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Quadratic Forms, Chebyshev Polynomials, and Geometric Inequalities

Titu Andreescu and Oleg Mushkarov

Abstract. We discuss a geometric inequality for $2n$-gons proved in [Mushkarov, O., Nikolov N. (2006). Semiregular polygons. *Amer. Math. Monthly*. 113(4): 339–344] and show how the algebraic inequality it is based on can be proved by using the standard theory of quadratic forms. In addition, we prove an "odd" version of the geometric inequality which leads to some interesting geometric problems for polygons with an odd number of sides.

1. INTRODUCTION. This note is motivated by the following geometric inequality proved in [2]:

Proposition. *Let $A_1, A_2, \ldots, A_{2n}$ $(n \geq 2)$ be arbitrary points in the plane. Denote by a_k the length of the segment A_kA_{k+1} $(1 \leq k \leq 2n)$ and by m_k the distance between the midpoints of the opposite segments A_kA_{k+1} and $A_{n+k}A_{n+k+1}(1 \leq k \leq n)$, where subscripts are taken modulo $2n$. Then*

$$\sum_{k=1}^{n}(a_k + a_{n+k})^2 \geq 4\tan^2\left(\frac{\pi}{2n}\right)\sum_{k=1}^{n} m_k^2. \tag{1}$$

Using complex numbers one can reduce (1) to the algebraic inequality

$$\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} x_k^2 \geq \left(\sum_{k=1}^{n-1} x_k x_{k+1}\right) - x_n x_1, \tag{2}$$

true for all real numbers $x_1, x_2, \ldots, x_n$. This is proved in [2, Lemma 2.2] by representing the difference of the expressions on the left- and right-hand sides as a sum of squares. We encourage the reader to do this for $n = 3$ and $n = 4$.

Our main purpose here is to show how the above algebraic inequality can be proved by using the standard theory of quadratic forms. This leads to finding the largest root of the equation $T_n(x) + 1 = 0$, where $T_n(x)$ is the nth Chebyshev polynomial of the first kind [3]. We also prove an "odd" version of the geometric inequality (1) which suggests some interesting geometric problems for polygons with an odd number of sides.

2. QUADRATIC FORMS. In this section, we recall some well-known facts from the theory of quadratic forms (see, e.g., [1]).

A quadratic form in the variables $x_1, x_2, \ldots, x_n$ is a homogeneous polynomial of second degree

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

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where $A = (a_{ij})$ is a symmetric real $n \times n$ matrix. Any such matrix has real eigenvalues, which means that the characteristic polynomial of A given by

$$\det(A - \lambda I) = 0,$$

where I is the $n \times n$ identity matrix, has n real roots. We next denote by λ_{max} and λ_{min} the maximal and minimal eigenvalues of A, respectively. Then it is well known [1] that the following inequalities

$$\lambda_{max} \sum_{i=1}^{n} x_i^2 \geq \sum_{i,j=1}^{n} a_{ij} x_i x_j \geq \lambda_{min} \sum_{i=1}^{n} x_i^2 \tag{3}$$

hold for all real numbers $x_1, x_2, \ldots, x_n$.

We use (3) to prove the inequality (1). To do this, we have to find the maximal eigenvalue of the symmetric matrix A corresponding to the quadratic form

$$2x_1x_2 + 2x_2x_3 + \cdots + 2x_{n-1}x_n - 2x_nx_1.$$

Note that

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Hence, we need to find the largest root of the characteristic polynomial of A which we will do in the next section.

3. CHEBYSHEV POLYNOMIALS.

We will show that the characteristic polynomial of the matrix A can be expressed by means of the so-called nth Chebyshev polynomial of the first kind [3]. This will allow us to find all the eigenvalues of A_n and, in particular, the maximal and minimal ones.

