# Semiregular Polygons

**Oleg Mushkarov and Nikolai Nikolov**

**1. INTRODUCTION.** This note is motivated by Problem 3 from IMO'2003, which states:

> Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\sqrt{3}/2$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

In fact, one can completely characterize the hexagons having the given property. Each is obtained from an equilateral triangle by cutting its corners at the same height.

It is tempting to conjecture that one can describe in a similar way the convex $2n$-gons ($n \geq 4$) whose pairs of opposite sides have the property that the distances between their midpoints are equal to $\cot(\pi/2n)/2$ times the sum of their lengths. We call these $2n$-gons *semiregular*. As we shall see, the semiregular $2n$-gons still have equal angles, but when $n \geq 5$ not all of them are obtained by cutting the corners from regular $n$-gons. This is a consequence of our main result, Theorem 3.1, which gives a complete characterization of the semiregular $2n$-gons for $n = 2, 3, \ldots$. The key point in its proof is a geometric inequality for arbitrary $2n$-gons, which in the case $n = 3$ gives another generalization of the Olympiad problem.

**2. AN INEQUALITY FOR $2n$-GONS.** Let $A_1, A_2, \ldots, A_{2n}$ ($n \geq 2$) be arbitrary points in the plane. Denote by $a_k$ the length of the segment $A_kA_{k+1}$ ($1 \leq k \leq 2n$) and by $m_k$ the distance between the midpoints of the opposite segments $A_kA_{k+1}$ and $A_{n+k}A_{n+k+1}$ ($1 \leq k \leq n$), where subscripts are taken modulo $2n$.

**Proposition 2.1.** *If $n \geq 2$, then the following inequality holds:*

$$\sum_{k=1}^{n}(a_k + a_{n+k})^2 \geq 4\tan^2\left(\frac{\pi}{2n}\right)\sum_{k=1}^{n} m_k^2. \tag{2.1}$$

*Proof.* We assume that the points $A_1, A_2, \ldots, A_{2n}$ are situated in the complex plane and denote by $z_k$ the complex number representing $A_k$. Set $w_k = z_{n+k} - z_k$. Then the triangle inequality gives

$$\begin{aligned}\sum_{k=1}^{n}(a_k + a_{n+k})^2 &= \sum_{k=1}^{n}(|z_k - z_{k+1}| + |z_{n+k} - z_{n+k+1}|)^2 \\ &\geq \sum_{k=1}^{n}|z_k - z_{k+1} - z_{n+k} + z_{n+k+1}|^2 \\ &= \left(\sum_{k=1}^{n-1}|w_{k+1} - w_k|^2\right) + |w_n + w_1|^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{n} m_k^2 &= \sum_{k=1}^{n}\left|\frac{z_k + z_{k+1}}{2} - \frac{z_{n+k} - z_{n+k+1}}{2}\right|^2 \\ &= \frac{1}{4}\left(\sum_{k=1}^{n-1}|w_k + w_{k+1}|^2\right) + \frac{1}{4}|w_n - w_1|^2.\end{aligned}$$

Hence it is enough to prove the inequality

$$\left(\sum_{k=1}^{n-1}|w_{k+1} - w_k|^2\right) + |w_n + w_1|^2 \geq \tan^2\left(\frac{\pi}{2n}\right)\left(\left(\sum_{k=1}^{n-1}|w_{k+1} + w_k|^2\right) + |w_n - w_1|^2\right). \tag{2.2}$$

Note that (2.2) becomes an identity for $n = 2$, so we assume that $n \geq 3$. Write $w_k = x_k + iy_k$. A simple calculation shows that (2.2) can be rewritten as

$$\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n}(x_k^2 + y_k^2) \geq \left(\sum_{k=1}^{n-1}(x_k x_{k+1} + y_k y_{k+1})\right) - x_n x_1 - y_n y_1,$$

which is a consequence of the following lemma:

**Lemma 2.2.** *For any integer $n \geq 3$ and any real numbers $x_1, x_2, \ldots, x_n$ the following inequality holds:*

$$\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} x_k^2 \geq \left(\sum_{k=1}^{n-1} x_k x_{k+1}\right) - x_n x_1. \tag{2.3}$$

