A_k is compact since φ_k is continuous in $\mathbf{R}^n$]. Let $k = 1, 2, 3, ...,$. It is easy to see that $A_{k+1} \subset A_k$. If we can prove that $\cap_{k=1}^{\infty} A_k = \emptyset$, then, because each A_k is compact, this will immediately give us that there exists a positive number h_C such that $A_k = \emptyset$ for $k \ge h_C$, i.e. that $\varphi_k(\xi) > C$ for all $|\xi| = 1$ and $k \ge h_C$. Moreover, since $g_k(x, \cdot)$ is p-homogeneous (that is $g_k(x, t\xi) = |t|^p g_k(x, \xi)$) we also have that φ_k is p-homogeneous. Therefore, $\varphi_k(\xi) > C\,|\xi|^p$ for all $\xi \ne 0$, $k \ge h_C$. Thus, because $g_k \le f_{chess}$ we get that

$$\varphi_{chess}(\xi) > C\,|\xi|^p$$

for all $k \ge h_C$, and, since C was arbitrarily chosen, the lower bound will follow.

It remains to prove that $\cap_{k=1}^{\infty} A_k = \emptyset$. Suppose on the contrary that $\cap_{k=1}^{\infty} A_k \ne \emptyset$. Then there exist a positive number C, a vector $\xi \in \mathbf{R}^n$, $|\xi| = 1$ and functions $u_k \in \mathrm{H}^{1,p}_{per}(\mathrm{Y})$, $k = 1, 2, 3, ...,$ such that

$$\varphi_k(\xi) = \inf_{v \in \mathrm{H}^{1,p}_{per}(\mathrm{Y})} \frac{1}{|Y|} \int_Y g_k(x, Dv + \xi)\, dx \le \frac{1}{|Y|} \int_Y g_k(x, Du_k + \xi)\, dx \le C,$$

i.e.,

$$k \frac{1}{|Y|} \int_{V \cap Y} |\xi + Du_k|^p\, dx + \frac{1}{|Y|} \int_{Y \setminus V} |\xi + Du_k|^p\, dx \le C. \tag{11}$$

Hence, we obtain that $Du_k \to -\xi$ on $V \cap Y$ and at the same time that $\{u_k\}$ is bounded in $H^{1,p}_{per}(Y)$. Consequently, there exists a subsequence $\{u_{t_k}\}$ converging weakly in $H^{1,p}_{per}(Y)$ to an element u_∞ with $Du_\infty = -\xi$ on V. This implies that

$$u_\infty(x) = -x \cdot \xi + u(x) \text{ on } V,$$

where u is constant on every unit-cube contained in V. We are going to show that u has to be constant on V, which is equivalent by saying that u is constant on $V \cap (Y + z)$ for every $z \in \mathbf{Z}^n$ with integer components. We prove this fact for $z = 0$. The other cases can be seen similarly. Let $u(\cdot) = a$ and $u(\cdot) = b$ on $[-1, 0]^n$ and $[0, 1]^n$, respectively, for some constants a and b. Let O be the set of all line-segments parallel to the vector $t = \sum_{i=1}^n e_i$ with endpoints in the plane sets $\{x \in [-1, 0]^n : x_1 = 0\}$ and $\{x \in [0, 1]^n : x_n = 0\}$. Furthermore, we let $x'_1, \ldots x'_n$ denote coordinates relative to an orthonormal coordinate system such that the x'_1-axis is parallel to t. For any segment $l \in O$ we have that

$$\inf_v \int_l \left|\frac{\partial v}{\partial x'_1}\right|^p dx'_1 = \int_l \left|\frac{a-b}{l}\right|^p dx'_1,$$

where infimum is taken over all $v \in H^{1,p}(l)$ such that $v = a$ and $v = b$ at each endpoint of l, respectively. This is easily seen by noting that the solution of the corresponding 1-dimensional Euler equation is such that $\partial v/\partial x'_1$ is constant. Observe also that if l has the endpoint $(0, x_2, \ldots, x_n)$, then $|l| = \sqrt{n}x_n$. It is obvious that we can find a spherical segment O_s with center at 0 and a radius r_s such that $O_s \subset O$. We obtain that

$$\int_O |Du|^p \, dx \geq \int_{l \in O} \left(\int_l \left|\frac{\partial u}{\partial x'_1}\right|^p dx'_1\right) dx'_2 \cdots dx'_n \geq$$

$$\geq \int_{l \in O} \left(\int_l \left|\frac{a-b}{l}\right|^p dx'_1\right) dx'_2 \cdots dx'_n = \int_O \left|\frac{a-b}{\sqrt{n}x_n}\right|^p dx \geq$$

$$\geq \int_{O_s} \left|\frac{a-b}{\sqrt{n}x_n}\right|^p dx \geq \text{const} \int_0^a \frac{1}{r^p} r^{n-1} dr = +\infty,$$

for $p \geq n$, i.e. $u \notin H^{1,p}(Y)$, unless $a = b$. Hence, we have that u is constant on V. But this gives that u_∞ is not Y-periodic, which is a contradiction, and we can conclude that our assumption is wrong, i.e. that $\cap_{h=1}^\infty A_h = \emptyset$. This completes the proof. ■

Remark 11. One of the problems in the above proof on the lower bound was to show that the function u (which is constant on every unit-cube contained in V) is constant on V. In the case $p > n$, this fact is a direct consequence of Sobolev Imbedding Theorem. Moreover, for the case $p = n = 2$ the lower bound is easily proved since we by the well known chess board formula have that $\varphi_k(\xi) = \sqrt{k}\,|\xi|^2$. Accordingly, the above proof on u is only necessary when $p = n \geq 3$.

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