Recall that the Chebyshev polynomials of the first kind are defined by the recurrence relation

$$T_0(x) = 1,\ T_1(x) = x,\ T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n = 2, 3, \ldots.$$

Thus

$$T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1, \ldots.$$

These polynomials also can be defined as the unique polynomials $T_n(x)$ satisfying the trigonometric identity

$$T_n(\cos\theta) = \cos n\theta, \tag{4}$$

which can be easily motivated by the trigonometric addition formula

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta.$$

Using the recursive definition of the Chebyshev polynomials of the first kind, it is easy to show that they have the following determinant form (see, e.g., [3]):

$$T_n(x) = \det \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2x \end{bmatrix}.$$

Denote by $A_n(x)$ the following $n \times n$ determinant:

$$A_n(x) = \det \begin{bmatrix} x & 1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 1 \\ -1 & 0 & 0 & 0 & \cdots & 1 & x \end{bmatrix}.$$

Lemma 1. *For all integers $n \geq 3$ the following identity holds*

$$A_n(2x) = 2T_n(x) + 2(-1)^n. \tag{5}$$

Proof. Let $\triangle_n(x)$ be the determinant obtained from $A_n(x)$ by replacing -1 with 0 in its first and nth row. Expanding $T_n(x)$ with respect to the entries of the first row, we obtain

$$T_n(x) = x \triangle_{n-1}(2x) - \triangle_{n-2}(2x). \tag{6}$$

Also, expanding the determinant $A_n(x)$ by the first row, we get

$$A_n(x) = x \triangle_{n-1}(x) - B_{n-1}(x) + (-1)^n C_{n-1}(x), \tag{7}$$

where

$$B_{n-1}(x) = \det \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x & 1 \\ -1 & 0 & 0 & 0 & \cdots & 1 & x \end{bmatrix},$$

$$C_{n-1}(x) = \det \begin{bmatrix} 1 & x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & x \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Note that expanding the determinant $B_{n-1}(x)$ by the first column involves two cofactors for the (1, 1) and $(1, n-1)$ entries. The first one is clearly $\triangle_{n-2}(x)$. The second one is equal to 1, since it is the determinant of a lower triangular matrix with 1's along the diagonal. Thus,

$$B_{n-1}(x) = \triangle_{n-2}(x) - (-1)^n. \tag{8}$$

Similarly, we get

$$C_{n-1}(x) = 1 - (-1)^n \triangle_{n-2}(x) \tag{9}$$

and (7)–(9) imply

$$A_n(x) = x \triangle_{n-1}(x) - 2 \triangle_{n-2}(x) + 2(-1)^n. \tag{10}$$

Now, using (6) and the last equality, we obtain the desired identity

$$A_n(2x) = 2T_n(x) + 2(-1)^n. \qquad \blacksquare$$

We are now ready to prove the following lemma.

Lemma 2. *The eigenvalues of the symmetric matrix A are*

$$\lambda_k = \cos\left(\frac{(2k-1)\pi}{n}\right), \quad 1 \le k \le n.$$

In particular, the maximal and the minimal eigenvalues of A are

$$\lambda_{max} = 2\cos\left(\frac{\pi}{n}\right),$$

$$\lambda_{min} = \begin{cases} -2\cos\left(\frac{\pi}{n}\right) & \textit{if } n \textit{ is even} \\ -2 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Note first that $T_n(-x) = (-1)^n T_n(x)$. (This identity is well known and follows, for example, from (4)). Then, by Lemma 1, the characteristic polynomial of A is related to the nth Chebyshev polynomial in the following way:

$$\det(A - \lambda I) = A_n(-\lambda) = 2(-1)^n \left(T_n\left(\frac{\lambda}{2}\right) + 1\right).$$

Using the trigonometric property $T_n(\cos\theta) = \cos n\theta$, we see that the eigenvalues of A are

$$\lambda_k = 2\cos\left(\frac{(2k-1)\pi}{n}\right), \quad 1 \le k \le n.$$

Note that if n is even, then the equation $T_n(x) = -1$ has $n/2$ double roots. In this case the maximal and minimal eigenvalues of A are

$$\lambda_{max} = 2\cos\left(\frac{\pi}{n}\right), \quad \lambda_{min} = 2\cos\left(\frac{(n-1)\pi}{n}\right) = -2\cos\left(\frac{\pi}{n}\right).$$