*Equality obtains if and only if*

$$x_k = \frac{\sin(k\pi/n)}{\sin(\pi/n)}x_1 + \frac{\sin((k-1)\pi/n)}{\sin(\pi/n)}x_n \quad (2 \leq k \leq n-1). \tag{2.4}$$

*Proof of Lemma 2.2.* Inequality (2.3) follows from the identity

$$\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n}x_k^2-\left(\sum_{k=1}^{n-1}x_kx_{k+1}\right)+x_nx_1$$
$$=\sum_{k=1}^{n-2}\frac{(x_k\sin((k+1)\pi/n)-x_{k+1}\sin(k\pi/n)+x_n\sin(\pi/n))^2}{2\sin(k\pi/n)\sin((k+1)\pi/n)}. \qquad (2.5)$$

To prove it, we just compare the coefficients of $x_k^2$ and $x_kx_{k+1}$ on both sides of (2.5). For example, the coefficients of $x_n^2$ on the two sides of (2.5) coincide because

$$\sum_{k=1}^{n-2}\frac{\sin^2(\pi/n)}{2\sin(k\pi/n)\sin((k+1)\pi/n)}=\sum_{k=1}^{n-2}\frac{\sin(\pi/n)}{2}\left(\cot\left(\frac{k\pi}{n}\right)-\cot\left(\frac{(k+1)\pi}{n}\right)\right)$$
$$=\frac{\sin(\pi/n)}{2}\left(\cot\left(\frac{\pi}{n}\right)-\cot\left(\frac{(n-1)\pi}{n}\right)\right)$$
$$=\cos\left(\frac{\pi}{n}\right).$$

Identity (2.5) shows that equality is attained in (2.3) if and only if

$$x_{k+1}=\frac{\sin((k+1)\pi/n)}{\sin(k\pi/n)}x_k+\frac{\sin(\pi/n)}{\sin(k\pi/n)}x_n \quad (1\le k\le n-2).$$

It then follows easily by induction on $k$ that $x_k$ is given by (2.4). ■

Concerning equality in (2.1), it follows from the proof of Proposition 2.1 that for $n=2$ it is attained only for parallelograms. If $n\ge 3$, then equality holds in (2.1) if and only if the opposite sides of the respective $2n$-gon $A_1A_2\dots A_{2n}$ are parallel and its main diagonals obey the following relations:

$$\overrightarrow{A_kA_{n+k}}=\frac{\sin(k\pi/n)}{\sin(\pi/n)}\overrightarrow{A_1A_{n+1}}+\frac{\sin((k-1)\pi/n)}{\sin(\pi/n)}\overrightarrow{A_nA_{2n}}. \qquad (2.6)$$

In particular, we obtain the following generalization of Problem 3 from IMO'2003:

**Corollary 2.3.** *Any convex hexagon for which equality holds in* (2.1) *is obtained from a triangle by cutting congruent triangles from its corners by means of lines parallel to their opposite sides.*

*Proof.* For a convex hexagon $A_1A_2A_3A_4A_5A_6$ equality is attained in (2.1) if and only if its opposite sides are parallel and

$$\overrightarrow{A_2A_5}=\overrightarrow{A_1A_4}+\overrightarrow{A_3A_6}$$

(see (2.6)). Writing the last identity as $\overrightarrow{A_1A_2}+\overrightarrow{A_3A_4}+\overrightarrow{A_5A_6}=0$ shows that

$$A_3A_6||A_1A_2||A_4A_5,\ A_1A_4||A_2A_3||A_5A_6,\ A_2A_5||A_3A_4||A_1A_6.$$

Denote by $A$, $B$, and $C$ the intersection points of the lines containing the sides $A_1A_2$ and $A_5A_6$, $A_1A_2$ and $A_3A_4$, and $A_3A_4$ and $A_5A_6$, respectively. Then it is easy to see that the triangles $A_1AA_6$, $BA_2A_3$, and $A_4A_5C$ are congruent. Conversely, it is not difficult to show that the construction described always produces a hexagon for which equality holds in (2.1). ■