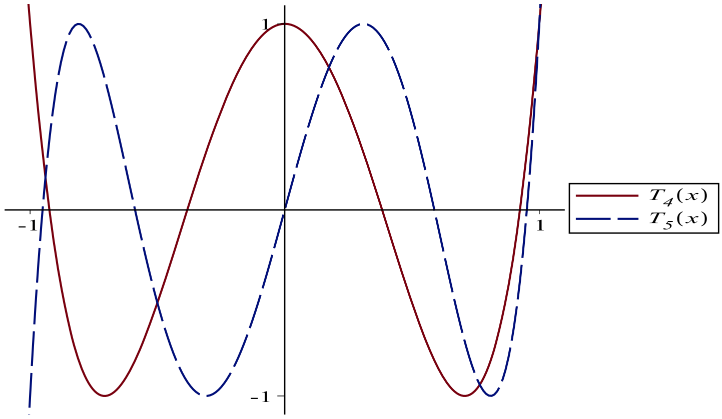

Figure 1. Graphs of the Chebyshev polynomials $T_4(x)$ and $T_5(x)$.

If n is odd, the equation $T_n(x) = -1$ has a single root equal to -1 and $(n-1)/2$ double roots (see Figure 1). In this case,

$$\lambda_{max} = 2\cos\left(\frac{\pi}{n}\right), \ \lambda_{min} = -2$$

and the lemma is proved. ∎

Hence, inequality (2) follows from inequality (3) and Lemma 2.

Remark. For n even, (3) and Lemma 2 imply the inequality

$$\cos\left(\frac{\pi}{n}\right)\sum_{k=1}^{n} x_k^2 \geq \left|\left(\sum_{k=1}^{n-1} x_k x_{k+1}\right) - x_n x_1\right|,$$

which is stronger than inequality (2). For n odd, (3) and Lemma 2 imply the inequality

$$\left(\sum_{k=1}^{2n} x_k x_{k+1}\right) - x_{2n+1}x_1 \geq -\sum_{k=1}^{2n+1} x_k^2,$$

which is obvious since it can be written in the form

$$(x_1 + x_2)^2 + (x_2 + x_3)^2 + \cdots + (x_{2n} + x_{2n+1})^2 + (x_{2n+1} - x_1)^2 \geq 0.$$

4. A GEOMETRIC INEQUALITY. In this section, we obtain an "odd" version of the inequality (1) and we discuss the notion of a semi-regular polygon for polygons with an odd number of sides.

Theorem 1. *Let $A_1, A_2, \ldots, A_{2n+1}$ $(n \geq 1)$ be arbitrary points in the plane, a_k be the length of the segment A_kA_{k+1} $(1 \leq k \leq 2n+1)$, and m_k be the distance between the midpoint of A_kA_{k+1} and the point A_{k+n+1}, where subscripts are taken modulo $2n+1$. Then the following inequality holds:*

$$\sum_{k=1}^{2n+1} a_k^2 \geq 4\tan^2\left(\frac{\pi}{4n+2}\right)\sum_{k=1}^{2n+1} m_k^2. \tag{11}$$

Proof. The desired inequality follows immediately from inequality (1) applied to $4n+2$ points $B_1, B_2, \ldots, B_{4n+2}$ such that $B_{2k} = B_{2k+1} = A_{k+1}$, $1 \le k \le 2n+1$. ■

We now discuss occurrence of equality in (11). Note first that for $n = 1$, (11) is an equality which follows by the median formula in a triangle. For $n \ge 2$, by the results in [**2**] (see identity (2.6)), we know that for a $2n$-gon $A_1A_2\ldots A_{2n}$ equality in (1) is attained if and only if its opposite sides are parallel and

$$\overrightarrow{A_kA_{n+k}} = \frac{\sin(k\pi/n)}{\sin(\pi/n)}\overrightarrow{A_1A_{n+1}} + \frac{\sin((k-1)\pi/n)}{\sin(\pi/n)}\overrightarrow{A_nA_{2n}}\,,\ 1 \le k \le n,$$

where $\overrightarrow{AB}$ denotes the vector from A to B.