**3. SEMIREGULAR 2*n*-GONS.** Recall that a convex $2n$-gon is *semiregular* if the distance between the midpoints of any two of its opposite sides is equal to $\cot(\pi/2n)/2$ times the sum of their lengths. The following theorem gives a complete characterization of all semiregular $2n$-gons:

**Theorem 3.1.** *A convex $2n$-gon $M_{2n}$ with side-lengths $a_1, a_2, \dots, a_{2n}$ is semiregular if and only if*

(i) $n = 2$ *and $M_4$ is a rhombus;*

(ii) $n \geq 3$*, all the angles of $M_{2n}$ are equal, and*

$$a_n = a_{n-1} - \sum_{k=1}^{n-2}(a_{k+1} - a_k)\frac{\sin(k\pi/n)}{\sin(\pi/n)},$$

$$a_{n+1} = a_{n-1} + \sum_{k=1}^{n-2}(a_{k+1} - a_k)\frac{\cos((2k+1)\pi/2n)}{\cos(\pi/2n)},$$

$$a_{n+k} = a_1 + a_{n+1} - a_k \quad (2 \leq k \leq n).$$

Before proceeding to the proof of Theorem 3.1 we record a useful lemma.

**Lemma 3.2.** *Let $ABC$ be a triangle with $\angle C \geq \pi/n$, and let $M$ be the midpoint of $AB$. Then*

$$AB \geq 2\tan\left(\frac{\pi}{2n}\right)CM,$$

*with equality if and only if $\angle C = \pi/n$ and $CA = CB$ in case $n \geq 3$.*

*Proof.* The law of cosines together with the AM-GM inequality gives

$$AB^2 = CA^2 + CB^2 - 2CA \cdot CB.\cos(\angle C) \geq (CA^2 + CB^2)\left(1 - \cos\left(\frac{\pi}{n}\right)\right).$$

Hence

$$4CM^2 = 2(CA^2 + CB^2) - AB^2 \leq \frac{2AB^2}{1 - \cos(\pi/n)} - AB^2 = \cot^2\left(\frac{\pi}{2n}\right)AB^2,$$

and Lemma 3.2 is proved. ■

*Proof of Theorem 3.1.* Given a semiregular $2n$-gon $M_{2n} = A_1A_2\dots A_{2n}$ denote by $B_k$ the intersection point of the segments $A_kA_{n+k}$ and $A_{k+1}A_{n+k+1} (1 \leq k \leq n)$. Then it is easy to see that

$$\sum_{k=1}^{n}\angle A_kB_kA_{k+1} = \pi.$$

Hence there is an index $l$ such that $\angle A_lB_lA_{l+1} \geq \pi/n$. Now applying Lemma 3.2 to the triangles $A_lB_lA_{l+1}$ and $A_{n+l}B_lA_{n+l+1}$ shows that $\angle A_lB_lA_{l+1} = \angle A_{n+l}B_lA_{n+l+1} = \pi/n$ and $A_lA_{n+l} = A_{l+1}A_{n+l+1}$ if $n \geq 3$. If $n = 2$, then $M_4$ is a parallelogram with perpendicular diagonals (i.e., $M_4$ is a rhombus). We assume henceforth that $n \geq 3$.

Then

$$\sum_{k=1,k\neq l}^{n} \angle A_k B_k A_{k+1} = \frac{(n-1)\pi}{n},$$

and proceeding in the same way as earlier we can conclude that $\angle A_k B_k A_{k+1} = \pi/n$ $(1 \le k \le n)$ and that all the main diagonals $A_k A_{n+k} (1 \le k \le n)$ have the same length. In particular, all the angles of $M_{2n}$ are equal. It follows that when $n \ge 3$ $M_{2n}$ is a semiregular $2n$-gon if and only if it has the following three properties:

- all the angles of $M_{2n}$ are equal;
- all the main diagonals of $M_{2n}$ have the same length;
- the angle between any two consecutive main diagonals of $M_{2n}$ is $\pi/n$.