If $n \ge 2$, consider $4n+2$ points $B_1, B_2, \ldots, B_{4n+2}$ such that $B_{2k} = B_{2k+1} = A_{k+1}$, $1 \le k \le 2n+1$. In this case, the above conditions reduce to the following equalities:

$$\overrightarrow{A_{k+1}A_{k+n+1}} = \frac{\sin\dfrac{2k\pi}{2n+1}}{\sin\dfrac{\pi}{2n+1}}\overrightarrow{A_1A_{n+2}} + \frac{\sin\dfrac{(2k-1)\pi}{2n+1}}{\sin\dfrac{\pi}{2n+1}}\overrightarrow{A_{n+1}A_1}\,,\ 1 \le k \le n;$$

$$\overrightarrow{A_{k+1}A_{k+n+2}} = \frac{\sin\dfrac{(2k+1)\pi}{2n+1}}{\sin\dfrac{\pi}{2n+1}}\overrightarrow{A_1A_{n+2}} + \frac{\sin\dfrac{2k\pi}{2n+1}}{\sin\dfrac{\pi}{2n+1}}\overrightarrow{A_{n+1}A_1}\,,\ 1 \le k \le n-1.$$

In particular, we obtain the following result.

Corollary. *Any convex pentagon $A_1A_2A_3A_4A_5$ for which equality holds in (11) is obtained from a parallelogram $A_2A_3A_4B$ by taking the vertices A_1 and A_5 on the rays $\overrightarrow{A_4B}$ and $\overrightarrow{A_2B}$, respectively, so that*

$$\frac{A_1A_4}{A_2A_3} = \frac{A_2A_5}{A_3A_4} = 2\cos\frac{\pi}{5}.$$

Proof. Note first that $\sin\frac{2\pi}{5} = 2\sin\frac{\pi}{5}\cos\frac{\pi}{5}$. Hence the identities above imply that for a convex pentagon $A_1A_2A_3A_4A_5$ equality is attained in (11) if and only if

$$\overrightarrow{A_2A_4} = 2\cos\frac{\pi}{5}\overrightarrow{A_1A_4} + \overrightarrow{A_3A_1};$$

$$\overrightarrow{A_2A_5} = 2\cos\frac{\pi}{5}\overrightarrow{A_1A_4} + 2\cos\frac{\pi}{5}\overrightarrow{A_3A_1};$$

$$\overrightarrow{A_3A_5} = \overrightarrow{A_1A_4} + 2\cos\frac{\pi}{5}\overrightarrow{A_3A_1}.$$

Now, it is easy to see that these identities are equivalent to

$$\overrightarrow{A_1A_4} = 2\cos\frac{\pi}{5}\overrightarrow{A_2A_3}\,,\ \overrightarrow{A_2A_5} = 2\cos\frac{\pi}{5}\overrightarrow{A_3A_4}.$$

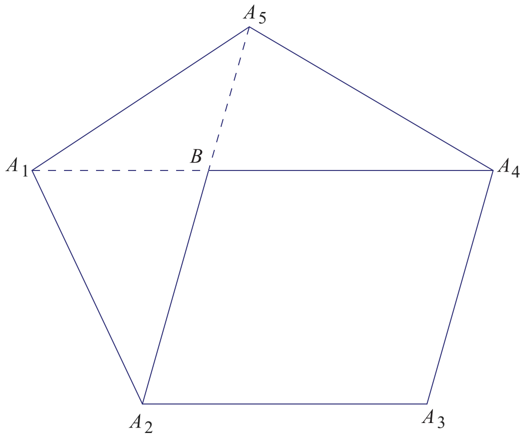

Figure 2. A pentagon satisfying equality in (11).

Hence, $A_1A_4 \parallel A_2A_3$ and $A_2A_5 \parallel A_3A_4$. Let B be the intersection point of the lines A_1A_4 and A_2A_5 (see Figure 2). Consider the parallelogram $A_2A_3A_4B$. Since

$$\frac{A_1A_4}{BA_4} = \frac{A_1A_4}{A_2A_3} = 2\cos\frac{\pi}{5} > 1,$$

the point B lies on the segment A_1A_4. Similarly, B lies on the segment A_2A_5 and the corollary is proved. ∎

Recall that a convex $2n$-gon is called semi-regular [2] if the distance between the midpoints of any two of its opposite sides is equal to $\cot(\pi/2n)/2$ times the sum of their lengths. These polygons have been completely characterized in [2, Theorem 3.1].