Denote by $z_k$ the complex number representing the vertex $A_k$, and set $r_k = a_k/a_1$. The three properties just listed are equivalent to the following relations:

$$z_{k+1} - z_k = (z_2 - z_1) r_k e^{\frac{i(k-1)\pi}{n}} \quad (1 \le k \le 2n), \tag{3.1}$$

$$z_{n+k} - z_k = (z_{n+1} - z_1) e^{\frac{i(k-1)\pi}{n}} \quad (1 \le k \le n). \tag{3.2}$$

Writing (3.1) as

$$z_{k+1} = z_1 + (z_2 - z_1) \sum_{j=1}^{k} r_j e^{\frac{i(j-1)\pi}{n}}$$

and using (3.2) leads to

$$\sum_{j=1}^{n} (r_{j+k} - r_j) e^{\frac{ij\pi}{n}} = 0. \tag{3.3}$$

Now subtracting the first identity in (3.3) ($k = 1$) from the $k$th one and then using the $(k-1)$th identity we get

$$r_{n+k} = r_1 + r_{n+1} - r_k. \tag{3.4}$$

On the other hand, taking into account the fact that the $r_k$ are real numbers, we see that identity (3.3) for $k = 1$ is equivalent to the following:

$$r_{n+1} - r_n = \sum_{k=1}^{n-1} (r_{k+1} - r_k) \cos\left(\frac{k\pi}{n}\right),$$

$$r_n - r_{n-1} = -\sum_{k=1}^{n-2} (r_{k+1} - r_k) \frac{\sin(k\pi/n)}{\sin(\pi/n)}.$$

These relations in tandem with (3.4) establish Theorem 3.1(ii). ■

To illustrate Theorem 3.1, we consider the cases $n = 3$ and $n = 4$.

**Example 1.** Given a semiregular hexagon $A_1 A_2 A_3 A_4 A_5 A_6$, we consider its enveloping triangles $B_1 B_2 B_3$ and $C_1 C_2 C_3$ determined by the lines containing the sides $A_1 A_2$,

$A_3A_4$, $A_5A_6$, and $A_2A_3$, $A_4A_5$, $A_6A_1$, respectively (see Figure 1). Theorem 3.1 implies that $B_1B_2B_3$ and $C_1C_2C_3$ are equilateral triangles with a common center and that their corresponding sides form angles of 60°. Conversely, one can easily show that the intersection of any two such triangles is a semiregular hexagon.

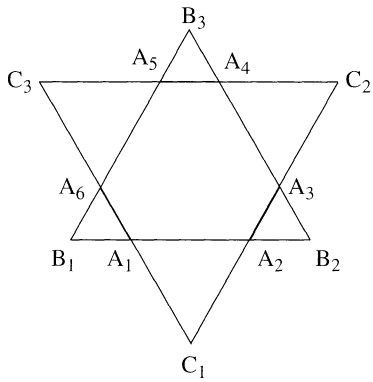


**Figure 1.**

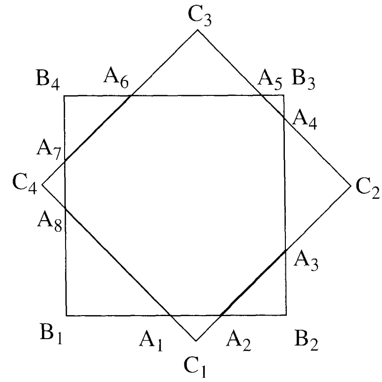


**Figure 2.**

**Example 2.** Given a semiregular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$, we consider its enveloping quadrilaterals $B_1B_2B_3B_4$ and $C_1C_2C_3C_4$ determined by the lines containing the sides $A_1A_2$, $A_3A_4$, $A_5A_6$, $A_7A_8$, and $A_2A_3$, $A_4A_5$, $A_6A_7$, $A_8A_1$, respectively (see Figure 2). Here Theorem 3.1 confirms that $B_1B_2B_3B_4$ and $C_1C_2C_3C_4$ are congruent squares whose corresponding sides meet at 45°. Conversely, the intersection of any two such squares is a semiregular octagon.

These two examples suggest the following question: Are the two enveloping $n$-gons of a semiregular $2n$-gon regular $n$-gons? We leave as an exercise to the reader to show that in general this is true only when $n = 3$ or $n = 4$.


*Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria*
*muskarov@math.bas.bg*
*nik@math.bas.bg*