Analogously, we call a convex $(2n+1)$-gon $A_1A_2\ldots A_{2n+1}$ semi-regular if the distance between the midpoint of every side A_kA_{k+1} and its opposite vertex A_{k+n+1} is equal to $\cot(\pi/4n+2)/2$ times the length of A_kA_{k+1}. So, it is natural to look for an "odd" analog of [2, Theorem 3.1]. However, as we will see, the "odd" case is very rigid because of the following theorem.

Theorem 2. *Every semi-regular polygon with an odd number of sides is a regular polygon.*

Proof. Our proof is similar to that of Theorem 3.1 in [2]. We will use the following inequality for a triangle.

Lemma 3. *Let ABC be a triangle with $\angle C \geq \pi/n$, and let M be the midpoint of AB. Then*

$$AB \geq 2\tan\left(\frac{\pi}{2n}\right)CM,$$

with equality if and only if $\angle C = \pi/n$ and $CA = CB$ in case $n \geq 3$.

Proof of Lemma 3. The law of cosines together with the AM-GM inequality gives

$$AB^2 = CA^2 + CB^2 - 2CA\cdot CB\cdot\cos\angle C \geq (CA^2+CB^2)\left(1-\cos\left(\frac{\pi}{n}\right)\right).$$

Hence,

$$4CM^2 = 2(CA^2 + CB^2) - AB^2 \leq \frac{2AB^2}{1 - \cos(\frac{\pi}{n})} = \cot^2\left(\frac{\pi}{2n}\right),$$

and Lemma 3 is proved. ■

Now consider a semi-regular $(2n+1)$-gon $A_1A_2\dots A_{2n+1}$. It is easy to see that

$$\sum_{k=1}^{2n+1} \angle A_kA_{k+n+1}A_{k+1} = \pi. \tag{12}$$

Indeed,

$$\angle A_kA_{k+n+1}A_{k+1} = \pi - \angle A_{k+n+1}A_kA_{k+1} - \angle A_{k+n+1}A_{k+1}A_k, \; 1 \leq k \leq n.$$

Note also that

$$\begin{aligned}&\angle A_{k+n+1}A_{k+1}A_k + \angle A_{k+n+2}A_{k+1}A_{k+2}\\ &\quad = \angle A_kA_{k+1}A_{k+2} + \angle A_{k+n+2}A_{k+1}A_{k+n+1}, \; 1 \leq k \leq n.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{k=1}^{2n+1} \angle A_kA_{k+n+1}A_{k+1}\\ &\quad = (2n+1)\pi - \sum_{k=1}^{2n+1} \angle A_kA_{k+1}A_{k+2} - \sum_{k=1}^{2n+1} \angle A_{k+n+2}A_{k+1}A_{k+n+1}.\end{aligned}$$

But

$$\sum_{k=1}^{2n+1} \angle A_kA_{k+1}A_{k+2} = (2n-1)\pi$$

and the identity (12) is proved.

Now it follows from (12) that there is an index l such that

$$\angle A_lA_{l+n+1}A_{l+1} \geq \frac{\pi}{2n+1}.$$

Hence, Lemma 3 implies $A_lA_{l+n+1} = A_{l+1}A_{l+n+1}$ and $\angle A_lA_{l+n+1}A_{l+1} \geq \dfrac{\pi}{2n+1}$. Then,

$$\sum_{k=1, k\neq l}^{2n+1} \angle A_kA_{k+n+1}A_{k+1} = \frac{2n}{2n+1}\pi$$

and proceeding in the same way we conclude that $\angle A_kA_{k+n+1}A_{k+1} = \frac{\pi}{2n+1}$ and $A_kA_{k+n+1} = A_{k+1}A_{k+n+1}$, $1 \leq k \leq 2n+1$. Now, it follows easily that all diagonals A_kA_{k+n+1} are equal, so the triangles $A_kA_{k+1}A_{k+n+1}$ are all congruent. Hence, all sides and all angles of $A_1 A_2 \ldots A_{2n+1}$ are equal and, therefore, we have a regular $(2n + 1)$-gon. ■

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